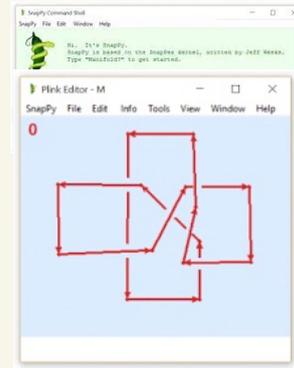
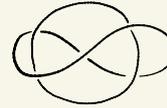


Reminder of Knot theory so far:

- Knots, links and their diagrams
- Knot invariants: tricolorability, Jones polynomial.
- Computational Knot theory:

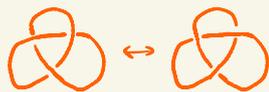


Today: "conjectures fuel mathematical fields"

6. The value of conjectures

In any math field:

Exploration \rightsquigarrow Pattern \rightsquigarrow Conjecture \rightsquigarrow Proof (understanding)



$$-t^4 + t^3 + t \leftrightarrow -\frac{1}{t^4} + \frac{1}{t^3} + \frac{1}{t}$$

$$t \leftrightarrow \frac{1}{t}$$

$$\langle X \rangle = A \langle \rangle (\rangle + A^{-1} \langle \rangle (\langle)$$

↓

$$\langle X \rangle = A^{-1} \langle \rangle (\rangle + A \langle \rangle (\langle)$$

... \rightsquigarrow More exploration (more informed this time)

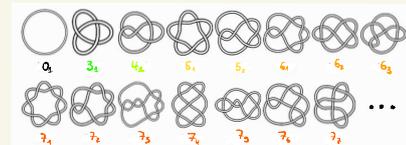
In knot theory history:



Peter Tait (1831-1901). Scottish physicist who thought maybe

Periodic Table of the Elements

1	2											3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18																																														
1	H																	2	He																																																						
3	Li	4	Be											5	B	6	C	7	N	8	O	9	F	10	Ne																																																
11	Na	12	Mg											13	Al	14	Si	15	P	16	S	17	Cl	18	Ar																																																
19	K	20	Ca											21	Sc	22	Ti	23	V	24	Cr	25	Mn	26	Fe	27	Co	28	Ni	29	Cu	30	Zn	31	Ga	32	Ge	33	As	34	Se	35	Br	36	Kr																												
37	Rb	38	Sr											39	Y	40	Zr	41	Nb	42	Hf	43	Ta	44	Hf	45	Ru	46	Rh	47	Pd	48	Ag	49	Cd	50	In	51	Sn	52	Sb	53	Te	54	Xe																												
55	Cs	56	Ba											57	La	58	Ce	59	Pr	60	Nd	61	Pm	62	Sm	63	Eu	64	Gd	65	Tm	66	Dy	67	Ho	68	Er	69	Tm	70	Yb	71	Lu	72	Hf	73	Ta	74	W	75	Re	76	Os	77	Ir	78	Pt	79	Au	80	Hg	81	Tl	82	Pb	83	Bi	84	Po	85	At	86	Rn
87	Fr	88	Ra											89	Ac	90	Th	91	Pa	92	U	93	Np	94	Pu	95	Am	96	Cm	97	Bk	98	Cf	99	Es	100	Fm	101	Md	102	No	103	Lr	104	Rf	105	Db	106	Sg	107	Bh	108	Hs	109	Mt	110	Ds	111	Rg	112	Cn	113	Nh	114	Fl	115	Mc	116	Lv	117	Ts	118	Og



(Spoiler: he was wrong)

Anyhow, he started the modern study of knots.

After much exploration, he posed three conjectures:

- Tait's first conjecture:

Any **reduced alternating** link diagram has the smallest number of crossings

- Tait's second conjecture:

Any two reduced **reduced alternating connected** diagrams for the same link have the same writhe.

- Tait's third conjecture:

Any two reduced **reduced alternating** diagrams for the same link are related by a sequence of **flypes**.

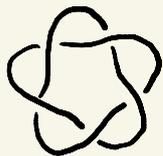
No one knew how to prove these until the discovery of the **Jones polynomial** (1984)

Proofs: 1987 (first and second), 1991 (third).

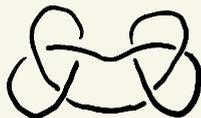
Alternating links:

Definition: A link diagram is **alternating** if following each strand's path, overcrossings and undercrossings alternate. A link is **alternating** if it has an alternating link diagram.

Example:



Nonexample:



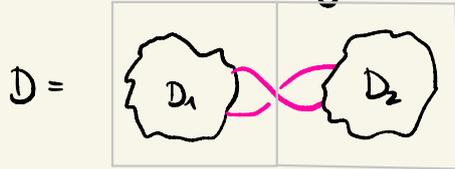
Fact: most prime knots are alternating:

# crossings	# prime knots	# alternating prime knots
3	1	1
4	1	1
5	2	2
6	3	3
7	7	7
8	21	18
9	49	41
10	165	123

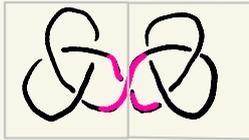
Q?

Reducibility, connectedness.

Definition: A link diagram D is **reduced** if we cannot find two arcs such that



Nonexample:



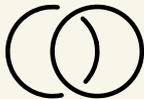
Example:



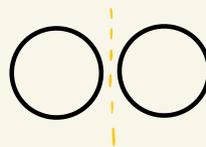
Notice if we had some twist  then we could untwist it, so then the trefoil could be represented with two crossings, which we can't do.

Definition: a link diagram is **connected** if ignoring (under/over) crossings, the diagram is connected.

Example:



Nonexample:



Q?

The first two conjectures, revisited:

- Tait's first conjecture:

Any reduced alternating link diagram has the smallest number of crossings

Application:



cannot be drawn with ≤ 5 crossings.

- Tait's second conjecture:

Any two reduced alternating connected diagrams for the same link have the same writhe.

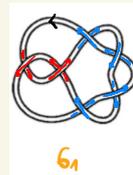
Application:



\neq



Proof:



writhe = -2,



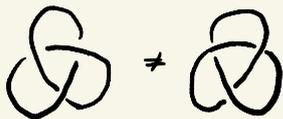
writhe = 0

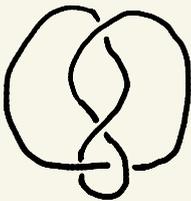
Q?

Amphichirality

Definition: A link is **amphichiral** if it is equivalent to its mirror image.

In other words: changing all crossings $+1 \leftrightarrow -1$ gives an equivalent link.

Nonexample: 

Example: 

<https://www.mi.sanu.ac.rs/vismath/sl/125.htm>

Q? Exercises