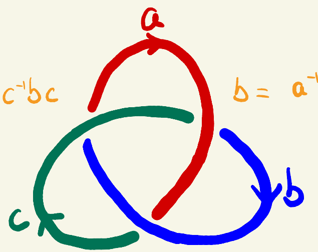
Reminder of last time:

- Groups by generators and relations

$$G = \langle \underbrace{a,b}_{\text{generators}} \mid \underbrace{ab=ba,\ a^2=1,\ b^2=1}_{\text{relations}} \rangle$$

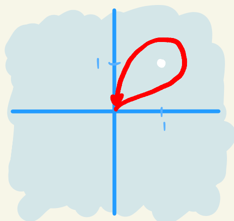
- Fundamental group of a link



$a = c^{-1}bc$ $\quad b = a^{-1}ca$ $\quad c = b^{-1}ab$

- Fundamental group of a space

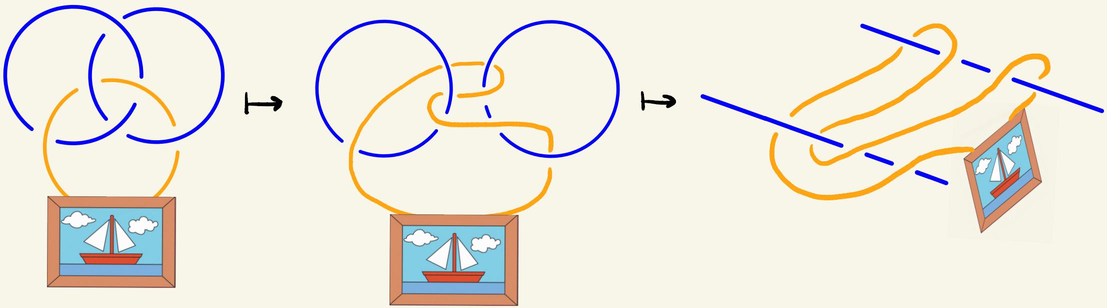
$$\pi_1(X) = \{\text{paths on } X \text{ starting and ending at } x_0,\ \text{up to homotopy}\}$$

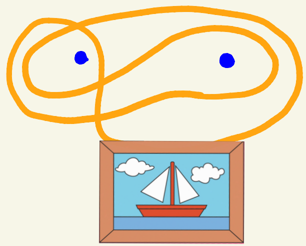


15. A party trick: Brunnian links

Poll: How to make the picture fall?

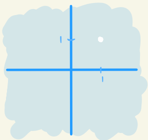
Explanation: Borromean link!

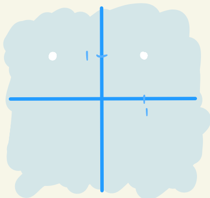




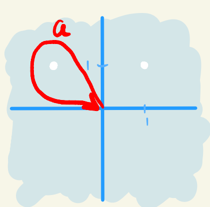
Q: How to generalize this trick?

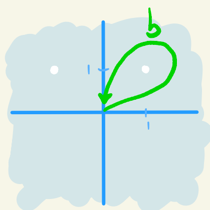
A: Last week's machinery! Let's solve the 2 pin case first

Recall $\pi_1\Big($  $\Big)$

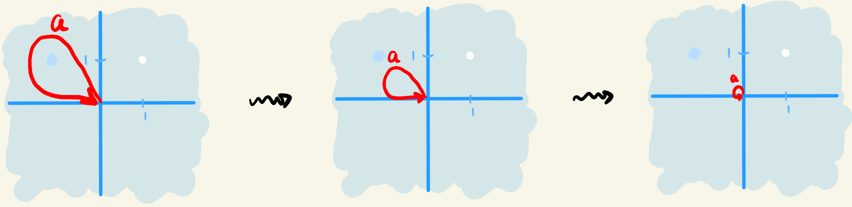
Now consider $X =$ 

We have two generators:





"Removing a pin" corresponds to adding a relation:



$\Rightarrow \quad a = 1$

Similarly, removing the other pin gives $b = 1$.

So we're looking for a word $\neq 1$ in a, b such that:

- If we set $a=1$, the word simplifies to 1.
- If we set $b=1$, the word simplifies to 1.

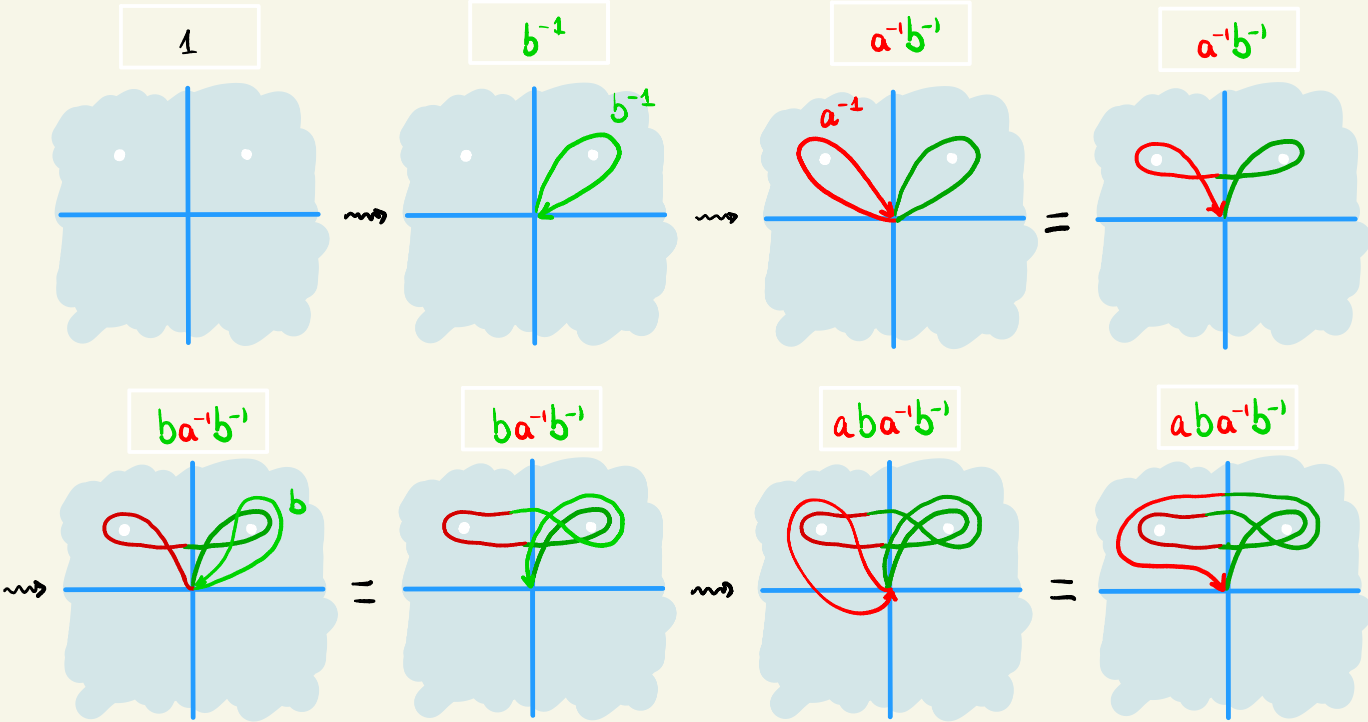
Guesses?

Enter the commutator: $[a, b] = aba^{-1}b^{-1}$

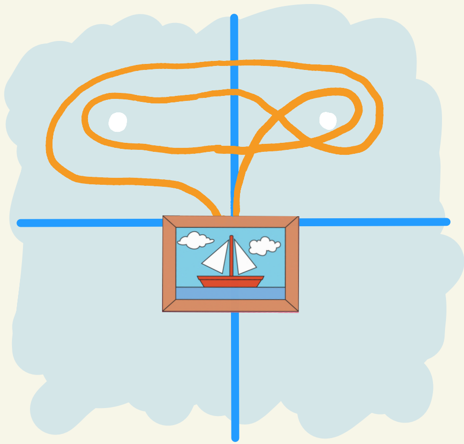
$$[a,b] \xrightarrow{a=1} 1b1^{-1}b^{-1} = 1$$

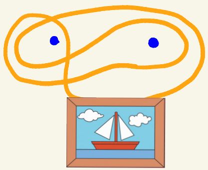
$$[a,b] \xrightarrow{b=1} a1a^{-1}1^{-1} = 1$$

Picture:



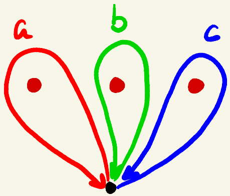
Upshot:



Remark: Same as  , but flipped.

The point: this approach generalizes:

Three pins:



$$[[a,b],c] = [a,b]\,c\,[a,b]^{-1}c^{-1} = aba^{-1}b^{-1}cbab^{-1}a^{-1}c^{-1}$$

This works:

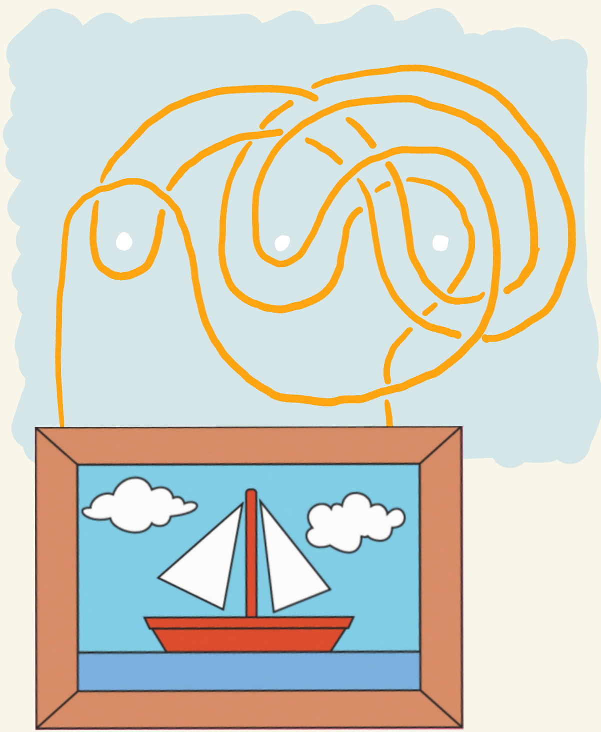
$$[[a,b],c] \xrightarrow{a=1} [[1,b],c] = [1,c] = 1$$

$$[[a,b],c] \xrightarrow{b=1} [[a,1],c] = [1,c] = 1$$

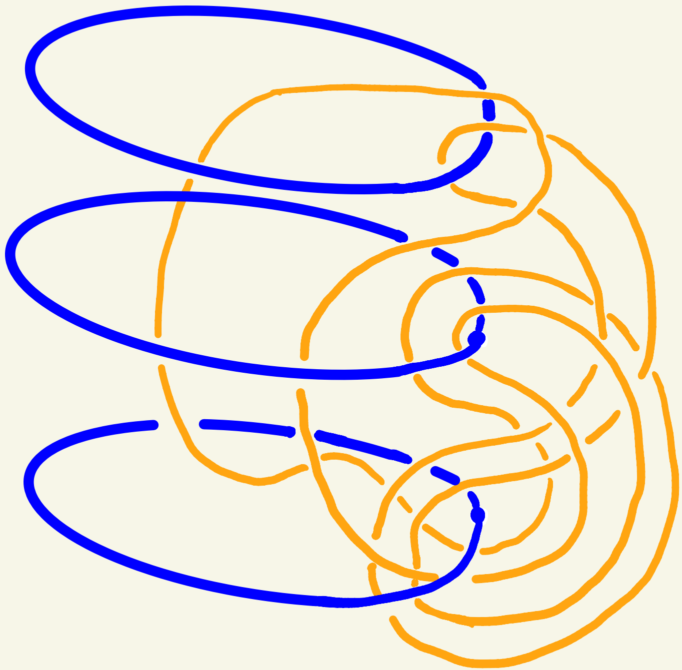
$$[[a,b],c] \xrightarrow{c=1} [[a,b],1] = 1$$

Q?

How this would look like:



... and back to links:



Brunnian link with 4 components

In the exercises: you will find Brunnian links yourselves!