The char p story

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What's an algebraic group

Let k be a field. An algebraic group G is a scheme of finite type over k together with morphisms of schemes

$$\mu : G \times G \to G \quad \iota : G \to G$$

satisfying the group axioms (i.e. a group object in Sch^0/k). Alternatively, to G we associate the functor

$$\tilde{G} : \mathrm{Alg}_k^0 \to \mathrm{Grp}$$

$$R \to G(R)$$

$$= \mathrm{Hom}_{\mathrm{Sch}/k}(\mathrm{Spec}(R), G)$$

and every such representable functor is representable by an algebraic group, which is determined up to isomorphism.

Linear algebraic groups

Some structure theory:

Theorem

Every algebraic group has a natural map $G \to \mathrm{Spec}(\mathcal{O}(G))$ whose kernel is an anti-affine* algebraic group.

Corollary

The rep theory of algebraic groups reduces to the affine case.

$$\text{Affine algebraic groups}/k \underset{\mathrm{Spec}}{\overset{\mathcal{O}}{\rightleftarrows}} \text{Hopf algebras}/k$$

Fact: Every affine algebraic group G has a finite dimensional faithful representation, that is, $G \leq \mathrm{Spec}(\mathcal{O}(\mathrm{GL}_n))$.

Linear algebraic groups

In what follows, assume G is also a connected variety.

Definition

Let $R(G)$ be the largest connected solvable normal subgroup variety of G. We say G is semisimple if $R(G)$ is trivial.

Similarly, G is reductive if $R_u(G)$ is trivial. Split* reductive groups have maximal tori, Borels, root systems... In fact almost all Lie theory carries over. For instance:

Theorem (Chevalley)

Split simple algebraic groups $\leftrightarrow$ Dynkin diagrams.

What about the simple modules?

Again {f.d. simples} $\leftrightarrow$ {dominant weights}.
However, in char p these are no longer tensor-indecomposable.

Definition

Let F_p be the Frobenius map $G \mapsto G$ sends $r \in G(R)$ to r^p.

For $q = p^r$, define the p-restricted (integral) dominant weights $\Lambda_q^+ = \{\lambda \in \Lambda : 0 \leq \langle \lambda, \alpha \rangle < q\} \subset \Lambda^+$.
For M be a G-module, let $M^{[i]}$ be M with twisted action: $gv := F_p^i(g)v$.
Writing $\lambda = \lambda_0 + \lambda_1 p + ... + \lambda_{m-1} p^{m-1}$ (λ_i p-restricted), we have:

Steinberg's Tensor Product Theorem

$$L(\lambda) \cong L(\lambda_0) \otimes_k L(\lambda_1)^{[1]} \otimes_k ... \otimes_k L(\lambda_{m-1})^{[m-1]}$$

Example

Let $G = \mathrm{SL}_2$ and consider the module $S^3(V) = \{x^3, x^2y, xy^2, y^3\}$ where V is the natural representation.
This has highest weight 3. Believe that this is $L(3)$.
We show that this equals $M = L(1) \otimes L(1)^{[1]}$, as predicted by Steinberg.

- A submodule of M is of the form $L(1) \otimes V$ ($\mathfrak{g}$-submodule).
- $V = \mathrm{Hom}_{\mathfrak{g}}(L(1), L(1) \otimes V)$
  $\hookrightarrow \mathrm{Hom}_{\mathfrak{g}}(L(1), L(1) \otimes L(1)^{[1]})$
  $= L(1)^{[1]}$ (as G-modules) $\square$

The general case for SL_2 follows by induction.

But why bother?

Note that

$$G^{F_q} = \{g \in G(k) : F_q(g) = g\}$$

is a finite group. The finite groups of this form are called **reductive groups of Lie type** $\rightsquigarrow$ most finite simple groups
Their modular representations are of great interest (local-global conjectures...).

Restriction Theorem (Brauer-Nesbitt)

- For each $\lambda \in X^+$ p^r-restricted, $L(\lambda)$ is simple as a kG^{F_q}-module, and these are pairwise nonisomorphic.
- Every kG^{F_q}-module arises in this way.

Example

Modules of A_5 over char 2.

$A_5 \cong \mathrm{SL}_2(4)$ so let $G = \mathrm{SL}_2$ over $k = \overline{\mathbb{F}_2}$.
2-restricted weights: $\{0, 1\}$
The restriction theorem implies:

Simple kA_5-modules:

- $L(0) \otimes L(0)^{[1]}$
- $L(0) \otimes L(1)^{[1]}$
- $L(1) \otimes L(0)^{[1]}$
- $L(1) \otimes L(1)^{[1]}$

$\rightsquigarrow$

g	$()$	(123)	(12345)	(13524)
ϕ_1	1	1	1	1
ϕ_{2a}	2	-1	$\zeta_5 + \zeta_5^4$	$\zeta_5^2 + \zeta_5^3$
ϕ_{2b}	2	-1	$\zeta_5^2 + \zeta_5^3$	$\zeta_5 + \zeta_5^4$
ϕ_4	4	1	-1	-1

Weyl modules

Question

How to construct $L(\lambda)$?

Two ways to define the Weyl module:

- Take a $\mathbb{Z}$-form for $L_{\mathbb{C}}(\lambda)$ and set $V(\lambda) = L_{\mathbb{Z}}(\lambda) \otimes_{\mathbb{Z}} k$. Write $\lambda^* = -w_0\lambda$.
- (Dual Weyl module) Define $W(\lambda) = \mathrm{ind}_{B^-}^G k_{-\lambda^*} = H^0(G/B^-, \mathcal{L}(\lambda))$, where $\mathcal{L}(\lambda)$ is the line bundle on the flag variety associated to the B^--module k_λ. Then $V(\lambda) = W(\lambda^*)^*$.

Theorem

$L(\lambda)$ is the unique simple quotient of $V(\lambda)$, as well as the unique simple submodule of $W(\lambda)$.

Example

For SL_2, $W(p) = S^p(V) = k\{x^p, x^{p-1}y, ..., y^p\}$. This has a submodule $k\{x^p, y^p\} = L(p) = L(1)^{[p]}$.

Kempf's vanishing theorem

Denote $H^i(\lambda) = H^i(G/B^-, \mathcal{L}(\lambda))$
As in char 0, the Euler characteristic $\chi(\lambda) = \sum_{i\geq 0} \mathrm{ch} H^i(\lambda)$ is given by the Weyl character formula.

Kempf's vanishing theorem

If $i > 0$, then $H^i(\lambda) = 0$.

Moreover $H^0(\lambda^*)^*$ has the "Verma universality property" that gives an embedding $V(\lambda) \hookrightarrow H^0(\lambda^*)^*$ so by character comparison these are isomorphic.

Rank of the contravariant form

Computing $\mathrm{ch} L(\lambda)$ is not hard (one at a time).
In fact $\dim L(\lambda)_\mu = \mathrm{rank}_p(T|_{V(\lambda)_\mu})$, the contravariant form.

Example

If $G = \mathrm{SL}_{n+1}$ and $\lambda = \lambda_1 + \lambda_n = \alpha_1 + \ldots + \alpha_n$, then the weights are $W\lambda \cup \{0\}$. A basis for $V(\lambda)_0$ is given by $\{f_{\alpha_1+\ldots+\alpha_i} f_{\alpha_{i+1}+\ldots+\alpha_n} v^+\}_{i=1\ldots n}$ and the contravariant form is

$$\begin{pmatrix} 2 & 1 & 1 & 1 & 1 \\ 1 & 2 & & 1 & 1 \\ \vdots & & \ddots & & \vdots \\ 1 & 1 & & 2 & 1 \\ 1 & 1 & \ldots & 1 & 2 \end{pmatrix} \rightsquigarrow \begin{pmatrix} n+1 & 0 & \ldots & 0 \\ 0 & 1 & & \vdots \\ \vdots & & \ddots & \\ 0 & \ldots & & 1 \end{pmatrix}$$

So $\mathrm{ch} L(\lambda) = \mathrm{ch} V(\lambda) - \epsilon_{p,n+1} e(0)$. (Multiples of I_{n+1} lie in $\mathfrak{sl}_{n+1}$.)

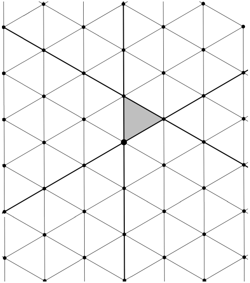
Alcoves and the linkage principle

Recall the BGG theorem: $[M(\lambda) : L(\mu)] \neq 0 \leftrightarrow \mu \uparrow \lambda$
The affine Weyl group is $W_p = W \ltimes \Phi^\vee$ and the dot action is defined as

- $w \cdot_p \lambda = w(\lambda + \rho) - \rho$
- $p\alpha \cdot_p \lambda = t + p\alpha$

$\rightsquigarrow$ Infinitesimal block decompositions
Define $\mu \uparrow_p \lambda$ by $\mu = (s_{\alpha_1} ... s_{\alpha_m}) \cdot_p \lambda$

and $(s_{\alpha_i} ... s_{\alpha_m}) \leq (s_{\alpha_{i-1}} ... s_{\alpha_m})$



Theorem (Strong linkage principle)

$[V(\lambda) : L(\mu)] \neq 0 \leftrightarrow \mu \uparrow_p \lambda$

The Steinberg module

Application

Linkage principle $\implies$ If $\lambda \in \overline{C^0}$ then $V(\lambda) = L(\lambda)$

The largest dimensional module in C^0 is
$\mathrm{St} := L((p-1)\lambda) = V((p-1)\lambda)$, of dimension $p^{|\Phi^+|}$.
By Steinberg's tensor product theorem,
$\mathrm{St}_r := L((p^r-1)\lambda) = \mathrm{St} \otimes \mathrm{St}^{[1]} \otimes \ldots \otimes \mathrm{St}^{[r-1]} = V((p^r-1)\lambda)$
$\implies$ St_r is the largest simple module for $G_r := G^{F_p^r}$.
St_r plays a central role in the finite dimensional theory.
Example: if $g \in G_r$ is a p'-element then

$$\chi_{\mathrm{St}_r}(g) = \begin{cases} |C_{G_r}(g)|_p, & \text{if } g \text{ is } p' \\ 0, & \text{otherwise} \end{cases}$$

Jantzen's p-sum formula

Analog of Jantzen's sum formula, and similar technique.

Jantzen's p-sum formula

$$\sum_{i>0} \mathrm{ch} V(\lambda)^i = \sum_{\alpha>0} \sum_{0<cp<\langle \lambda+\rho, \alpha^\vee \rangle} \nu_p(cp) \chi(s_{\alpha,cp} \cdot \lambda)$$

As in the category $\mathcal{O}$ case, this gives $\mathrm{ch} L(\lambda)$ for all p-restricted λ but for small rank: A_2, B_2, C_2, G_2, A_3.

Translation and reflection functors

In $\mathrm{Rep}(G) = \bigoplus \mathrm{Rep}_\lambda(G)$ we can also define translation functors: $T_\mu^\lambda = \mathrm{pr}_\lambda(L(\nu) \otimes -)$ where ν is the dominant weight in the orbit of $\lambda - \mu$.

These give equivalences of categories similar to the ones for category $\mathcal{O}$.

A glimpse into categorification

Consider $\mathrm{Rep}_0(G)$. For each s_i choose μ "on the s_i-wall" and define $\Theta_i = T_\nu^0 T_0^\nu$. Then taking K_0 and identifying $[V(w \cdot_p 0)]$ with $1 \otimes w \in sgn \otimes_{\mathbb{Z}W} \mathbb{Z}W_p$, we get $[\Theta_i] = [(1 + s_i)]$.

$\rightsquigarrow$ $\mathrm{Rep}_0(G)$ categorifies the anti-spherical module for W_p.

Tilting modules

Definition

A G-module is tilting if it has a Weyl and a dual Weyl filtrations.

Fact: there is one indecomposable tilting $T(\lambda)$ for each highest weight λ of $\mathrm{Rep}_0(G)$.
The problem of finding $\{\mathrm{ch}T(\lambda)\}$ is equivalent to that of finding $\{\mathrm{ch}L(\lambda)\}$:

Propostion

Suppose $p \geq 2h-2$. Then if λ is p-restricted

$$(T(\tilde{\lambda}) : V(\mu)) = [V(\mu) : L(\lambda)]$$

where $\tilde{\lambda} = 2(p-1)\rho + w_0\lambda$

Tilting modules

Another similarity between $T(\lambda)$ and $P(\lambda)$:

Theorem

Let $w = s_1 ... s_t$ be a reduced expression. Then $\lambda = w \cdot_p 0$ appears as a summand of $\Theta_{s_1} ... \Theta_{s_t} T(0)$ (again tilting) with multiplicity 1. Every other summand $T(\mu)$ has $\mu < \lambda$.

Category $\mathcal{O}$?

- Rational representations are defined to be finite dimensional.
- Weyl module has the role of the Verma module (among other parallel notions).
- $\mathrm{Rep}(G) \not\leftrightarrow \mathrm{Rep}(\mathfrak{g}) = \mathrm{Rep}(\mathcal{U}(\mathfrak{g}))$
  $\mathrm{Rep}(G) \leftrightarrow \mathrm{Rep}(\mathrm{Dist}(G))$

Definition

Let X be an affine scheme over k, and $x \in X(k)$. Define $\mathrm{Dist}_n(X, x) = \mathrm{Hom}_k(\mathcal{O}_{X,x}/\mathfrak{m}_x^{n+1}, k)$. The **algebra of distributions** with support at x is the algebra $\mathrm{Dist}(X) := \bigcup_{\geq 0} \mathrm{Dist}_n(X, x)$.
If G is an algebraic group, $\mathrm{Dist}(G) := \mathrm{Dist}(G, 1)$

This is a filtered associative algebra over k.

Example

Take the origin of the affine line $x \in \mathbb{A}^1 = \mathrm{Spec}(k[t]])$. Then
$\mathrm{Dist}_n(X, x) = \mathrm{Hom}_k(k[t]_{(t)}/(t)^n, k) = \mathrm{Hom}_k(k[t]/(t)^n, k)$
This has a basis γ_r sending $t^m \mapsto \delta_{r,m}$. If $\mathrm{char}(k) = 0$, we can identify $\gamma_r = \frac{1}{r!}(\frac{\partial}{\partial t})^r$.
So $\mathrm{Dist}(\mathbb{A}^1, x)$ consists of derivations of any order.

In general the associated graded has pieces $(\mathfrak{m}_x/\mathfrak{m}_x^2)^*$, $(\mathfrak{m}_x^2/\mathfrak{m}_x^3)^*$...
Fact: the divided powers version of the PBW basis is a -form for $\mathrm{Dist}(G)$.
Problems: $\mathrm{Dist}(G)$ is not Noetherian, the center is unknown...

Humphreys:

"There is no likely analogue of the BGG category for the hyperalgebra [...] my current understanding is that the char p theory for G is essentially finite dimensional and requires deep geometry to resolve."

Lusztig's conjecture

Question

Is there an analog of the KL conjecture?

Let $\lambda = -w \cdot_p 0$.
Jantzen's condition: $\langle w\rho, \alpha_0^\vee \rangle \leq p(p-h+2)$, where h is the Coxeter number*.

Lusztig's conjecture (1979)

Assume $p \geq h$ and w as above. Then

$$\mathrm{ch}L(\lambda) = \sum_{y \leq w} (1)^{l(w)-l(y)} P_{y,w}(1) \mathrm{ch}V(-y \cdot_p 0)$$

where $P_{y,w}$ are the KL polynomials.

p-Kazhdan Lusztig polynomials

For p potentially very large the conjecture was proven by Andersen-Jantzen-Soergel (1994).
However, the conjecture was proven to be **false** (even for p exponential in the rank) in 2016 by Williamson's landmark paper *Schubert calculus and torsion explosion*.

Conjecture (Riche-Williamson, 2018)

Assume $p > h$. Then $(T(w \cdot_p 0) : V(y \cdot_p 0)) = P^p_{y,w}(1)$

They prove it in type A.

Theorem (Riche-Williamson (after work by Achar, Makisumi), 2019)

The above conjecture holds.

Billiards conjecture

Problem

The p-KL are hard to compute. *"This is not the end of the story"*

The tilting characters appear to have some deep structure, according to Lusztig-Williamson (2017), based on some computations for SL_3.

Lusztig-Williamson

"The conjecture can be interpreted as saying these characters are governed by a discrete dynamical system ("billiards bouncing in alcoves")"

`https://www.youtube.com/watch?v=Ru0Zys1Vvq4`

(Main) references

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- Jens Carsten Jantzen, *Representations of algebraic groups*, second ed., 2003.
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Thanks!