Symmetric groups and the Heisenberg category

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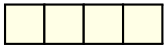
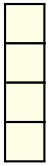
χ_{triv}  $\sum_{I: i_1 \leq i_2 \leq i_3 \leq i_4} x_{i_1} x_{i_2} x_{i_3} x_{i_4}$

χ_{sgn}  $\sum_{I: i_1 < i_2 < i_3 < i_4} x_{i_1} x_{i_2} x_{i_3} x_{i_4}$

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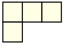
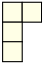
χ_{std}  $\sum_{\substack{I: i_1 \leq i_2 \leq i_3 \\ i_1 < i_4}} x_{i_1} x_{i_2} x_{i_3} x_{i_4}$

$\chi_{\text{std}} \otimes \chi_{\text{sgn}}$  $\sum_{\substack{I: i_1 \leq i_2 \\ i_1 < i_3 < i_4}} x_{i_1} x_{i_2} x_{i_3} x_{i_4}$

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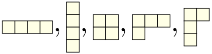
$K_0(\mathbb{C}S_4\text{-mod}) = \mathbb{Z} \cdot \Big\{$  $\Big\}$

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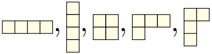
$K_0(\mathbb{C}S_4\text{-mod}) = \mathbb{Z} \cdot \{$  $\}$

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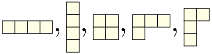
$K_0(\mathbb{C}S_4\text{-mod}) = \mathbb{Z} \cdot \{$  $\}$

Since $\mathrm{Vect_k}^{\mathrm{fd}}$ is monoidal (has $\otimes$), its K_0 is a ring. The multiplication agrees with that of $\mathbb{Z}$: $[V \otimes W] = [V][W]$. This is a first example of **categorification**.

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Theorem

The map ψ is an isomorphism of $\mathbb{C}$-algebras.

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Definition

The Heisenberg algebra Heis is the $\mathbb{C}$-algebra generated by a_m, b_m subject to $a_m b_n = b_n a_m + b_{n-1} a_{m-1}$, with $a_0 = b_0 = 1$.

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Heis acts on $\bigoplus_{n\geq 0} \mathbb{C}S_n$-mod via

$$a_m \mapsto \bigoplus_n \mathrm{Res}_{S_n}^{S_{n+m}} \left(\boxed{1}\boxed{2}\boxed{\ldots}\boxed{m} \otimes -\right), \; b_m \mapsto \bigoplus_n \mathrm{Ind}_{S_n}^{S_{n+m}} \left(\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline \ldots \\ \hline m \\ \hline \end{array} \otimes -\right)$$

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Passing to $K(-)$, this is the **Fock space representation** of Heis on Sym.

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Khovanov (2010): diagrammatic construction of $\mathcal{H}eis$.

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These act diagonally with eigenvalues in $\mathbb{F}_p$, and we can decompose $\mathrm{Ind} = \bigoplus_{i \in \mathbb{F}_p} \mathrm{Ind}_i$ and $\mathrm{Res} = \bigoplus_{i \in \mathbb{F}_p} \mathrm{Res}_i$.

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Broué's abelian defect group conjecture holds for the symmetric groups.

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To reconcile this point of view with (more general) Heisenberg categorification, see Brundan, Savage and Webster's *Heisenberg and Kac-Moody categorical actions* (2020).

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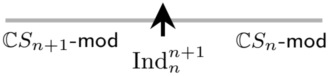

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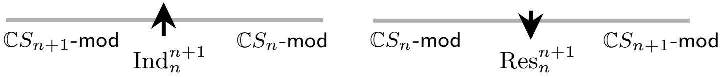

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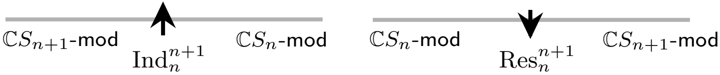

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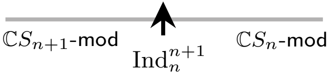

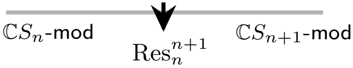

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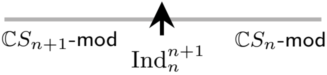

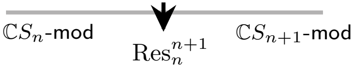

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Maps between functors are here **bimodule maps**.

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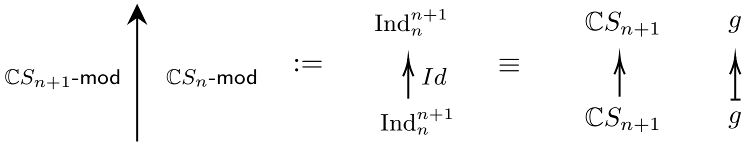

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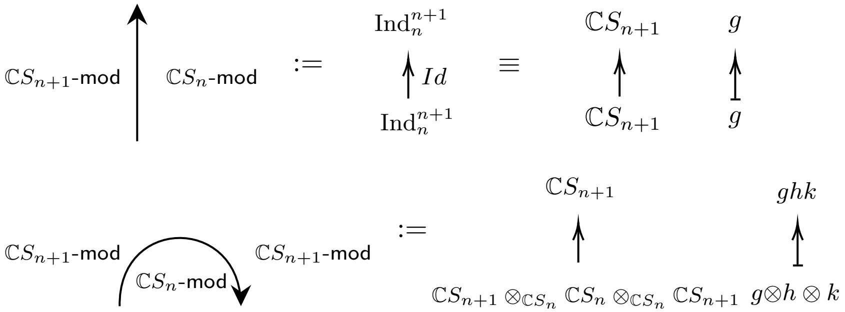

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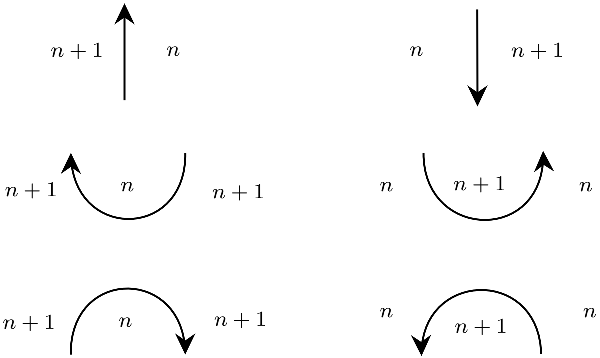

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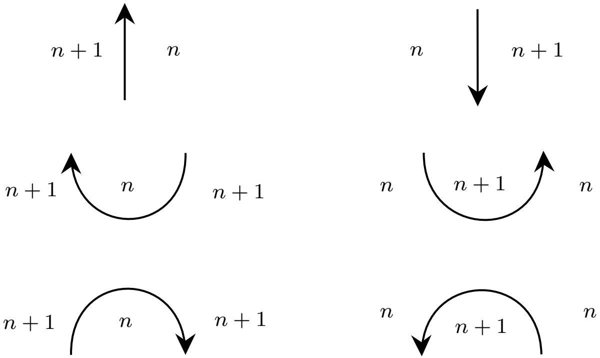

These maps are the biadjointness endomorphisms of Ind_n^{n+1} and Res_n^{n+1}.

Isotopy relations

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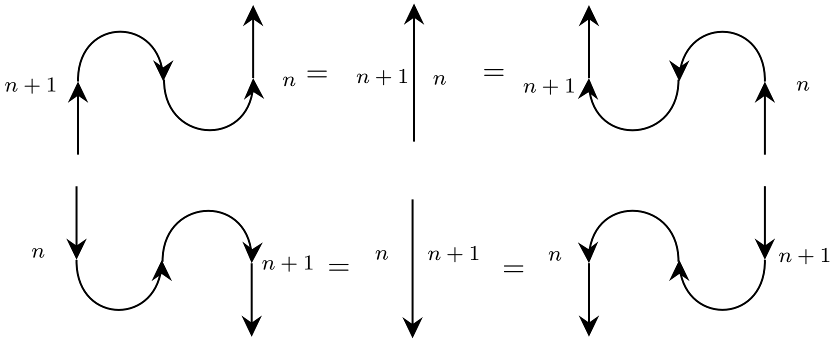

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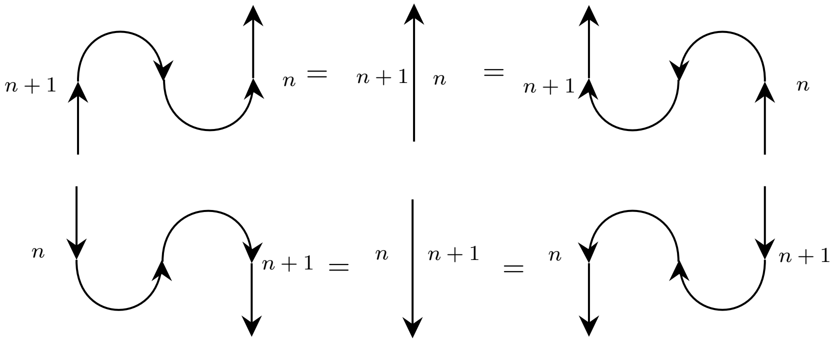

These equalities are equivalent to the statement: Ind_n^{n+1} and Res_n^{n+1} are biadjoint.

More maps

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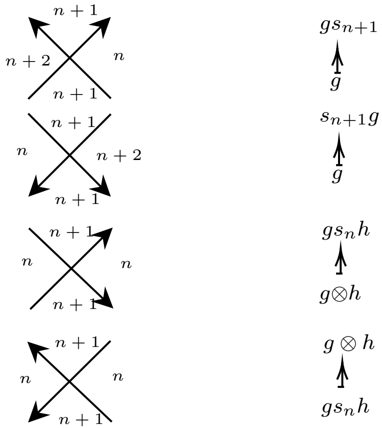
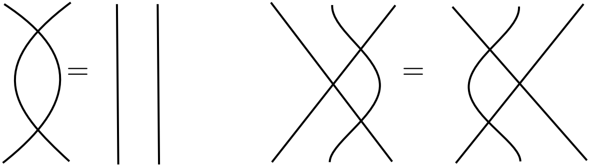


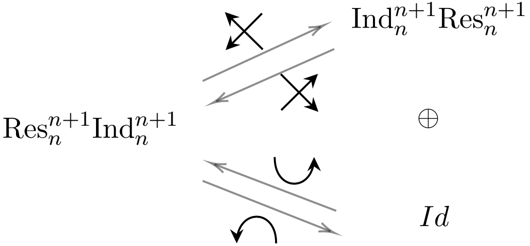
Cross relations

The first two clearly satisfy the "quadratic" and "braid" relations:



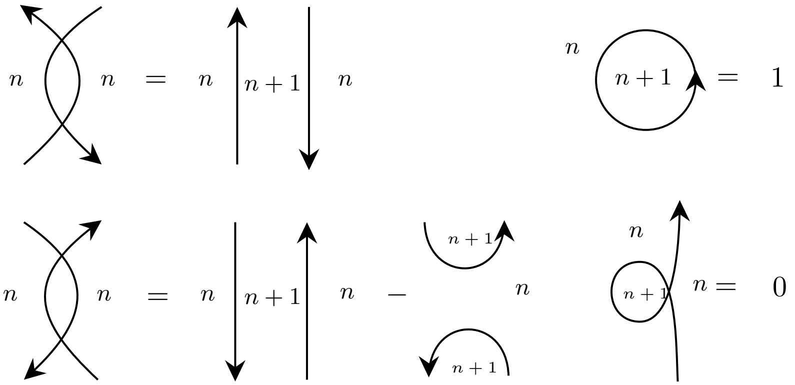
Cross relations

The other two realize the isomorphism in Mackey's Theorem:



Cross relations

The different compositions yield “Mackey” relations



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Theorem (Khovanov, 2010)

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The algebra $\mathrm{Heis}_{\mathbb{Z}}$ *embeds in the Grothendieck ring of* $\mathcal{H}eis$.

Theorem (Brundan, Savage, Webster, 2018)

This embedding is in fact an isomorphism.

Further developments

So far we have been considering induction from S_n to S_{n+1}

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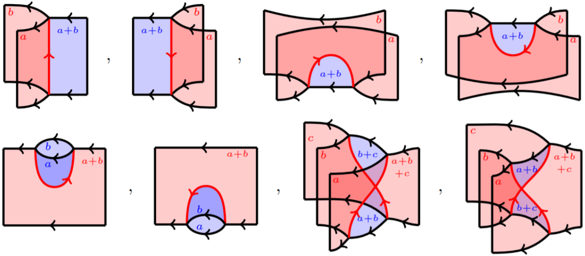

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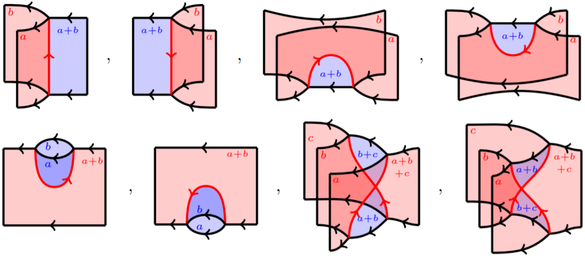

Developing this theory is an open question.

Thanks for your attention!

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- Mikhail Khovanov, *Heisenberg algebra and a graphical calculus*, 2010.
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