# Introduction to Hecke algebras and Affine Hecke algebras

## 1. Motivation (Hecke algebras in nature)

- Definition

A Coxeter system $(W, S)$ is a group and a finite set $S \subset W$ such that $W = \langle S \mid R \rangle$, where the set of relations is:

- $s^2 = 1$ $\forall s \in S$ "quadratic"
- $\underbrace{sts\ldots}_{m_{st}} = \underbrace{tst\ldots t}_{m_{st}}$ $\forall s,t \in S$ "braid"

Example: $W$ = Weyl group, $S$ = simple reflections

The Hecke algebra associated to $(W, S)$ is the unital associative algebra $H = H(W)$ over $\mathbb{Z}[v, v^{-1}]$ generated by the symbols $\{ \delta_s : s \in S \}$ such that

- $\delta_s^2 = (v^{-1} - v)\delta_s + 1$ "quadratic" $\leftrightarrow (\delta_s - v^{-1})(\delta_s + v) = 0$
- $\underbrace{\delta_s \delta_t \cdots}_{m_{st}} = \underbrace{\delta_t \delta_s \cdots}_{m_{st}}$ "braid"

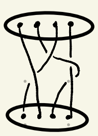
Note that specializing to $v = 1$ one gets the group algebra $\mathbb{C}[W]$

But why?

- Braid groups

The Braid group is the group generated by $\{T_s : s \in S\}$ subject only to the braid relations.

$$B_n \to H \to W$$

Type A: 

Representations of the braid group that factor through $W$ have $\varphi(T_s)^2 = 1$.
Consider representations that satisfy the deformed $\varphi(T_s)^2 + p\varphi(T_s) + r \underset{\text{scale}}{\rightsquigarrow} \varphi(T_s)^2 = (q^{-1} - q)\varphi(T_s) + 1$

(many choices for a presentation, all isom.)

- Number theory

$(G, K)$ $\Rightarrow$ $H(G/\!/K)$ = $(K \times K)$-invariant continuous functions $G \to \mathbb{C}$ of compact support.

$G$: unimodular, locally compact top group; $K$: closed subgp

Algebra structure: convolution $(u, v) \mapsto \int_G u(g)\, v(g^{-1}x)\, dg$

Example: $G = GL_2(\mathbb{Q})$, $K = GL_2(\mathbb{Z})$ $\rightsquigarrow$ $H(G/\!/K)$ = ring of Hecke operators on modular forms

(hence "Hecke", although it was Iwahori who introduced them)

- **Finite groups**

  Consider a finite group $G \geq B$, and an irrep $\psi$ of $H$.

Now how does $\mathrm{Ind}_B^G(\psi)$ decompose? "Unipotent principal series reps" $\mathrm{End}_{\mathbb{C}G}(\mathrm{Ind}_B^G \psi)$ will contain information (e.g., irrep $\Leftrightarrow$ dim=1).

Clearly: $\{\text{irreps in } \mathrm{Ind}_B^G \psi\} \overset{1:1}{\longleftrightarrow} \{\text{irreps of } ``\mathcal{H}(G,B,\psi)"\}$. More precisely:

$$\mathrm{End}_{\mathbb{C}G}(\mathrm{Ind}_B^G\psi) = \mathrm{Hom}_{\mathbb{C}G}(\mathrm{Ind}_B^G\psi, \mathrm{Ind}_B^G\psi) = \mathrm{Hom}_{\mathbb{C}B}(\mathrm{Res}_B^G \mathrm{Ind}_B^G \psi, \psi) = \bigoplus_{B\backslash G/B} \mathrm{Hom}_{\mathbb{C}B}(\mathrm{Ind}^B_{B\cap B^g} \mathrm{Res}^B_{B\cap B^g} \psi^g, \psi)$$

Setting $\psi = 1$, $\mathrm{Ind}_B^G 1 = \bigoplus_{G/B} Bg \leftrightarrow (B\times 1)$-invariant functions $G \to \mathbb{C}$

$\mathrm{End}_{\mathbb{C}G}(\mathrm{Ind}_B^G 1) \leftrightarrow (B\times B)$-invariant functions $G \to \mathbb{C}$

Note that the algebra structure is again given by convolution: $(u * v)(x) = \frac{1}{|B|}\sum_{g\in G} u(g)\,v(g^{-1}x)$

Groups of Lie type: $G$ (e.g. $GL_n(\mathbb{F}_{p^k})$), $B$ Borel $\rightsquigarrow \mathcal{H}(G,B,1) = \mathcal{H}_q(W)\big/_{(q=p^k)}$

Another heuristic for why $\mathcal{H}(GL_n(\mathbb{F}_q), B, 1)$ deforms $S_n$:

$\#$flags in $\mathbb{F}_q^n = \#$ordered basis in $\mathbb{F}_q^n = q^{\binom{n}{2}} \cdot \frac{q^n-1}{q-1} \cdot \frac{q^{n-1}-1}{q-1} \cdot \ldots \cdot \frac{q-1}{q-1} = q^{\binom{n}{2}}(q^{n-1}+\ldots+1)(q^{n-2}+\ldots+1) \cdot \ldots \cdot (1)$

If $q=1$, this is $n!$, and since $GL_n(\mathbb{F}_q) \curvearrowright G/B$ freely + trans $\rightsquigarrow$ "$GL_n(\mathbb{F}_1) = S_n$", "$B = 1$", "$\mathcal{H}(GL_n(\mathbb{F}_1), 1, 1) = S_n$".

- **Quantum groups**

  Classical Schur-Weyl duality: $\mathfrak{gl}_n \curvearrowright V^{\otimes r} \curvearrowleft S_r$ (actions centralize each other) $\rightsquigarrow \{\mathfrak{gl}_n\text{-irreps in } V^{\otimes r}\} \overset{1:1}{\longleftrightarrow} \{S_r\text{-irreps in } V^{\otimes r}\}$ ($\overset{1:1}{\longleftrightarrow}$ partitions of $r$)

  Quantum Schur-Weyl duality: $U_q(\mathfrak{gl}_n) \curvearrowright V_q^{\otimes r} \curvearrowleft \mathcal{H}(S_n) \rightsquigarrow$ idem (q-deformed actions)

- **Kazhdan-Lusztig theory**

Two $\mathbb{Z}[v,v^{-1}]$-bases:

- Standard: $\{\delta_x : x\in W\}$ (Here we take a reduced expression $\underline{x} = s_1 \cdots s_n$ and define $\delta_x = \delta_{s_1} \cdots \delta_{s_n}$)

- Kazhdan-Lusztig: $\{b_x : x \in W\}$ characterized by: $\begin{cases} \overline{b_x} = b_x & \text{(KL involution: } \overline{\delta_s} := \delta_s^{-1} = \delta_s + (v-v^{-1}),\ \overline{v} := v^{-1} \text{ extended multiplicatively)} \\ b_x = \delta_x + \sum_{y<x} h_{yx}\,\delta_y & \text{for some } h_{yx} \in v\,\mathbb{Z}[v] \quad \text{"degree bound"} \end{cases}$

Let $\lambda$ be dominant, $M(\lambda) := U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_\lambda$, $L(\lambda) :=$ simple module of h.w. $\lambda$. The KL conjecture says:

$$[M(y\cdot 0) : L(x\cdot 0)] = h_{yx}|_{v=1}.$$

- **Link invariants**

  Traces on $\mathcal{H} \rightsquigarrow$ Alexander, Jones polys

- **Categorical actions**: Hecke category $\curvearrowright \mathcal{O}_0$

- **Modular rep theory**: $\mathcal{H}|_{q=\sqrt[p]{1}}$ (type A) decomposition matrices $\overset{\text{closely linked}}{\rightsquigarrow}$ decomp matrices of $S_n$ in char $p$.

## 2. Representation theory of $H$ for $W$ finite.

What does $H$-mod look like? Spoiler: just like $W$-mod. However, we can specialize $q$ to any element of $\mathbb{C}^\times$, and the representation categories will be different.

Let $z \in \mathbb{C}^\times$. We will denote $H_z := \mathbb{Z}[q^{\pm 1}]/(q-z) \otimes_{\mathbb{Z}[q^{\pm 1}]} H$.

First question: for what values of $z$ is $H_z$ ss?

**Def** (Trace form): If $A$ is a f.diml. $K$-algebra, denote $L_x : A \to A$, $a \mapsto xa$. Then the trace form is $(\,,\,) : A \times A \to K$, $(x,y) = \mathrm{Tr}(L_x L_y)$.

**Rmk:** The trace can be defined for Hecke algebras even if they are not f.diml.

**Prop**: $A$ is ss $\Leftrightarrow$ $(\,,\,)$ is nondegenerate

**Proof**:

$\Rightarrow$) Nondegeneracy can be checked by passing to $\bar{K}$. Now $A \otimes_K \bar{K}$ is a product of matrix algebras over $\bar{K}$. These are simple and hence contain no double-sided ideals, in particular the radical of the form restricted to each is $0$.

$\Leftarrow$) Recall $A$ Artinian $\Rightarrow$ $J(A)$ = largest nilpotent right ideal.

Now if $j \in J(A)$, $ja$ is nilpotent for all $a \in A$. Pass to $\bar{K}$, upper triangularize the action of $ja$ $\rightsquigarrow$ $\mathrm{Tr}(ja) = 0$ $\forall a$ $\Rightarrow$ $j \in \mathrm{rad}((\,,\,))$. $\square$

Let $R = K[q^{\pm 1}]$. Assume $A$ is an $R$-algebra, finite as an $R$-module. For $f \in K^\times$, denote $A_f = A \otimes_R R/(q-f)$.

**Prop**: If $A_f$ is ss, then $A$ is ss.

**Proof**: The discriminant of the trace form on $A$ is $D(q)$, so if $D(q)|_{q=f} \neq 0$, $D(q) \neq 0$.

**Cor**: The generic Hecke algebra is semisimple.

**Proof**: The specialization to $q=1$ is $\mathbb{C}[W]$, which is semisimple.

Moreover, we have the following stronger result:

**Theorem** (Tits' Deformation thm): If $H_z, H_{z'}$ are semisimple, then $H_z \cong H_{z'}$ abstractly.

**Proof**: By the previous proposition, the discriminant on $H$ is nonzero, so $H \otimes \overline{\mathbb{C}(q)}$ is a product of matrix algebras over $\overline{\mathbb{C}(q)}$, of dimensions $n_1, \dots, n_k$, "the numerical invariants". It suffices to show the numerical invariants for $H_z$ are the same: $n_1, \dots, n_k$.

(so that $H_z \cong H \otimes \overline{\mathbb{C}(q)}|_{q=z} \cong H \otimes \overline{\mathbb{C}(q)}|_{q=z'} \cong H_{z'}$) (omit)

Adjoin formal variables $x_w$ for $w \in W$ and consider $H_{\overline{\mathbb{C}(q)}} \otimes \overline{\mathbb{C}(q)}(x_w : w \in W)$, in order to write a "generic element" $a = \sum_w x_w \delta_w$. Let $P(t)$ be its char poly, say $P(t) = \prod P_i(t)^{e_i}$ in $\overline{\mathbb{C}(q)}[t, x_w : w \in W]$ is the decomp into irreds.

Since $H_{\overline{\mathbb{C}(q)}} \otimes \overline{\mathbb{C}(q)}(x_w : w \in W) = \prod M_{n_i}(\overline{\mathbb{C}(q)}(x_w : w \in W))$, it has a basis $\{E_{ij}^\ell\}$ for each entry in each summand.

So write $a = \sum_{i,j,\ell} y_{ij}^\ell E_{ij}^\ell$ for $y_{ij}^\ell \in \overline{\mathbb{C}(q)}(x_w : w \in W)$. The change of basis matrix has entries in $\overline{\mathbb{C}(q)}$ so $\overline{\mathbb{C}(q)}(x_w : w \in W) = \overline{\mathbb{C}(q)}(y_{ij}^\ell)$. In this basis,

$$P(t) = \prod_\ell \det(t\,\mathrm{id} - y_{ij}^\ell)^{n_\ell}$$

Now specialize $y_{ij}^\ell$ so that the $\det(t\,\mathrm{id} - y_{ij}^\ell)$ are irred and distinct. Then $P_\ell(t) = \det(t\,\mathrm{id} - y_{ij}^\ell)$ and $e_\ell = n_\ell = \deg P_\ell(t)$.

Now consider the generic element $a = \sum x_w \delta_w \in H_z \otimes_{\mathbb{C}} \mathbb{C}(x_w : w \in W)$. By the same argument, since $H_z$ is ss, its char poly, $P_\ell(t)|_{q=z}$, has its irreducible factors appearing with multiplicity = degree. Since $n_\ell = \deg P_\ell(t)|_{q=z}$, the $P_\ell(t)|_{q=z}$ must be irred and distinct. Hence the $n_\ell$ are the numerical invariants for $H_z$ too. ∎

Conclusion: $H_z$ for generic $z$ is isomorphic to $\mathbb{C}[W]$.

When exactly? Whenever $z \notin \{D(q) = 0\}$. This amounts to: $H_z$ is ss. iff $z^{2\cdot \ell(w_0)} \sum_{w \in W} z^{2\ell(w)} \neq 0$.
For type A, this amounts to: $H_z$ is ss. iff order$(z) > n$ or if $n \geq 3$, $z = 0$ also works.

# 3. Affine Hecke algebras and their representations

Reference for 2nd half: MIT-Northeastern 2017 DAHAEHA seminar notes.

Definition: not quite $H(W,S)$ for $W$ affine, but close.

Motivation:

- Reductive p-adic groups

  Example: $G = GL_n(\mathbb{Q}_p)$, $\underset{\text{Iwahori subgroup}}{I \subseteq GL_n(\mathbb{Z}_p)} \Rightarrow C_c(I\backslash G/I) = \mathcal{H}(W_{aff}, q=p)$. "Iwahori spherical algebra"

- K-theory

  $G$ complex ss simply connected Lie group, $\mathcal{N}$ = nilpotent cone, $\tilde{\mathcal{N}} \to \mathcal{N}$ Springer resolution, $Z := \tilde{\mathcal{N}} \times_{\mathcal{N}} \tilde{\mathcal{N}}$ "Steinberg variety"

  Then $K^G(Z) = \mathbb{Z}[W_{aff}]$ and $K^{G\times\mathbb{C}^\times}(Z) = H_{aff}$

- First step to understand Cherednik algebras

Immediate problem: Tits' deformation argument fails. Rep theory of specializations becomes much more involved.

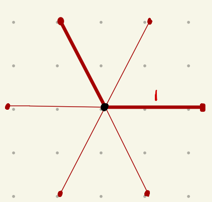
(For the rest I use a single reference: MacDonald, AHAs and orthogonal polynomials)

Affine stuff

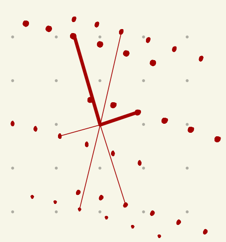
Fix a finite irreducible reduced system $R \subset V$. Here $V$ has an inner product $(\,,\,)$. Embed $V \subset V \oplus \mathbb{C}\delta$, with $\delta \perp V$.

Then the associated affine root system is $R^a := \{\alpha + n\delta : n \in \mathbb{Z}\} \subset F$. $\quad \delta : V \to \mathbb{C}$, $x \mapsto 1$

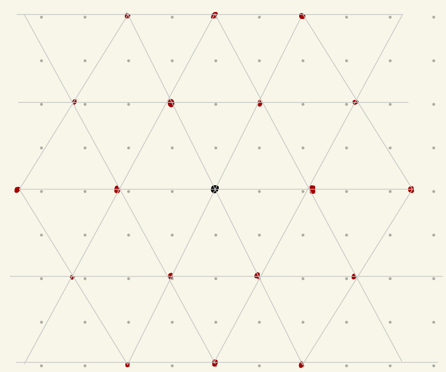
Writing $\alpha^\vee = \frac{2\alpha}{(\alpha,\alpha)}$, we have $Q := \sum_{\alpha\in R} \mathbb{Z}\alpha$ root lattice $Q^\vee := \sum_{\alpha^\vee\in R^\vee} \mathbb{Z}\alpha^\vee$ coroot lattice.



$R$



$R^a$



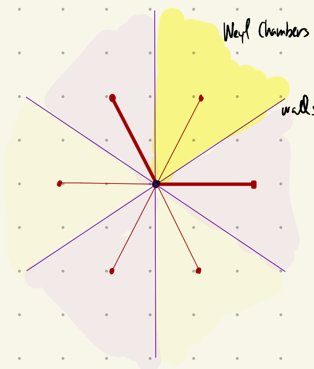
$Q$

Then $W = \langle s_\alpha : \alpha \in R\rangle$, $W^a = \langle s_a : a \in R^a\rangle$.

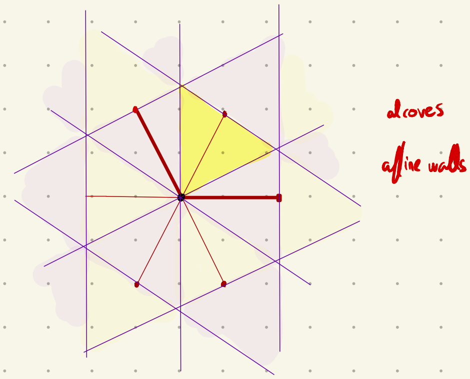
For $v \in V$, denote $t(v) : V \to V$, $x \mapsto x+v$. Then $t(v)(\alpha) = \alpha - (v,\alpha)\delta$.

Then $t(Q^\vee) \trianglelefteq W^a$ and $W^a = W \ltimes t(Q^\vee)$. "affine Weyl group"

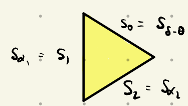
Zero locus of $R$:




Zero locus of $R^\vee$:

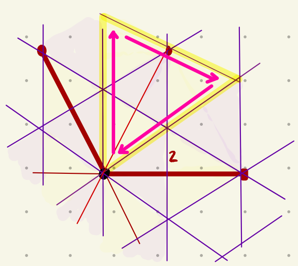


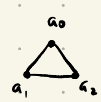

Consider the "fundamental alcove". This is an $n$-simplex with $n+1$ walls given by affine walls corresponding to some $a_0, a_1, \dots, a_n \in R^a$. These are the simple affine roots. Note $\alpha_i = a_i$. Fact: if $\theta \in R^+$ is the highest root, $a_0 = -\theta + \delta$.

$A =$ 


Notice that $W^{ae} := W \ltimes t(P^\vee)$ (dominant coweights) also acts on $R^a$. $t(\lambda)(a) = a - \underbrace{(\lambda, a)}_{\in \mathbb{Z}}\delta$.

This admits a length function extending that of $W^a$ and $W$. Note that $\Omega := \{w \in W^{ae} : \ell(w) = 0\} = \{w \in W^{ae} : wA = A\}$ is a finite group.




$\Omega = C_3$ 


We have $\Omega = P^\vee / Q^\vee$ (Dynkin diagram automorphisms) and $W^{ae} = \Omega \ltimes W^a$. Note that $\Omega \curvearrowright A \leadsto \Omega \curvearrowright$ affine simple roots, so if $\pi_r(a_i) = a_j$, $\pi_r s_i \pi_r^{-1} = s_j$, hence the semidirect product.

## Braid groups

The Braid group of a Coxeter system is the group generated by $\{T_w : w \in W\}$, subject only to the braid relation.
The braid relation is equivalent to: $T_w T_{w'} = T_{ww'}$ whenever $\ell(ww') = \ell(w) + \ell(w')$.

Define the affine braid group $B^a$ as that of $(W^a, I)$, and the extended affine braid group $B^{ae}$ as that of $(W^{ae}, I)$ (with its length function)

$B^{ae}$ has two important subgroups.

- The elements $T_\pi$ with $\pi \in \Omega$ form a subgroup of $B^{ae}$ isom to $\Omega$ (obvs), and $B^{ae} = \Omega \ltimes B^a$, where if $\pi_r(a_i) = a_j$, $\pi_r T_i \pi_r^{-1} = T_j$.
- For $\lambda \in P^\vee_+$, define $Y^\lambda = T_{t(\lambda)}$, for $\mu - \nu \in P^\vee$, define $Y^{\mu-\nu} = Y^\mu (Y^\nu)^{-1}$. These generate a copy of $P^\vee$.

**Proposition:** $T_1, \dots, T_n, Y^\lambda : \lambda \in P^\vee$ generate $B^{ae}$ as a group (notice the absence of $T_0$)

If $(\lambda, \alpha_i) = 0$ then $T_i Y^\lambda = Y^\lambda T_i$
$(\lambda, \alpha_i) = 1$ then $T_i Y^\lambda = Y^\lambda T_i^{-s_i\lambda}$ $\quad (Y^\lambda = T_i Y^{s_i\lambda} T_i)$

(idea: reduce to $\lambda \in P_+^\vee$ and use properties of the length function)

The previous proposition leads to a presentation of $B^{ae}$ reminiscent of $W^{ae} = W \ltimes t(P^\vee)$:

$$B^{ae} = \left\langle T_1, \dots, T_n, Y^{P^\vee} \;\middle|\; \begin{array}{ll} T_i Y^\lambda = Y^\lambda T_i & (\alpha_i, \lambda) = 0 \\ T_i Y^\lambda = Y^\lambda T_i^{-s_i\lambda} & (\alpha_i, \lambda) = 1 \end{array} \right\rangle$$

We can finally state:

**Definition (AHA):** The **affine Hecke algebra** $H(W^{ae})$ is the quotient of the group algebra of $B^{ae}$ by the Hecke relations: $(T_i - q)(T_i + q^{-1}) = 0$

How do the $T_i$ and $Y^\lambda$ interact in $H(W^{ae})$?

**Lemma:** $T_i Y^\lambda - Y^{s_i(\lambda)} T_i = (q - q^{-1}) \dfrac{Y^{s_i\lambda} - Y^\lambda}{Y^{-\alpha_i^\vee} - 1}$

*Proof:* A calculation shows that if this holds for $Y^\lambda$ and $Y^\mu$, then it holds for $Y^{-\lambda}$ and $Y^{\lambda+\mu}$. So two cases to check:

- $(\lambda, \alpha_i) = 0 \Rightarrow$ this says $T_i Y^\lambda = Y^\lambda T_i$.
- $(\lambda, \alpha_i) = 1 \Rightarrow$ this says $T_i Y^\lambda - Y^{s_i(\lambda)} T_i = (q - q^{-1}) Y^\lambda$

$$\overset{\text{(Prop)}}{=} T_i Y^\lambda - T_i^{-1} Y^\lambda = T_i Y^\lambda - (T_i^{-1} + q - q^{-1}) Y^\lambda \text{ as desired } \square$$

We have given two presentations of $B^{ae}$: Coxeter ($B^{ae} = \Omega \ltimes B^a$) and Bernstein ($B^{ae} = \langle T_i, Y^\lambda | \dots \rangle$). This implies the following.

**Prop:**
- $H(W^{ae}) \cong \Omega \ltimes H(W^a)$, therefore $\{T_w : w \in W^{ae}\}$ is a $\mathbb{C}$-basis for $H(W^{ae})$.
- The subalgebra generated by $T_i$ (including 0) is isom to $H(W^a, S)$
  $T_i$ (not including 0) " $H(W, S)$

Q: Basis for the second presentation?

Fact: as $\mathbb{C}$-v.s., $H(W, S) \otimes_{\mathbb{C}} \mathbb{C} Y^{P^\vee} \xrightarrow{\sim} H(W^{ae})$, so $\{T_w Y^\lambda : w \in W, \lambda \in P^\vee\}$ is another basis.
$x \otimes y \longmapsto xy$

This map allows us to construct many representations of $H(W^{ae})$: for $E$ a rep of $H(W, S)$, $\operatorname{Ind} E := H(W^{ae}) \otimes_{H(W)} E$
As a $\mathbb{C} Y^{P^\vee}$-module, $\operatorname{Ind} E = \mathbb{C} Y^{P^\vee} \otimes E$.
In particular, if $E = \mathbb{C}$ by specializing $q = \tau$, we get $\mathbb{C} Y^{P^\vee}$ "Polynomial representation"
Now the last lemma implies that $T_i$ acts by $\tau s_i + (\tau - \tau^{-1}) \dfrac{s_i - 1}{Y^{-\alpha_i^\vee} - 1}$ in $\mathbb{C} Y^{P^\vee}$

Remarks:
- One can modify this action to $\beta: T_i \mapsto \tau s_i + (\tau - \tau^{-1}) \dfrac{s_i - 1}{X^{a_i} - 1}$ now acting on $\mathbb{C}[X]$ (group algebra of the weight lattice)
  $\pi_r \longmapsto \pi_r$

  This is called Cherednik's basic representation.
- Both of these representations are faithful.
- In fact DAHAs can be defined as the 2-parameters $(q, \tau)$ subalgebra of $\operatorname{End}_{\mathbb{C}}(\mathbb{C}[X])$ gen by $X^\lambda \beta(T_w)$ $(\lambda \in P, w \in W^{ae})$.

We finish by computing the center of the AHA.

Theorem: $Z(H(W^{ae})) = (\mathbb{C}Y^{P^\vee})^W$.

Proof: $\supseteq$) Easy: if $f = f(Y) \in (\mathbb{C}Y^{P^\vee})^W$, $T_i f - f T_i = 0$ (since $T_i Y^\lambda - Y^{s_i(\lambda)} T_i = (q - q^{-1}) \dfrac{Y^{s_i\lambda} - Y^\lambda}{Y^{-\alpha_i^\vee} - 1}$ by our lemma).
Since $\mathbb{C}Y^{P^\vee}$ is commutative and the $T_i$ and the $Y^\lambda$ generate, $f \in Z(H(W^{ae}))$.

$\subseteq$) By the lemma, if $f \in \mathbb{C}Y^{P^\vee}$, $T_i f - s_i f T_i = g(Y) \in \mathbb{C}Y^{P^\vee}$.
If $f$ is central, $(f - s_i f) T_i = g(Y) \underset{\text{basis}}{\Rightarrow} f = s_i f$.

So it suffices to see that $f \in \mathbb{C}Y^{P^\vee}$. Now under the polynomial representation $\mathbb{C}Y^{P^\vee}$, $f$ commutes with all Laurent polynomials, hence its image is a Laurent polynomial. But the polynomial representation is faithful so $f \in \mathbb{C}Y^{P^\vee}$. $\square$