0. Introduction

This is my personal (yet explicit) account of the second part of the paper, which concerns mostly the following.

Theorem 10.1. Let V be a f.dim'l. $\mathfrak{g}$-irrep with hw λ. Then there is an exact sequence of $\mathfrak{g}$-modules

$$0 \leftarrow V \leftarrow C_0^V \leftarrow C_1^V \leftarrow \cdots \leftarrow C_s^V \leftarrow 0$$

where $s = \dim \mathfrak{n}$, $\quad C_k = \bigoplus_{w \in W^{(k)}} M_{w(\lambda+\delta)}$ $\quad$ where $W^{(k)} = \{w \in W : \ell(w) = k\}$

$M_{\lambda+\rho}$ = Verma of h.w. λ

This is now known as a BGG resolution, and part of its significance lies in that it categorifies Kostant's multiplicity formula, and hence essentially categorifies Weyl's character formula. Indeed, taking formal characters in the resolution we get

$$\mathrm{ch}(V) = \sum_{k\geq 0} (-1)^k \,\mathrm{ch}\, C_k^V = \sum_{k \geq 0} (-1)^k \sum_{\substack{\mu \in \Sigma \\ \mu+\delta \text{ dominant}}} (C_k^V : M_{\mu+\rho})\, \mathrm{ch}\, M_{\mu+\rho}$$

(Σ = simple roots)

$$= \sum_{\substack{w \in W \\ w\lambda+\rho \text{ dom.}}} \underbrace{\left(\sum_{k\geq 0} (-1)^k (C_k^V : M_{w\lambda+\rho}) \right)}_{= (-1)^{\ell(w)} \sum_{k \geq 0} (-1)^k (C_k^V : M_{w\lambda+\rho})} \mathrm{ch}\, M_{w\lambda+\rho}$$

so that $a(\lambda, \mu) = \sum_{k\geq 0} (-1)^k (C_k^V : M_{\mu+\delta})$ for $\mu+\delta$ dominant.

$a(\lambda,\mu)$: coefficients of change of basis, defined in the introduction of the Part III Essay

Therefore $\mathrm{ch}\, V = \sum_{\mu \text{ dominant}} a(\lambda,\mu) \sum_{w \in W} (-1)^{\ell(w)}\, \mathrm{ch}\, M_{w\mu+\rho}$, from which Kostant's multiplicity follows.

8. The category $\mathcal{O}$

This part summarizes some basic properties of the category $\mathcal{O}$, most of which appear in the Part III essay. I state the facts that are not mentioned explicitly in it.

<u>Central characters</u>: define $\Theta = \mathrm{Hom}_{\mathbb{C}\text{-alg}}(Z(\mathfrak{g}), \mathbb{C})$

Also given M any $\mathfrak{g}$-module, denote $\Theta(M) = \{\theta \in \Theta : \theta \text{ is a central character of } M\}$

$= \{\theta \in \Theta : \exists m \in M \text{ s.t. } zm = \theta(z)m \text{ for all } z \in Z(\mathfrak{g})\}$

$= \{\theta \in \Theta$ appearing in the block decomposition of $M = \bigoplus_\theta M^\theta$

where $M^\theta = \{m \in M : (z - \theta(z))^n m = 0 \text{ for some } n \geq 0\}$

Warning: the Verma modules are taken shifted so in this paper there is no dot action, hence if we write $\Theta(M_\chi) = \{\theta_\chi\}$, we get $\theta_{\chi_1} = \theta_{\chi_2} \Leftrightarrow \chi_1 \in \underbrace{W\chi_2}_{\text{usual action}}$

Verma, h.w. $\chi - \rho$

<u>Fact</u>: for $\chi, \psi \in \mathfrak{h}^*$,

Jantzen: iff $\chi \uparrow \psi$

$$\mathrm{Hom}(M_\chi, M_\psi) = \begin{cases} \mathbb{C} & \text{iff } \begin{cases} \bullet \text{ There is a sequence } \chi = \sigma_{\gamma_k} \cdots \sigma_{\gamma_1} \psi \\ \bullet\ \chi_{i-1} - \chi_i = n_i \gamma_i,\ n_i \geq 0 \end{cases} \\ 0 & \text{o/w} \end{cases}$$

$(\chi_i = \sigma_{\gamma_i} \cdots \sigma_{\gamma_1} \psi)$

if nonzero, it is an inclusion i.e. "only Verma submodules"

In particular $M_\chi \longrightarrow M_\psi$ nonzero $\Rightarrow \chi \in W\psi$ and $\chi \leq \psi$ (but this is not sufficient)

Next they denote the Bruhat order by $\leq$ and state the following variation of the above fact:

If $\chi \in D$ (is integral dominant)

$\mathrm{Hom}(M_{w_1\chi}, M_{w_2\chi}) = \mathbb{C} \Leftrightarrow w_1 \leq w_2$

↳ note, submod of M_χ (a posteriori)

Seeing how this follows requires some Weyl grp combinatorical facts, which follow directly from Prop. 4.1. in the Essay. (Lemmas 8.10, 8.11)

They finally state Theorem 8.12 : $[M_\psi : L_\chi] \neq 0$ iff $\chi \uparrow \psi$

[This is hard, the Appendix proves it but better see the cohomological, self-contained proof in Jantzen II.6.]

Corollary: If $\chi \in D$, the Jordan-Hölder decomp of $M_{w\chi}$ only has terms $L_{w'\chi}$ with $w' \leq w$ and (of course) $[M_{w\chi} : L_{w\chi}] = 1$

Warning: even if $\psi \in D$, M_ψ may contain submodules not generated by sums of $M_{w\psi}$'s.

9. Lie algebra cohomology

This part summarizes some basic facts about Lie algebra cohomology.

- $\mathfrak{a}$ is any complex Lie algebra. Throughout, M, N are $U(\mathfrak{a})$-modules.
- $\tau : \mathfrak{a} \to \mathfrak{a}$, $X \mapsto -X$ (this extends to $U(\mathfrak{a}) \to U(\mathfrak{a})$)
- N^τ is the module N with $U(\mathfrak{a})$-multiplication $X \cdot n = \tau(X) n$.

Some homological algebra facts:

a) $\mathrm{Ext}^i(M,N)^* = \mathrm{Tor}_i(N^*, M)$ (here $(\ )^*$ denotes $\mathbb{C}$-linear dual)

b) $\mathrm{Tor}_i(N^*, M) = \mathrm{Tor}_i(M^\tau, (N^\tau)^*)$

Now $H^i(\mathfrak{a}, M)$ is defined as $\mathrm{Ext}^i(\mathbb{C}, M)$ ($\mathbb{C}$ trivial $\mathfrak{a}$-rep). It is computed using the Chevalley-Eilenberg resolution $V(\mathfrak{a})$ of $\mathbb{C}$:

$C_k = U(\mathfrak{a}) \otimes_{\mathbb{C}} \Lambda^k \mathfrak{a}$ Since $\Lambda^k\mathfrak{a}$ is a free $\mathbb{C}$-module, this is a free (left) $U(\mathfrak{a})$-module

with differential $d_k : C_k \to C_{k-1}$

given by $d_k(X \otimes X_1 \wedge \dots \wedge X_k) = \sum_{i=1}^{k} (-1)^{i+1} (X X_i \otimes X_1 \wedge \dots \wedge \hat{X}_i \wedge \dots X_k) + \sum_{1 \le i < j \le k} (-1)^{i+j} (X \otimes [X_i, X_j] \wedge X_1 \wedge \dots \wedge \hat{X}_i \wedge \dots \wedge \hat{X}_j \wedge \dots \wedge X_k)$

Denoting $\varepsilon : C_0 \to \mathbb{C}$, $X \in U(\mathfrak{a}) \mapsto$ coefficient of 1, we get a free complex $0 \to C_{\dim \mathfrak{a}} \to \dots \to C_1 \to C_0 \to \mathbb{C} \xrightarrow{\varepsilon} 0$

To see that it is exact, take a Lie group whose Lie algebra is $\mathfrak{a}$. Then $V(\mathfrak{a})$ is the dual of the de Rham complex of formal analytic forms. This also works for the relative case below.

Applying $\mathrm{Hom}_{U(\mathfrak{a})}(-, M)$ and taking cohomology groups we get $\mathrm{Ext}^i(\mathbb{C}, M) = H^i(\mathfrak{a}, M)$

Relative Lie algebra cohomology

Now consider a subalgebra $\mathfrak{p} \subset \mathfrak{a}$ and view $\mathfrak{a}/\mathfrak{p}$ as a $\mathfrak{p}$-representation θ. We set analogously $D_k = U(\mathfrak{a}) \otimes_{U(\mathfrak{p})} \Lambda^k(\mathfrak{a}/\mathfrak{p})$

and $d_k(X \otimes X_1 \wedge \dots \wedge X_k) = \sum_{i=1}^{k} (-1)^{i+1} (X X_i \otimes X_1 \wedge \dots \wedge \hat{X}_i \wedge \dots X_k) + \sum_{1 \le i < j \le k} (-1)^{i+j} (X \otimes [X_i, X_j] \wedge X_1 \wedge \dots \wedge \hat{X}_i \wedge \dots \wedge \hat{X}_j \wedge \dots \wedge X_k)$

(lifts of elements in $\Lambda^k(\mathfrak{a}/\mathfrak{p})$)

and we set $\varepsilon : D_0 \to \mathbb{C}$, $X \otimes 1 \mapsto$ coefficient of 1 in X. This complex is denoted $V(\mathfrak{a}, \mathfrak{p})$.

The paper gives a purely algebraic proof that this is a free resolution. The strategy is: the PBW filtration on $U(\mathfrak{a})$ induces a filtration on D_k which is compatible with the differential, so we may take the complex $\mathrm{Gr}\, V(\mathfrak{a}, \mathfrak{p})$ and show that it is exact in every degree. Now by PBW $D_k^{(\ell)} / D_k^{(\ell-1)} = S_{\ell-k}(\mathfrak{a}/\mathfrak{p}) \otimes \Lambda^k(\mathfrak{a}/\mathfrak{p})$ ($\hookrightarrow (\ell-k)$-part in $\mathrm{Sym}(\mathfrak{a}/\mathfrak{p})$) and now $d_k^{(\ell)}$ coincides with the differential of the Koszul complex, showing that $\mathrm{Gr}\, V(\mathfrak{a}, \mathfrak{p})$ and therefore $V(\mathfrak{a}, \mathfrak{p})$ is exact.

In the rest of the section, they take $\mathfrak{a} = \mathfrak{g}$ semisimple, $\mathfrak{p} = \mathfrak{b}$ Borel subalgebra.

Lemma 9.3 is obvious: $\mathfrak{b}\text{-mod} \to \mathfrak{g}\text{-mod}$, $V \mapsto V^{\mathfrak{g}} := U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} V$ is exact and $(\mathbb{C}_\chi)^{\mathfrak{g}} = M_{\chi+\delta}$

($U(\mathfrak{g})$ is a free $U(\mathfrak{b})$-mod, generated by $\mathfrak{n}^-$-monomials)

Definition 9.4: Ψ finite set of weights. M is said to be of type Ψ iff it has a (Verma flag / standard filtration) $0 = M^{(0)} \subset \dots \subset M^{(\ell)} = M$ with $M^{(i+1)}/M^{(i)} = M_{\psi_i}$ and $\Psi = \{\psi_0, \dots, \psi_\ell\}$.

Lemma 9.5: Let N be a f.dim'l. $\mathfrak{h}$-diagonalizable $\mathfrak{b}$-module and $\Psi(N) = \{\varphi + \rho\}_{\varphi \text{ weights of } N}$. Then $N^{\mathfrak{g}}$ is of type $\Psi(N)$

Proof: $N = N_{(\ell)} \supset N_{(\ell-1)} \supset \dots \supset N_{(0)} = 0$ with $N_{(i)}/N_{(i-1)} = \mathbb{C}_\varphi$ for some weight of N. Therefore we get

$$N^{\mathfrak{g}} = N^{\mathfrak{g}}_{(\ell)} \supset N^{\mathfrak{g}}_{(\ell-1)} \supset \dots \supset N^{\mathfrak{g}}_{(0)} = 0 \quad \text{with} \quad \underbrace{N^{\mathfrak{g}}_{(i)}/N^{\mathfrak{g}}_{(i-1)} = \left(N_{(i)}/N_{(i-1)}\right)^{\mathfrak{g}}}_{\text{exactness}} = \underbrace{(\mathbb{C}_\varphi)^{\mathfrak{g}} = M_{\varphi+\rho}}_{\text{definition of Verma module}} \text{ as desired } \square.$$

It follows from this that D_k is of type $\Psi(\Lambda^k(\mathfrak{g}/\mathfrak{b}))$.

Next we study the "principal block part" of D_k i.e. if $\theta = \theta_\rho$ then we study the (still exact since the projection is exact) complex $V(\mathfrak{g}, \mathfrak{b})_\theta$:

Proposition 9.6: Write $\Psi_k = \{w\rho \mid w \in W^{(k)}\}$ Then $(D_k)_\theta$ is of type Ψ_k.

Proof:

Lemma 9.7: Let M of type Ψ and $\theta \in \Theta$. Then M_θ is of type $\Psi_\theta := \{\text{weights } \varphi \in \Psi \text{ s.t. } \theta_\varphi = \theta\}$ * (the latter are the same as asking the weights to be in the same Weyl orbit by Harish-Chandra)

Proof: easy: if $0 \subset M^{(0)} \subset \dots \subset M^{(k)} = M$ is a Verma flag, applying the (exact) projection functor gives a filtration of M_θ and $M^{(i)}_\theta / M^{(i-1)}_\theta = (M_{\varphi_i})_\theta = \begin{cases} M_{\varphi_i} & \text{if } \theta = \theta_{\varphi_i} \\ 0 & \text{o/w} \end{cases}$ ✓

It follows from this that $(D_k)_\theta$ is of type $\Psi(\Lambda^k(\mathfrak{g}/\mathfrak{b}))_\theta$, and for 9.6. we want to show this is Ψ_k.

Now for a subset $\Phi \subset \Delta$, write $|\Phi| = \sum_{\gamma \in \Phi} \gamma$. Then since the weights of $\mathfrak{g}/\mathfrak{b}$ are Δ_-, $\Psi(\Lambda^k(\mathfrak{g}/\mathfrak{b})) = \{\rho - |\Phi| : \operatorname{card}\Phi = k,\ \Phi \subset \Delta_+\}$ so that

$$\Psi(\Lambda^k(\mathfrak{g}/\mathfrak{b}))_{\theta_\rho} \underset{*}{=} \{\rho - |\Phi| : \operatorname{card}\Phi = k,\ \Phi \subset \Delta_+,\ \rho - |\Phi| \in W\rho\}$$

Writing $\Phi_w = \{\gamma \in \Delta_+ \mid \gamma \subset w\Delta_-\}$, $\operatorname{card}\Phi_w = k$ (standard) and whenever $\Phi \subset \Delta_+$, $w \in W$ and $\rho - w\rho = |\Phi|$, we have $\Phi = \Phi_w$

Therefore it follows that

$$\Psi(\Lambda^k(\mathfrak{g}/\mathfrak{b}))_{\theta_\rho} = \{\rho - |\Phi| : \exists w \in W \text{ s.t. } |\Phi| = \rho - w\rho,\ \operatorname{card}\Phi = k\}$$

$$= \{w\rho : w \in W,\ \ell(w) = k\}$$

$$= \Psi_k, \text{ proving Prop. 9.6.}$$

Proof: by induction on $\ell(w)$. If $w = e$ then $\Phi = \emptyset = \Phi_e$ ✓

If $\ell(w) > 0$, write $w = s_{\alpha_1} \cdots s_{\alpha_{\ell(w)}}$ reduced expression. Write also $w' = s_{\alpha_1} w$.

Then $|s_{\alpha_1}\Phi| = s_{\alpha_1}\rho - w'\rho = \rho - w'\rho - \alpha$

and so $\rho - w'\rho = |s_{\alpha_1}\Phi \cup \{\alpha_1\}|$.

Claim: $\alpha_1 \in \Phi$. Pf: otherwise $s_{\alpha_1}\Phi \cup \{\alpha_1\}$ is contained in Δ_+ (Note α_1 is simple so $s_{\alpha_1}(\text{positive})$ is positive $\neq \alpha_1$)

Then by ind. hyp., $s_{\alpha_1}\Phi \cup \{\alpha_1\} = \Phi_{w'}$ i.e. $\alpha_1 \in w'\Delta_-$, a contradiction since $\alpha_1 \in s_{\alpha_1} \cdots s_{\alpha_{\ell(w)}}\Delta_-$ (since $\ell(s_{\alpha_1}w) < \ell(w)$) (This proves the claim)

Finally $\alpha \in \Phi$ so by induction, $\Phi \setminus \{\alpha_1\} = s_{\alpha_1}\Phi_{w'}$ i.e. $\Phi = s_\alpha \Phi' \cup \{\alpha\} = \Phi_w$ ✓

The last part of this section concerns the following theorem:

Theorem 9.9: Let V be a f.dim'l. $\mathfrak{g}$-irrep with h.w. λ. Then there exists an exact sequence of $U(\mathfrak{g})$-modules

$$0 \leftarrow V \leftarrow B_0^V \leftarrow B_1^V \leftarrow \dots \leftarrow B_s^V \leftarrow 0 \quad \text{where } s = \dim \mathfrak{n}_- \text{ and } B_k^V \text{ is of type } \Psi_k(\lambda) = \{w(\lambda+\rho) \mid w \in W^{(k)}\}$$

The strategy is: we have found such a sequence for the case $\lambda = 0$, i.e. we have $B_\bullet^{\mathbb{C}}$. Then put $B_k^V := (B_k^{\mathbb{C}} \otimes V)_{\theta_{\lambda+\rho}}$

Since $-\otimes V$ is exact (as it is the right adjoint of $-\otimes V^*$ by finite dimensionality) and $(-)_\theta$ is exact, we only need to show B_χ^V is of type $\Psi_\chi(\lambda)$.

Lemma 9.10 is Theorem 3.1 in the essay. It states that if $\chi \in \mathfrak{h}^*$ and V is f.diml., $M_\chi \otimes V$ is of type $\{\lambda+\chi\}_{\lambda \text{ weight of } V}$

It follows that B_χ^V is of type $\{\lambda_i + w\rho \mid \lambda_i \text{ weight of } V, w \in W^{(\alpha)}, \lambda_i + w\rho \sim \lambda+\rho\}$

Finally, note (9.11) that if $\underset{(\cdot)}{w'(\lambda_i + w\rho) = \lambda+\rho}$ then $w'\lambda_i \le \lambda$ (since λ is h.w.) and $w'w\rho \le \rho$ (since ρ is dominant) $\Rightarrow$ $w'\lambda_i = \lambda$, $w'w\rho = \rho$ $\Rightarrow$ $w'w = 1$ and $\lambda_i = w\lambda$ uniquely determined, has multiplicity 1 as it is a conjugate of the highest weight

so that (·) holds

$\mathrm{Stab}_W(\rho) = \{1\}$

so that B_χ^V is of type $\{w(\lambda+\rho) \mid w \in W^{(\kappa)}\}$ as desired.

Corollary: Bott's theorem. If V is a $\mathfrak{g}$-irrep, then $\dim H^i(\mathfrak{n}_-, V) = \operatorname{card} W^{(i)}$.

Proof: $H^i(\mathfrak{n}_-, V) = \mathrm{Ext}^i_{\mathfrak{n}_-}(\mathbb{C}, V) = \mathrm{Tor}_i^{\mathfrak{n}_-}(V^*, \mathbb{C})^* = \mathrm{Tor}_i^{\mathfrak{n}_-}(\mathbb{C}, \underbrace{(V^\tau)^*}_{V_1 :=})^*$

homological algebra facts

Now tensoring the resolution $B_\bullet^{V_1}$ by $\mathbb{C}$ gives

$$0 \leftarrow B_0^{V_1}/\mathfrak{n}_- B_0^{V_1} \leftarrow \cdots \leftarrow B_s^{V_1}/\mathfrak{n}_- B_s^{V_1} \leftarrow 0, \quad s = \dim \mathfrak{n}_-$$

By the theorem, the $\mathfrak{h}$-decomposition of each term is $\bigoplus_{w \in W^{(i)}} (\mathbb{C}w_i)_{w(\lambda+\rho)}$, so $\dim B_i^{V_1}/\mathfrak{n}_- B_i^{V_1} = \operatorname{card} W^{(i)}$ and since

some eigenvectors

$dB_i^{V_1} \subset U(\mathfrak{n}_-) B_{i-1}^{V_1}$, the differentials are zero and the result follows $\square$

10. Resolution of a f.diml. $\mathfrak{g}$-module

This part proves a stronger version of Thm 9.9, namely one where $B_k^\vee$ is replaced by just a direct sum $\bigoplus_{w \in W^{(k)}} M_{w(\lambda+\rho)}$:

Theorem 10.1: let V be a f.diml. $\mathfrak{g}$-irrep with highest weight λ. Then there exists an exact sequence of $\mathfrak{g}$-modules:

$$0 \leftarrow V \xleftarrow{\varepsilon} C_0^\vee \xleftarrow{d_1} C_1^\vee \leftarrow \dots \leftarrow C_s^\vee \leftarrow 0 \qquad \text{where } s = \dim \mathfrak{n}_-,\ C_k = \bigoplus_{w \in W^{(k)}} M_{w(\lambda+\rho)}$$

(ε is the quotient map).

In order to construct d_i, note (§8 or otherwise) that each $M_{w\chi}$ is a submodule of M_χ (χ dominant), and (8.7) any map $M_{w_1\chi} \to M_{w_2\chi}$ is a multiple of the inclusion map for $w_1 < w_2$ (and zero otherwise). It follows that d_i is given by a matrix $d_i = (d^{(i)}_{w_1,w_2})_{\substack{w_1 \in W^{(i)} \\ w_2 \in W^{(i-1)}}}$ (of complex numbers).

Recall that $w_1 \to w_2$ iff $w_1 = s_\alpha w_2$ for some simple root α and $l(w_1) = l(w_2)+1$.

We make some observations about the Weyl group. Suppose we have $w_1 \to w' \to w_2$. Then by the Exchange lemma (Essay, Prop 4.1.(3)) if $w_1 = s_{\alpha_1} \cdots s_{\alpha_l}$ is a reduced expression, then w_2 is of the form $s_{\alpha_1} \cdots \hat{s}_{\alpha_i} \cdots \hat{s}_{\alpha_j} \cdots s_{\alpha_l}$ for some i,j, and otherwise no such arc exists. It follows that between any two elements of length 2 apart we either have no arcs or we have a square (by another application of the Exchange lemma)

$$\begin{tikzcd}[column sep=small]
& \underbrace{s_{\alpha_1} \cdots \hat{s}_{\alpha_i} \cdots s_{\alpha_l}}_{w_a'} \arrow[dr] & \\
w_1 = s_{\alpha_1} \cdots s_{\alpha_l} \arrow[ur] \arrow[dr] & (i<j) & s_{\alpha_1} \cdots \hat{s}_{\alpha_i} \cdots \hat{s}_{\alpha_j} \cdots s_{\alpha_l} = w_2 \\
& \underbrace{s_{\alpha_1} \cdots \hat{s}_{\alpha_j} \cdots s_{\alpha_l}}_{w_b'} \arrow[ur] &
\end{tikzcd}$$

Now for each arrow $\underbrace{s_{\alpha_1} \cdots s_{\alpha_l}}_{w} \longrightarrow \underbrace{s_{\alpha_1} \cdots \hat{s}_{\alpha_i} \cdots s_{\alpha_l}}_{w'}$ define a sign $s(w,w') = (-1)^{i+1}$. Then for each square as above we have

$$\begin{tikzcd}
& w_a' \arrow[dr, "(-1)^j"] & \\
w_1 \arrow[ur, "(-1)^{i+1}"] \arrow[dr, "(-1)^{j+1}"'] & & w_2 \\
& w_b' \arrow[ur, "(-1)^{i+1}"'] &
\end{tikzcd}$$

i.e. it anticommutes. This proves lemmas 10.3 and 10.4

Alternative, much longer, proofs are provided in §11.

and allows us to define d_i by $d^{(i)}_{w,w'} = s(w,w')$. It is clear that $d^{(i)} \circ d^{(i+1)} = 0$.

The last three lemmas prove the exactness of the sequence.

Now exactness at V is obvious (ε is the quotient map) and exactness at C_0 states

Exactness at C_0: The paper cites Harish-Chandra (1951) but I don't know which theorem, so I record this proof from Humphreys' book on the category $\mathcal{O}$. We need to show that the maximal submodule $N_{\lambda+\rho} \subseteq M_{\lambda+\rho}$ is the sum of the $M_{s_\alpha(\lambda+\rho)}$ for α simple.

- Write $\alpha_1, \dots, \alpha_l$ for the simple roots and $n_i = \frac{2\langle \lambda, \alpha_i \rangle}{\langle \alpha_i, \alpha_i \rangle}$. Write $M_{\lambda+\rho} = U(\mathfrak{g})/I$ and let $J = \langle I, y_i^{n_i+1} \rangle_{i=1,\dots,\ell}$. Now if $X := U(\mathfrak{g})/J$ is finite dimensional, since it is a highest weight module with quotient V, it will follow that $J = N_{\lambda+\rho}$. To see finite dimensionality, it clearly suffices to prove that the y_i act locally nilpotently on X. Now X is spanned by (the cosets of) the possible monomials $y_{i_1} \cdots y_{i_r}$. Since root strings have length ≤ 4, $(\mathrm{ad}\, y_{i_j})^4 (y_{i_k}) = 0$ for each pair (j,k). Now if a coset $y_{i_1} \cdots y_{i_r}$ is killed by y_i^p then the longer monomial $y_{i_0} y_{i_1} \cdots y_{i_\ell}$ is killed by y_i^{p+3}, and by induction y_i acts locally nilpotently.

- Now observe that $M_{s_\alpha(\lambda+\rho)}$ is the submodule generated by $y^{n_i+1} v^+$. (v^+ maximal vector in $M_{\lambda+\rho}$) $\square$

Back to the paper, we now prove exactness inductively, so assume we have proved exactness at $C_0, C_1, \dots, C_{i-1}$ and let $K = \operatorname{Ker} d_i$. In order to prove $d_{i+1}(C_{i+1}) = K$, they prove three lemmas:

Lemma 10.5: let C be a free $U(\mathfrak{n}_-)$-module with generators $f_1, \dots, f_n$ and $\xi : C \to K$ a $U(\mathfrak{n}_-)$-module homomorphism such that each $\xi(f_i)$ is a weight vector. Then if $\bar{\xi} : C/\mathfrak{n}_- C \to K/\mathfrak{n}_- K$ is surjective, ξ is surjective too.

Lemma 10.6: $\bar{d}_{i+1} : C_{i+1}/\mathfrak{n}_- C_{i+1} \to K/\mathfrak{n}_- K$ is an injection

Lemma 10.7: $\dim C_{i+1}/\mathfrak{n}_- C_{i+1} = \dim K/\mathfrak{n}_- K < \infty$

$\Rightarrow$ $\bar{d}_{i+1}$ is an isomorphism, so by 10.5, d_{i+1} is surjective as desired.

Proof of 10.5:
Suppose $\bar{\xi}$ is surjective but ξ is not. Assume wlog that $\bar{\xi}(f_1), \dots, \bar{\xi}(f_t)$ are a basis of $K/\mathfrak{n}_- K$. Let ψ be a weight of K maximal so that $\psi' > \psi \Rightarrow \psi' \in \mathrm{Im}(\xi)$. Let $f \in K$ have weight ψ and write $\bar{f} = f + \mathfrak{n}_- K = \sum_{i=1}^{t} c_i \bar{\xi}(f_i)$ then since $\bar{f}$ is an $\mathfrak{h}$-eigenvector of weight ψ and the $\bar{\xi}(f_i)$ are <u>linearly independent</u>, each $\xi(f_i)$ with $c_i \neq 0$ must have weight ψ (the paper sort of omits this). Now $g := f - \sum c_i \xi(f_i)$ is a weight vector in $\mathfrak{n}_- K$, so $g = \sum_{\gamma \in \Delta_+} E_{-\gamma} g_\gamma$ where $E_{-\gamma} g_\gamma$ has weight ψ, hence g_γ has weight $\psi + \gamma$ and therefore $g_\gamma \in \mathrm{Im}\, \xi$ by maximality, hence $f \in \mathrm{Im}(\xi)$.

Proof of 10.6: since $\{ f_{w\chi} \mid w \in W^{(i+1)} \}$ is a basis for C_{i+1}, $\{ \bar{f}_{w\chi} \mid w \in W^{(i+1)} \}$ is a basis for $C_{i+1}/\mathfrak{n}_- C_{i+1}$. Since the $\bar{f}_{w\chi}$ have different eigenvalues, it suffices to show that $\bar{d}_{i+1}(\bar{f}_{w\chi}) \neq 0$ for any $w \in W^{(i+1)}$.
This is shown in two sublemmas:

Lemma 10.6.a. The composition factors of K are of the form $L_{w\chi}$, $l(w) > i$.

Proof: using 9.9, let $B^\vee_\bullet$ be the resolution of V and recall that $JH(B_j^\vee) = \bigcup_{w \in W^{(j)}} JH(M_{w\chi}) = JH(C_j^\vee)$
Now note that we have exact sequences

$$\begin{tikzcd}
0 & V \arrow[l] & C_0^\vee \arrow[l] \arrow[d, phantom, "\updownarrow \text{same JH}"] & \cdots \arrow[l] & C_i^\vee \arrow[l] \arrow[d, phantom, "\updownarrow \text{same JH}"] & K \arrow[l] \arrow[d, phantom, "\Rightarrow \updownarrow \underline{\text{same JH}}"] & 0 \arrow[l] \\
0 & V \arrow[l] & B_0^\vee \arrow[l] & \cdots \arrow[l] & B_i^\vee \arrow[l] & \underbrace{\operatorname{Ker}(B_i^\vee \to B_{i-1}^\vee)}_{K_B :=} \arrow[l] & 0 \arrow[l]
\end{tikzcd}$$

Thus $JH(K_B) \subset JH(B_{i+1}^\vee) = \bigcup_{w \in W^{(i+1)}} JH(M_{w\chi})$ and we are done by the Corollary in §8.

Lemma 10.6.b: Let $w_0 \in W$ and $M \in \mathcal{O}$. Assume $\ell(w) \geq \ell(w_0)$ for all $L_{w\chi} \in JH(M)$. Let $\tau : M_{w_0\chi} \to M$ be a homomorphism s.t. $\tau(f_{w_0\chi}) \neq 0$. Then $\bar{\tau}(\overline{f_{w_0\chi}})$ in $M/_{n_- M}$ is $\neq 0$.

Note that applying this to $M = K$, $\tau = d_{i+1}$ and using 10.6.a for the assumption $\ell(w) \geq \ell(w_0) = i+1$, we get $\overline{d_{i+1}}(\overline{f_{w_0\chi}}) \neq 0$, concluding the proof of 10.6.

Proof of 10.6.b: By induction on card $JH(M)$. Let $f \in M$ of weight $\psi - \rho$ with ψ maximal and $N \subset M$ the submodule generated by f, so that N is a quotient of M_ψ.
If $\tau(f_{w_0\chi}) \notin N$ then applying the result to M/N gives the result by induction hypothesis. So assume $\tau(f_{w_0\chi}) \in N$. Then obviously $L_{w_0\chi} \in JH(N) \subset JH(M_\psi)$, so by the Corollary in §8, $\psi = w_1\chi$ where $w_1 \geq w_0$. Now since $L_\psi \in JH(N) \subset JH(M)$, by hypothesis $\ell(w_1) \geq \ell(w_0)$ so we must have $w_0 = w_1$. Finally since ψ is maximal in M, $\tau(f_{w_0\chi}) \notin n_- M$, and we are done. ◻

Proof of Lemma 10.7: Let $f_1, \dots, f_n \in K$ be weight vectors mapping to a basis of $K/_{n_- K}$ and consider the map

$$C := \bigoplus_{i=1}^{n} U(n_-) g_i \longrightarrow K \qquad g_i \longmapsto f_i$$

of $U(n_-)$-modules. The induced map $C/_{n_- C} \to K/_{n_- K}$ is clearly surjective, (in fact an isomorphism by dimension) so by 10.5, $C \twoheadrightarrow K$.

Now complete the free resolution

$$0 \leftarrow V \leftarrow C_0^\vee \leftarrow \dots \leftarrow C_i^\vee \xleftarrow{\theta} C \xleftarrow{\eta} D \leftarrow \dots$$

and henceforth denote $\overline{(\;)} := 1 \otimes_{U(n_-)} (\;)$, so that we get

$$(*) \quad \dots \to \bar{D} \xrightarrow{\bar{\eta}} \bar{C} \xrightarrow{\bar{\theta}} \bar{C}_i^\vee \to \bar{C}_{i-1}^\vee \to \dots$$

By definition, $\mathrm{Tor}_i^{n_-}(\mathbb{C}, V) = \dfrac{\mathrm{Ker}\,\bar{\theta}}{\mathrm{Im}\,\bar{\eta}}$.

Now from $(*)$, $\bar{D} \xrightarrow{\bar{\eta}} \bar{C} \xrightarrow{\bar{\tau}} \bar{K} \to 0$ is exact, and since $\bar{\tau}$ is an isom, $\bar{\eta} = 0$.

On the other hand, from $(*)$, $\bar{C} \xrightarrow{\bar{\theta}} \bar{C}_i^\vee \xrightarrow{\bar{d}_i} \overline{\mathrm{Ker}(d_{i-1})} \to 0$ is exact. By lemma 10.6, $\bar{d}_i$ is an isom and $\bar{\theta} = 0$.

We conclude that $\mathrm{Tor}_i^{n_-}(\mathbb{C}, V) = \bar{C} = K/_{n_- K}$. But by Bott's theorem

$\dim \mathrm{Tor}_i^{n_-}(\mathbb{C}, V) = \mathrm{card}\, W^{(i)} = \dim \left(C_i/_{n_- C_i} \right)$, and the lemma 10.7 is proved.

This concludes the proof of Thm 10.1.