Lecture 5

Recap:

- Characterized linear independence, spanning, bases in terms of rank:

  $v_1, \ldots, v_n \in \mathbb{R}^m$ are linearly independent $\Leftrightarrow$ $\operatorname{rank}\begin{pmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{pmatrix} = \#\text{vectors } (=n)$

  $v_1, \ldots, v_n \in \mathbb{R}^m$ are a spanning set $\Leftrightarrow$ $\operatorname{rank}\begin{pmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{pmatrix} =$ "dimension" of the space $(=m)$
  $(\dim \mathbb{R}^m = m.)$

  $v_1, \ldots, v_n \in \mathbb{R}^m$ are a basis $\Leftrightarrow$ $m = n = \operatorname{rank}\begin{pmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{pmatrix}$.

- Defined linear transformations.

  - A function $T: \mathbb{R}^n \to \mathbb{R}^m$ given by $T\left(\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}\right) = \begin{pmatrix} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n \end{pmatrix}$ for some fixed $\underbrace{a_{ij}}_{\text{Matrix of the transformation}} \in \mathbb{R}$.

  - Equivalently, a function $T: \mathbb{R}^n \to \mathbb{R}^m$ given by $T\left(\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}\right) = x_1 a_1 + x_2 a_2 + \ldots + x_n a_n$
  for some fixed vectors $a_1, \ldots, a_n \in \mathbb{R}^m$

  - Equivalently, a function $T: \mathbb{R}^n \to \mathbb{R}^m$ such that $T(v+w) = T(v) + T(w)$, $T(\lambda v) = \lambda T(v)$ $\quad \forall v, w \in \mathbb{R}^n, \lambda \in \mathbb{R}$

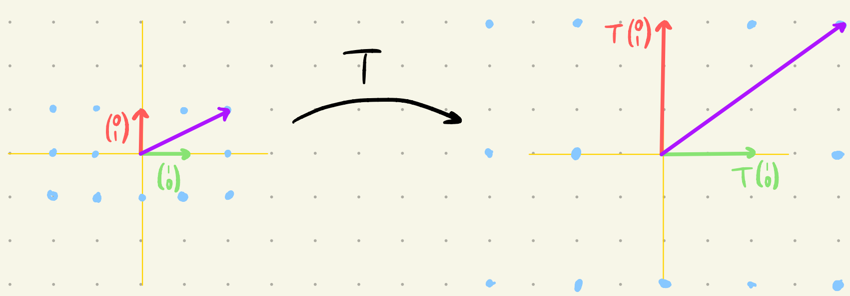
Today: More on linear transformations and matrix products

Examples of linear transformations $\mathbb{R}^2 \to \mathbb{R}^2$ (good to keep in mind).

Example 1: Scaling

Consider the linear transformation $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto \begin{pmatrix} 2x_1 \\ 3x_2 \end{pmatrix} = \begin{pmatrix} 2x_1 + 0x_2 \\ 0x_1 + 3x_2 \end{pmatrix}$. Its associated matrix is $\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$.

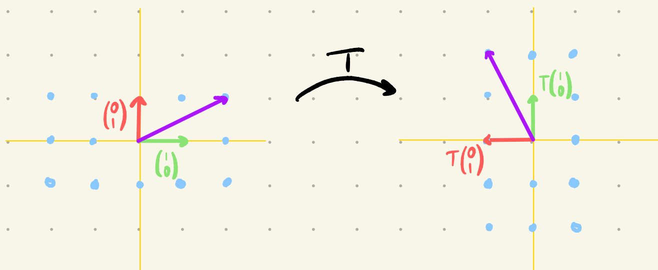
This matrix scales the x-coordinates by 2 and the y-coordinates by 3.



Example 2: Rotation:

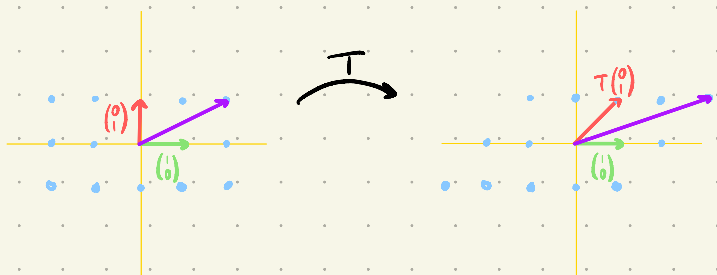
Consider $T\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix} = \begin{pmatrix} 0x_1 - x_2 \\ x_1 + 0x_2 \end{pmatrix}$ Its matrix is $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

This matrix is a counterclockwise $90°$ rotation.



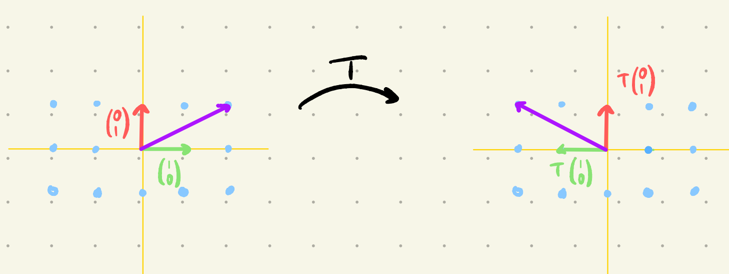
Example 3: Shear:

$$T\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ x_2 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$



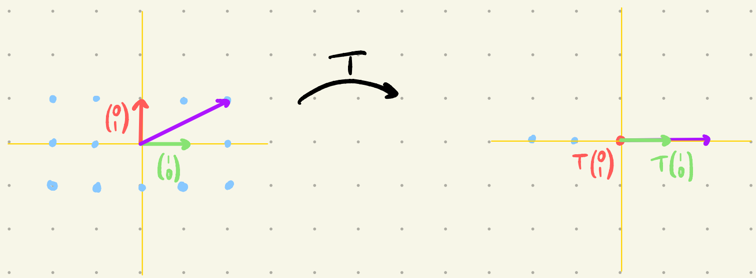
Example 4: Reflection:

$$T\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -x_1 \\ x_2 \end{pmatrix} \rightsquigarrow \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$



Example 5: Projection

$$T\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right) = \begin{pmatrix} x_1 \\ 0 \end{pmatrix}.$$



Observe: this is the only one that we've seen that "loses information".

Example 6: $T\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. This loses all of the information.

<u>Composing linear transformations</u>

Suppose $T_1 : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ and $T_2 : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ are given by:

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad\qquad B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

(rotation) (shear)

and I want to compose them one after the other.

Crucial observation: $T_2 \circ T_1$ is linear. Proof: $T_2 \circ T_1(v+w) \overset{T_1 \text{ linear}}{=} T_2(T_1(v) + T_2(w)) \overset{T_2 \text{ linear}}{=} T_2 \circ T_1(v) + T_2 \circ T_1(w)$.

$$T_2 \circ T_1(\lambda v) = T_2(\lambda T_1(v)) = \lambda T_2 \circ T_1(v) \quad \square$$

Then, what is the matrix of $T_2 \circ T_1$?

Idea: only have to look at where $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ go.

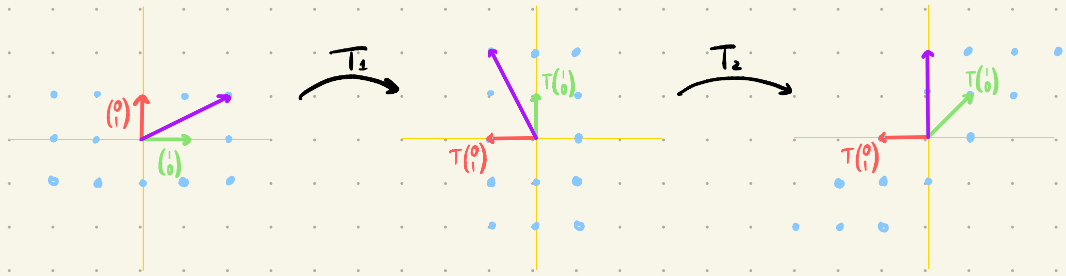
Now $T_1\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ (first column of A)

and $T_2\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Similarly, $T_1\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$

$$T_2\left(\begin{pmatrix} -1 \\ 0 \end{pmatrix}\right) = -T_2\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) = -\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}.$$

Geometrically:



So the matrix for $T_2 \circ T_1$ is $\begin{pmatrix} | & | \\ T_2 \circ T_1\left(\binom{1}{0}\right) & T_2 \circ T_1\left(\binom{0}{1}\right) \\ | & | \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$

In general, if $T_1 : \mathbb{R}^{n_1} \to \mathbb{R}^{n_2}$ and $T_2 : \mathbb{R}^{n_2} \to \mathbb{R}^{n_3}$ are linear transformations with matrices A and B, then the matrix for the linear transformation $T_2 \circ T_1 : \mathbb{R}^{n_1} \to \mathbb{R}^{n_3}$ is

$$\begin{pmatrix} | & | & & | \\ T_2 \circ T_1 \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} & T_2 \circ T_1 \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix} & \cdots & T_2 \circ T_1 \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \\ | & | & & | \end{pmatrix}$$

$$= \begin{pmatrix} | & | & & | \\ T_2 \begin{pmatrix} \text{1st column} \\ \text{of } A \end{pmatrix} & T_2 \begin{pmatrix} \text{2nd column} \\ \text{of } A \end{pmatrix} & \cdots & T_2 \begin{pmatrix} \text{last column} \\ \text{of } A \end{pmatrix} \\ | & | & & | \end{pmatrix}$$

$$= \begin{pmatrix} | & | & & | \\ B \cdot \begin{pmatrix} \text{1st column} \\ \text{of } A \end{pmatrix} & B \begin{pmatrix} \text{2nd column} \\ \text{of } A \end{pmatrix} & \cdots & B \begin{pmatrix} \text{last column} \\ \text{of } A \end{pmatrix} \\ | & | & & | \end{pmatrix}$$

We have arrived at a convincing definition for the product of two matrices:

Definition 1: Let $A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n_1} \\ a_{21} & & & \vdots \\ \vdots & & & \vdots \\ a_{n_2 1} & - & - & a_{n_2 n_1} \end{pmatrix}$ $\leftrightarrow T_1 : \mathbb{R}^{n_1} \to \mathbb{R}^{n_2}$

$B = \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n_2} \\ \vdots & & & \\ b_{n_3 1} & b_{n_3 2} & \cdots & b_{n_3 n_2} \end{pmatrix}$ $\leftrightarrow T_2 : \mathbb{R}^{n_2} \to \mathbb{R}^{n_3}$

Then their product is defined as

$$BA = \begin{pmatrix} B \cdot \begin{pmatrix} a_{11} \\ \vdots \\ a_{n_2 1} \end{pmatrix} & B \begin{pmatrix} a_{12} \\ \vdots \\ a_{n_2 2} \end{pmatrix} & \cdots & B \begin{pmatrix} a_{1n_1} \\ \vdots \\ a_{n_2 n_1} \end{pmatrix} \end{pmatrix}$$

$T_2 \circ T_1 : \mathbb{R}^{n_1} \to \mathbb{R}^{n_3}$

Example 6:

$$\overbrace{\begin{pmatrix} 1 & 2 & -3 \\ 2 & -1 & 0 \end{pmatrix}}^{T_2 : \mathbb{R}^3 \to \mathbb{R}^2} \cdot \overbrace{\begin{pmatrix} 1 & 1 & 1 & 0 \\ -1 & 0 & -1 & 0 \\ 0 & -1 & 1 & 1 \end{pmatrix}}^{T_1 : \mathbb{R}^4 \to \mathbb{R}^3} = \begin{pmatrix} 1\cdot 1 + 2\cdot(-1) - 3\cdot 0 & 1\cdot 1 - 3\cdot(-1) & 1\cdot 1 + 2\cdot(-1) - 3\cdot 1 & 1\cdot 0 + 2\cdot 0 - 3\cdot 1 \\ 2\cdot 1 - 1\cdot(-1) + 0\cdot 0 & 2\cdot 1 - 1\cdot 0 + 0\cdot(-1) & 2\cdot 1 - 1\cdot(-1) + 0\cdot 1 & 2\cdot 0 - 1\cdot 0 + 0\cdot 1 \end{pmatrix}$$

$$= \underbrace{\begin{pmatrix} -1 & 4 & -4 & -3 \\ 3 & 2 & 3 & 0 \end{pmatrix}}_{T_2 \circ T_1 : \mathbb{R}^4 \to \mathbb{R}^2}$$

Remark: matrix multiplication is not commutative:

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$$
$$\neq$$
$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$$

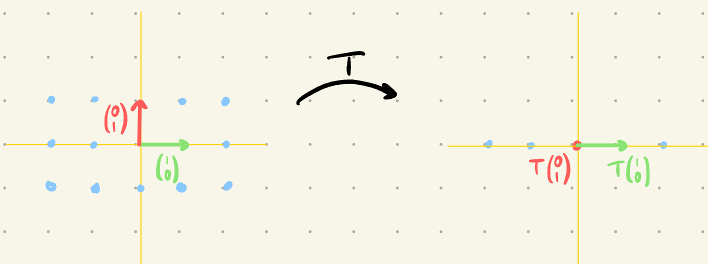
Invertibility

Definition 2: A linear transformation T has an inverse iff there exists another linear transformation T^{-1}, such that $T \circ T^{-1} =$ identity map and $T^{-1} \circ T =$ identity map.

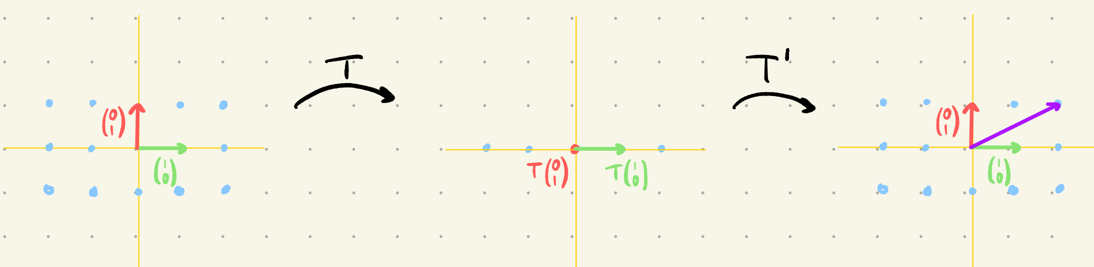
(Identity map: $f(x) = x$, matrix is $\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix}$)

Sometimes, linear tranformations can be inverted. For instance, the inverse of a counterclockwise $90°$ rotation is a clockwise $90°$ rotation. Similarly, scalings can be inverted. However, projections cannot, as the following example illustrates.

Example 7: Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be given by $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$.



Then, if T is invertible, we would have a linear transformation T' such that:



The problem is: $T\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$, so $T'\circ T\begin{pmatrix}0\\1\end{pmatrix} = T'\left(T\begin{pmatrix}0\\1\end{pmatrix}\right) = T'\begin{pmatrix}0\\0\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix} \neq \begin{pmatrix}0\\1\end{pmatrix}$.

So such a T' cannot exist!

In general, if T "kills" some vector, then T cannot have an inverse.

Another problem: the image of each element of $\mathbb{R}^2$ under T is a multiple of $\begin{pmatrix}1\\0\end{pmatrix}$, so in particular

we cannot have $T\circ T'\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}0\\1\end{pmatrix}$, since $T\circ T'\begin{pmatrix}0\\1\end{pmatrix} = \underbrace{T\left(T'\begin{pmatrix}0\\1\end{pmatrix}\right)}_{\text{image of an element under } T}$.

We will come back to these ideas later. In order to play around with inverses, it will be useful to learn how to compute them.

<u>Fact</u>: Only square matrices have inverses (we will prove this soon).

<u>How to find the inverse of a matrix</u> (if it exists)

Suppose we want to find the inverse of A, the matrix of a linear transformation $T: \mathbb{R}^n \to \mathbb{R}^n$.

Then we seek another square matrix A' such that $AA' = \mathrm{Id}$

In particular, $A\underbrace{A'\cdot\begin{pmatrix}1\\0\\\vdots\\0\end{pmatrix}}_{\text{first column of } A'} = \begin{pmatrix}1\\0\\\vdots\\0\end{pmatrix} \xrightarrow{\text{Gaussian elim}}$ Can figure out the first column of A'!

Similarly, n linear systems will give all of A', column by column. If no inverse exists, one of the systems will be incompatible.

<u>Example 8</u>: Let us find the inverse of $\begin{pmatrix}0 & -1\\1 & 2\end{pmatrix}$.

$\begin{pmatrix}0 & -1\\1 & 2\end{pmatrix}\cdot\underbrace{A'}_{\text{unknown}}\cdot\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix}$ Get a system:

$\left(\begin{array}{cc|c}0 & -1 & 1\\1 & 2 & 0\end{array}\right) \xrightarrow{\mathrm{I}\leftrightarrow\mathrm{II}} \left(\begin{array}{cc|c}1 & 2 & 0\\0 & -1 & 1\end{array}\right) \xrightarrow{\mathrm{II}\to-\mathrm{II}} \left(\begin{array}{cc|c}1 & 2 & 0\\0 & 1 & -1\end{array}\right) \xrightarrow{\mathrm{I}\to\mathrm{I}-2\cdot\mathrm{II}} \left(\begin{array}{cc|c}1 & 0 & 2\\0 & 1 & -1\end{array}\right) \Rightarrow \begin{pmatrix}2\\-1\end{pmatrix}$ is the 1st column

$\begin{pmatrix}0 & -1\\1 & 0\end{pmatrix}\cdot A'\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}0\\1\end{pmatrix}$

$\left(\begin{array}{cc|c}0 & -1 & 0\\1 & 2 & 1\end{array}\right) \xrightarrow{\mathrm{I}\leftrightarrow\mathrm{II}} \left(\begin{array}{cc|c}1 & 2 & 1\\0 & -1 & 0\end{array}\right) \xrightarrow{\mathrm{II}\to-\mathrm{II}} \left(\begin{array}{cc|c}1 & 2 & 1\\0 & 1 & 0\end{array}\right) \xrightarrow{\mathrm{I}\to\mathrm{I}-2\cdot\mathrm{II}} \left(\begin{array}{cc|c}1 & 0 & 1\\0 & 1 & 0\end{array}\right) \Rightarrow \begin{pmatrix}1\\0\end{pmatrix}$ is the 2nd column

$\Rightarrow A^{-1} = \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}$. let's check this: $\begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix} \cdot \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ ✓

$\begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ ✓

<u>Remark</u>: Notice that solving the 2 systems required the same steps. We can put these steps together as follows:

$$\left(A \;\middle|\; \begin{matrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{matrix} \right) \overset{\text{Gaussian elim.}}{\rightsquigarrow} \left(\operatorname{rref}(A) \;\middle|\; A' \right)$$

If $\operatorname{rref}(A) = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$, then the inverse of A is A'. Otherwise, A has no inverse.

Observation: once you have computed A^{-1}, you never have to do Gaussian elim to solve $Ax=b$ ever again! Simply, $Ax = b \Rightarrow A^{-1}Ax = A^{-1}b \Rightarrow x = A^{-1}b$.

1. What is the matrix of the tranformation $\mathbb{R}^2 \to \mathbb{R}^2$ which reflects along the line $y=x$? What about the projection onto that same line? (Hint: where do the basis vectors go?) Are both of them invertible? Find the inverses if they exist.
2. "Draw" the linear tranformations for $A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ and AB as in the lecture. Check that AB sends the basis vectors to the same images as composing the other two.
3. Find the inverse of the matrix $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$.