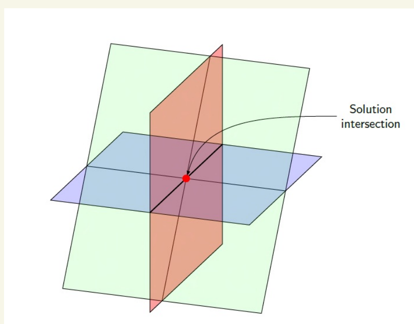
Lecture 2:

Recap:
- Systems of linear equations
  $\updownarrow$
- Augmented matrices
- RREFs and Gaussian elimination

Today: Systems with infinitely many solutions, rank of a matrix.

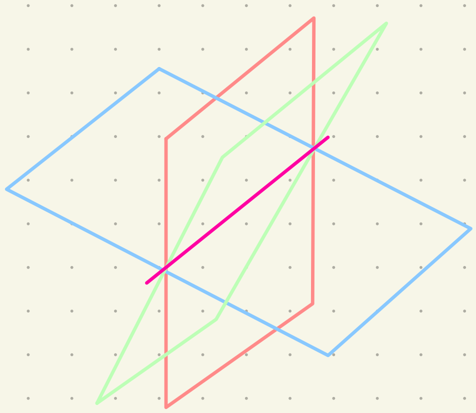
Recall that last time we had situations like

3 equations $\longleftrightarrow$ 3 planes $\longleftrightarrow$ 1 intersection point



However one could also have a situation like

3 equations $\longleftrightarrow$ 3 planes $\longleftrightarrow$ intersection is a line



Then, the solution set will have <u>free variables</u>, to account for all the possible values.

How does it work in terms of the augmented matrix? Suppose that, after Gaussian elimination we obtain:

$$\left(\begin{array}{ccccc|c} 1 & 3 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Then the free variables will correspond to the non-pivot columns. In this case,

$$\left.\begin{array}{l} x_2 = t \ , \quad t \in \mathbb{R} \\ x_4 = s \ , \quad s \in \mathbb{R} \end{array}\right\} \text{free variables}$$

$$\left.\begin{array}{l} x_1 = -3t + s \\ x_3 = 2 - 2s \\ x_5 = -3 \end{array}\right\} \text{determined by the free variables}$$

In other words, the solution set is $\{(-3t+s, t, 2-2s, s, -3) \mid t, s \in \mathbb{R}\}$

Definition 1: a linear system of equations is <u>consistent</u> (or "compatible") iff it has one (or more) solutions.

It is <u>inconsistent</u> iff it has no solution.

<u>Theorem 1</u> Let M be the augmented matrix of a linear system, and let A be an RREF for M. Then:

1) The system is consistent if and only if A has no row of the form $(0 \cdots 0 \mid 1)$

2) The system has a unique solution if and only if A is of the form

$$\left(\begin{array}{cccc|c} 1 & & & & * \\ & 1 & & & * \\ & & \ddots & & \vdots \\ & & & 1 & * \\ & & & & 0 \\ & & & & \vdots \\ & & & & 0 \end{array}\right)$$

3) The solution set for a consistent system has s free variables, where

$s = \#$ non-pivot columns to the left of the divider.

(We have seen examples of each)

Definition 2: the **rank** of a matrix A is the number of pivots in an RREF for A.

Corollary 1: Let $M = \left(\begin{array}{c|c} A & \begin{matrix} * \\ * \\ \vdots \\ * \end{matrix} \end{array}\right)$ be an augmented m-by-$(n+1)$ matrix for a linear system, with coefficient matrix A. Then,

m: # equations, n = #variables

1) $\operatorname{rank}(A) \leq m$ and $\operatorname{rank}(A) \leq n$

2) If the system is inconsistent, $\operatorname{rank}(A) < m$

3) If the system has a unique solution, $\operatorname{rank}(A) = n$.

4) If the system has infinitely many solutions, $\operatorname{rank}(A) < n$

Proof: Let $M' = \left(\begin{array}{c|c} C' & \begin{matrix} * \\ \vdots \\ * \end{matrix} \end{array}\right)$ be an RREF for M. (Note that C' is also an RREF for C)

1)
$$M' = m\left\{\left(\begin{array}{cccc|c} \multicolumn{4}{c|}{\overbrace{\hphantom{1 \quad 1 \quad 1 \quad 1}}^{n}} & \\ 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ 0 & & & 1 & \end{array}\right)\right. \Rightarrow \text{\# pivots in } C' \text{ is} \leq n \text{ and } \leq m.$$

2) Inconsistent $\overset{\text{Theorem 1}}{\Rightarrow}$ some row in M' looks like $(0 \cdots 0 \mid 1)$

$\Rightarrow$ some row in C' is $(0 \cdots 0)$ (has no pivot)

$\Rightarrow$ #pivots is $< m$

3) Unique solution $\overset{\text{Theorem 1}}{\Rightarrow}$ B looks like
$$\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \\ & 0 & & \end{pmatrix}$$

$=$ #pivots $= n$.

4) Infinitely many solutions $\Rightarrow$ there are some free variables

$\overset{\text{Theorem 1}}{\Rightarrow}$ some column in A doesn't have a pivot: $B = \underbrace{\begin{pmatrix} 1 & \ddots & * & & \\ & & * & & \\ & & * & & \\ & & 0 & 0 & 1 \cdots \end{pmatrix}}_{n}$

$\Rightarrow$ #pivots $< n$.

Example 1: find the rank of the matrix $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$. How many solutions can the system associated to the augmented matrix $M = \left(\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 4 & 5 & 6 & 15 \\ 7 & 8 & 9 & 24 \end{array}\right)$ have?

Rank: Step 1: put A in rref.

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \xrightarrow{\mathrm{II} \to \mathrm{II} - 4 \cdot \mathrm{I}} \begin{pmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 7 & 8 & 9 \end{pmatrix}$$

$$\xrightarrow{\mathrm{III} \to \mathrm{III} - 7 \cdot \mathrm{I}} \begin{pmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & -6 & -12 \end{pmatrix}$$

$$\xrightarrow{\mathrm{II} \to \frac{1}{-3} \cdot \mathrm{II}} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & -6 & -12 \end{pmatrix}$$

$$\xrightarrow{\mathrm{I} \to \mathrm{I} - 2 \cdot \mathrm{II}} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & -6 & -12 \end{pmatrix}$$

$$\xrightarrow{\mathrm{III} \to \mathrm{III} + 6 \cdot \mathrm{II}} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

Step 2: Two pivots $\Rightarrow$ rank$(A) = 2$.

Step 3:

Number of solutions: if we're handed the system $\left(\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 4 & 5 & 6 & 15 \\ 7 & 8 & 9 & 24 \end{array}\right)$, this has a solution: $(x,y,z) = (1,1,1)$. Since the rank of A is $2 < 3 = \#$variables, the system must have <u>infinitely many</u> solutions.

Example 2: How many solutions does $M = \left(\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 4 & 5 & 6 & 0 \\ 7 & 8 & 9 & 1 \end{array}\right)$ have?

No obvious solutions $\Rightarrow$ Need to use Gaussian elimination.

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 4 & 5 & 6 & \\ 7 & 8 & 9 & \end{array}\right) \xrightarrow{\mathrm{II} \to \mathrm{II} - 4 \cdot \mathrm{I}} \left(\begin{array}{ccc|c} 1 & 2 & 3 & \\ 0 & -3 & -6 & 0 \\ 7 & 8 & 9 & \end{array}\right)$$

$$\xrightarrow{\mathrm{III} \to \mathrm{III} - 7 \cdot \mathrm{I}} \begin{pmatrix} 1 & 2 & 3 & \\ 0 & -3 & -6 & 0 \\ 0 & -6 & -12 & \end{pmatrix}$$

$$\xrightarrow{\mathrm{II} \to \frac{1}{-3} \cdot \mathrm{II}} \begin{pmatrix} 1 & 2 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & -6 & -12 & 1 \end{pmatrix}$$

$$\xrightarrow{\mathrm{I} \to \mathrm{I} - 2 \cdot \mathrm{II}} \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & -6 & -12 & 1 \end{pmatrix}$$

$$\xrightarrow{\mathrm{III} \to \mathrm{III} + 6 \cdot \mathrm{II}} \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \Rightarrow \text{ System is } \underline{\text{inconsistent}}$$

Example 3: How many solutions does $M = \left(\begin{array}{cc|c} 1 & 2 & 5 \\ 3 & 4 & 6 \end{array}\right)$ have?

$\underbrace{\quad\quad}_{A}$

Step 1: rank(A)

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \xrightarrow{\mathrm{II} \to \mathrm{II} - 3 \cdot \mathrm{I}} \begin{pmatrix} 1 & 2 \\ 0 & -2 \end{pmatrix} \xrightarrow{\mathrm{II} \to \frac{1}{-2} \mathrm{II}} \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \xrightarrow{\mathrm{I} \to \mathrm{I} - 2 \mathrm{II}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Step 2: two pivots $\Longrightarrow$ rank$(A) = 2$

Step 3: rank(A) = # variables $\Rightarrow$ There is exactly one solution.

Example 4: For what values of λ does the matrix $A = \begin{pmatrix} 2 & -1 \\ 1 & \lambda \end{pmatrix}$ have rank 1?

Step 1: RREF: $\begin{pmatrix} 2 & -1 \\ 1 & \lambda \end{pmatrix} \xrightarrow{\mathrm{I} \to \frac{1}{2} \mathrm{I}} \begin{pmatrix} 1 & -\frac{1}{2} \\ 1 & \lambda \end{pmatrix} \xrightarrow{\mathrm{II} \to \mathrm{II} - \mathrm{I}} \begin{pmatrix} 1 & -\frac{1}{2} \\ 0 & \lambda + \frac{1}{2} \end{pmatrix}$

If $\lambda + \frac{1}{2} \neq 0$, then $\xrightarrow{\mathrm{II} \to \frac{1}{\lambda + \frac{1}{2}} \mathrm{II}} \begin{pmatrix} 1 & -\frac{1}{2} \\ 0 & 1 \end{pmatrix}$ <u>Two pivots</u>. This cannot happen if rank$(A) = 1$.

Thus $\lambda = -\frac{1}{2}$, in which case rref$(A) = \begin{pmatrix} 1 & -\frac{1}{2} \\ 0 & 0 \end{pmatrix}$ and indeed rank$(A) = 1$.

Discussion: 2×2 matrices

Take $M = \left(\begin{array}{cc|c} a & b & e \\ c & d & f \end{array}\right)$, with $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

If $\operatorname{rank}(A) = 0$ then there are no pivots, we didn't have a nonzero row to begin with.

$\Rightarrow A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

Then, the system has a solution iff $(e,f) = (0,0)$, in which case $(x,y) = (t,s)$ is a solution for each $t \in \mathbb{R}$, $s \in \mathbb{R}$.

If $\operatorname{rank}(A) = 1$, we have a single pivot, so $\operatorname{rref}(A) = \begin{pmatrix} 1 & * \\ 0 & 0 \end{pmatrix}$ or $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

Starting with $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and assuming $a \neq 0$ (other cases are similar), we get

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \xrightarrow{I \to \frac{1}{a} I} \begin{pmatrix} 1 & \frac{b}{a} \\ c & d \end{pmatrix} \xrightarrow{II \to II - c \cdot 1} \begin{pmatrix} 1 & \frac{b}{a} \\ 0 & d - \frac{bc}{a} \end{pmatrix}$$

Since $\operatorname{rank}(A) = 1$, $\lambda = d - \frac{bc}{a}$ must be zero, as otherwise we would get $\cdots \xrightarrow{II \to \frac{1}{\lambda} II} \begin{pmatrix} 1 & \frac{b}{a} \\ 0 & 1 \end{pmatrix}$ two pivots

We see that $(c, d) = (c \cdot a, c \cdot b)$, a multiple of the first row.

If $\operatorname{rank}(A) = 2$, the same computation shows that $d - \frac{bc}{a} \neq 0$ and the second row is not a multiple of the first row.

This is a lot to keep track of, especially with larger matrices. We will develop a theoretical framework to understand this data.

Main take-aways:

- 2×2 matrices have full rank if and only if a certain quantity $ad - bc$ is $\neq 0$.
- 2×2 matrices cannot have full rank if one is a multiple of another.

A look ahead: we can think of linear systems as follows:

$$\left.\begin{aligned} x + 2y &= 3 \\ \underbrace{x - 3y} &= -2 \end{aligned}\right\} \quad (*)$$

Can see this as a function

$$\begin{pmatrix} x \\ y \end{pmatrix} \to \boxed{f} \to \begin{pmatrix} x+2y \\ x-3y \end{pmatrix}$$

Let $\mathrm{Im}(f) = \{ \text{images of } f \} = \{ f(\begin{pmatrix} r \\ s \end{pmatrix}) \mid r, s \in \mathbb{R} \}$

Then the system $(*)$ has a solution translates to: is $\begin{pmatrix} 3 \\ -2 \end{pmatrix}$ in the image of f?

The main character in this course will be functions such as

$$\begin{pmatrix} x \\ y \end{pmatrix} \xrightarrow{f} \begin{pmatrix} x+2y \\ x-3y \end{pmatrix}$$

These tuples are called "vectors" and f is an example of a linear transformation.

As we have seen, these are in correspondence with matrices. In this case,

$$f \longleftrightarrow \begin{pmatrix} 1 & 2 \\ 1 & -3 \end{pmatrix}$$

Extra questions: (if you haven't seen this before): can you find f^{-1}? In other words, another function f^{-1} such that $f^{-1}\left(f\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} x \\ y \end{pmatrix}$ for all $x, y \in \mathbb{R}$?

What is the matrix associated to $f \circ f$ (f composed with itself)?

In-class exercise session:

1. Find the solutions of the system with augmented matrix $\left(\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 4 & 5 & 6 & 15 \\ 7 & 8 & 9 & 24 \end{array}\right)$. Verify that there's infinitely many.

2. Find a value of λ for which the matrix $\begin{pmatrix} -2 & -1 & 2 \\ 4 & 2 & -4 \\ 1 & \lambda & 1 \end{pmatrix}$ has rank 1.

   Are there other values?

3. Find the solution to the system $\left(\begin{array}{cc|c} 1 & 2 & a \\ 3 & 4 & b \end{array}\right)$ for each $a, b \in \mathbb{R}$.