Lecture 1

Systems of linear equations

Example 1: let's solve the following system of equations:

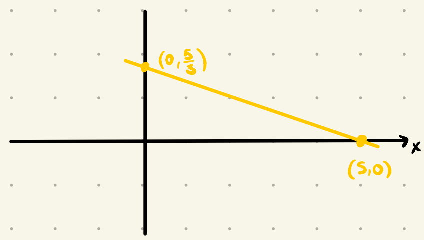
$$\left.\begin{aligned} x + 3y &= 5 \\ 2x - y &= 3 \end{aligned}\right| \xrightarrow{\mathrm{II} \to \mathrm{II} - 2\cdot \mathrm{I}} \left.\begin{aligned} x + 3y &= 5 \\ -7y &= -7 \end{aligned}\right|$$

$$\xrightarrow{\mathrm{II} \to \frac{1}{-7}\cdot \mathrm{II}} \left.\begin{aligned} x + 3y &= 5 \\ y &= 1 \end{aligned}\right|$$

$$\xrightarrow{\mathrm{I} \to \mathrm{I} - 3\cdot \mathrm{II}} \left.\begin{aligned} x \quad &= 2 \\ y &= 1 \end{aligned}\right| \qquad \text{Done!}$$

Why linear?

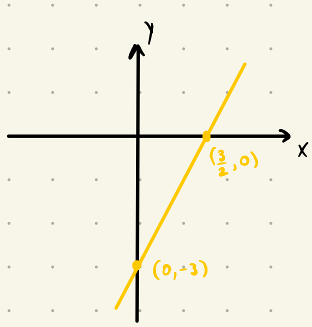
$x + 3y = 5$ is the equation for a line:



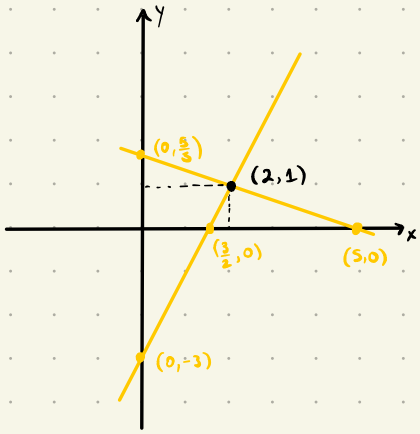
$x = 0 \Rightarrow y = \frac{5}{3}$

$y = 0 \Rightarrow x = 5$

Same for $2x - y = 3$:



Then the solution of the system is precisely the intersection of the two lines:



This "algorithm" also works for systems with more variables and more equations:

Example 2

$$\left.\begin{aligned} x + 2y + 3z &= 39 \\ x + 3y + 2z &= 34 \\ 3x + 2y + z &= 26 \end{aligned}\right| \xrightarrow{\mathrm{II} \to \mathrm{II} - \mathrm{I}} \left|\begin{aligned} x + 2y + 3z &= 39 \\ y - z &= -5 \\ 3x + 2y + z &= 26 \end{aligned}\right|$$

$$\xrightarrow{\mathrm{III} \to \mathrm{III} - 3 \cdot \mathrm{I}} \left.\begin{aligned} x + 2y + 3z &= 39 \\ y - z &= -5 \\ -4y - 8z &= -91 \end{aligned}\right|$$

$$\xrightarrow{\mathrm{III} \to \mathrm{III} + 4 \cdot \mathrm{II}} \left.\begin{aligned} x + 2y + 3z &= 39 \\ y - z &= -5 \\ -12z &= -111 \end{aligned}\right|$$

$$\xrightarrow{\mathrm{II} \to \mathrm{II} - 2 \cdot \mathrm{I}} \left.\begin{aligned} x + 5z &= 49 \\ y - z &= -5 \\ -12z &= -111 \end{aligned}\right|$$

$$\xrightarrow{\mathrm{III} \to \frac{1}{-12}\mathrm{III}} \quad \left.\begin{aligned} x \qquad\quad + 5z &= 49 \\ y - z &= -5 \\ z &= 9.25 \end{aligned}\right|$$

$$\xrightarrow{\mathrm{II} \to \mathrm{II} + \mathrm{III}} \quad \left.\begin{aligned} x \qquad\quad + 5z &= 49 \\ y \qquad &= 4.25 \\ z &= 9.25 \end{aligned}\right|$$

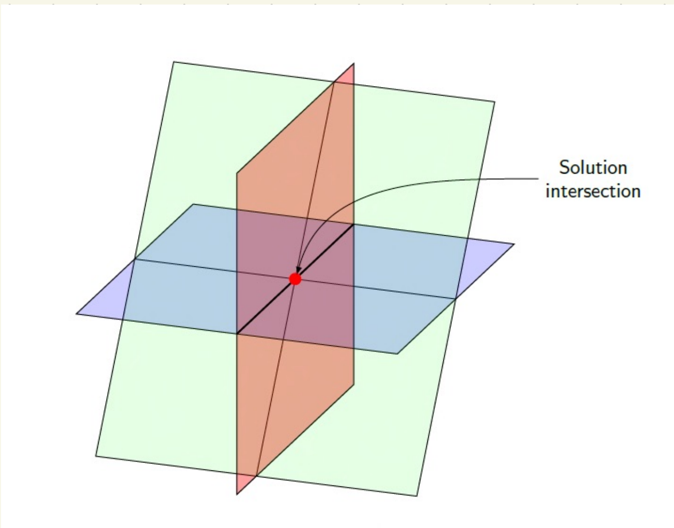
$$\xrightarrow{\mathrm{I} \to \mathrm{I} - 5\,\mathrm{III}} \quad \left.\begin{aligned} x \qquad\quad &= 2.75 \\ y \qquad &= 4.25 \\ z &= 9.25 \end{aligned}\right|$$

$\Rightarrow (x, y, z) = (2.75, 4.25, 9.25)$ is the unique solution to the system.

<u>Remark 1</u>: geometrically, an equation like $x + 2y + 2z = 2$ is the equation of a plane in $\mathbb{R}^3$.

"Usually", two planes intersect in a line, and three planes intersect in a point (see below).

In example 2, the three equations determine three planes which intersect in the point $(2.75, 4.25, 9.25)$.

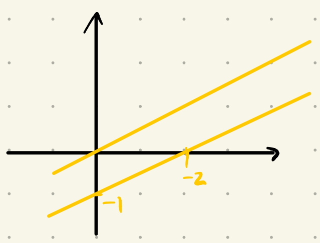


Remark 2: Sometimes the system doesn't have a unique solution, or even any solutions at all.

Example 3: (No solutions)

$$\begin{aligned} x - 2y &= 2 \\ -2x + 4y &= 0 \end{aligned} \quad \xrightarrow{\mathrm{II} \to \mathrm{II} + 2\mathrm{I}} \quad \begin{aligned} x - 2y &= 3 \\ \underbrace{0 + 0 = 6}_{\text{Impossible! No solutions}} \end{aligned}$$

Geometrically:



The lines are parallel.

Example 4: (Infinitely many solutions)

$$\begin{aligned} x + y + z &= 4 \\ 2x + 3z &= 6 \\ 2y - z &= 2 \end{aligned} \quad \xrightarrow{\mathrm{II} \to \mathrm{II} - 2 \cdot \mathrm{I}} \quad \begin{aligned} x + y + z &= 1 \\ -2y + z &= -2 \\ 2y - z &= 2 \end{aligned}$$

$$\xrightarrow{\mathrm{III} \to \mathrm{III} + \mathrm{II}} \quad \begin{aligned} x + y + z &= 1 \\ -2y + z &= -2 \\ 0 &= 0 \end{aligned}$$

$$\xrightarrow{\mathrm{II} \to \frac{1}{-2}\mathrm{II}} \quad \begin{aligned} x + y + z &= 1 \\ y - \tfrac{1}{2} z &= 1 \\ 0 &= 0 \end{aligned}$$

$$\xrightarrow{\mathrm{I} \to \mathrm{I} - \mathrm{II}} \quad \begin{aligned} x + \tfrac{3}{2} z &= 0 \\ y - \tfrac{1}{2} z &= 1 \\ 0 &= 0 \end{aligned}$$

Can't progress further. Any value for z will give a valid solution.

Therefore, we say that z is a free variable, because it may take any real value. The variables x, y are determined (in this case) by the value of z. The set of solutions is

$$\{ (-\tfrac{3}{2}t, 1+\tfrac{1}{2}t, t) \underbrace{|}_{\text{"such that"}} t \in \mathbb{R} \}$$

[Geometrically: three planes intersecting in a line]

Theorem 1: A system of linear equations either has

- 1 solution
- No solutions
- Infinitely many solutions

Remark 3: Notice that we have been performing operations on systems of equations as if the entries were on a table. Moreover, we don't really need to write "x", "y", "z" over and over.

Definition 1: An m-by-n matrix is a table of real numbers with m rows and n columns:

$$m=3 \left\{ \overbrace{\begin{pmatrix} 2 & -3 & 4 & 2.5 \\ 1.1 & 0 & -1 & 2 \\ -3 & 1 & 0 & 1 \end{pmatrix}}^{n=4} \right.$$

Definition 2 An augmented matrix is a matrix with an extra column and a divider:

$$\left(\begin{array}{cccc|c} 2 & -3 & 4 & 2.5 & 1 \\ 1.1 & 0 & -1 & 2 & -2 \\ -3 & 1 & 0 & 1 & 3 \end{array}\right)$$

To each augmented matrix corresponds a unique system of linear equations and vice-versa:

$$\left(\begin{array}{cccc|c} 2 & -3 & 4 & 2.5 & 1 \\ 1.1 & 0 & -1 & 2 & -2 \\ -3 & 1 & 0 & 1 & 3 \end{array}\right) \longleftrightarrow \begin{array}{l} 2x - 3y + 4z + 2.5w = 1 \\ 1.1x + 0y - 1z + 2w = -2 \\ -3x + 1y + 0z + 1w = 3 \end{array}$$

Then, as before, we can solve the system by performing row operations to the augmented matrix:

- Add a multiple of a row to another row
- Multiply a row by a nonzero number
- Swap two rows

} the three "elementary row operations"

Example 1 revisited:

$$\begin{aligned} x + 3y &= 5 \\ 2x - y &= 3 \end{aligned} \longrightarrow \left(\begin{array}{cc|c} 1 & 3 & 5 \\ 2 & -1 & 3 \end{array}\right)$$

In all of our examples, we obtained an augmented matrix of the form

$$\left(\begin{array}{ccccccccccc|c} 1 & * & 0 & * & * & * & 0 & * & * & 0 & * & * \\ 0 & 0 & 1 & * & * & * & 0 & * & * & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & * & * & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * & * \end{array}\right)$$

Such matrices are said to be in row-reduced echelon form (RREF). More formally:

Reduced row-echelon form
A matrix is said to be in *reduced row-echelon form* (rref) if it satisfies all of the following conditions:

a. If a row has nonzero entries, then the first nonzero entry is a 1, called the *leading* 1 (or *pivot*) in this row.

b. If a column contains a leading 1, then all the other entries in that column are 0.

c. If a row contains a leading 1, then each row above it contains a leading 1 further to the left.

Condition c implies that rows of 0's, if any, appear at the bottom of the matrix.

Theorem 2: Any matrix can be put into RREF by elementary row operations.

Definition: The process of turning a matrix into RREF by performing row operations is called Gaussian elimination.

In-class exercise session:

1. Write down the augmented matrix associated to the following system of linear equations:

$$\begin{aligned} x + y &= 5 \\ x + z &= 7 \\ y + z &= 8 \end{aligned}$$

2. Perform Gaussian elimination on the matrix you obtained in 1 in order to solve the system. Check that your solution satisfies the system of equations.