Mock midterm solutions:

1. (20 points) For the following matrices, determine if the corresponding linear transformation is injective, whether it is surjective, and whether it has an inverse. Justify your answers. In the case that it has an inverse, compute it.

(a) $\begin{pmatrix} 1 & 0 & -1 \\ 2 & 1 & 0 \\ 2 & 1 & 1 \end{pmatrix}$

(b) $\begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$

(c) $\begin{pmatrix} 2 & 1 & -1 \\ 3 & 2 & 1 \end{pmatrix}$

(d) $\begin{pmatrix} 1 & 2 & -1 & 0 \\ 0 & 2 & 2 & 4 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

Note: this question has several different possible solutions, I picked these to showcase different ways to answer them.

1. a) Let T be the linear transformation associated to $A = \begin{pmatrix} 1 & 0 & -1 \\ 2 & 1 & 0 \\ 2 & 1 & 1 \end{pmatrix}$

$$\operatorname{Ker}(T) = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} : T\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\}$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 2 & 1 & 0 & 0 \\ 2 & 1 & -1 & 0 \end{array}\right) \xrightarrow[\mathrm{III} \to \mathrm{III} - 2\cdot \mathrm{I}]{\mathrm{II} \to \mathrm{II} - 2\cdot \mathrm{I}} \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 1 & -3 & 0 \end{array}\right) \xrightarrow{\mathrm{III} \to \mathrm{III} - \mathrm{II}} \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & -1 & 0 \end{array}\right) \xrightarrow{\mathrm{III} \to -\mathrm{III}} \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right) \xrightarrow[\mathrm{I} \to \mathrm{I} - \mathrm{III}]{\mathrm{II} \to \mathrm{II} + 2\cdot \mathrm{III}} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

$\Rightarrow \operatorname{Ker}(T) = \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\} \Rightarrow T$ is injective.

The previous part shows also $\operatorname{rank}(T) = 3 = \dim(\mathbb{R}^3)$, so T is surjective.

Since T is injective and surjective, it is invertible.

Inverse:

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 & 1 & 0 \\ 2 & 1 & -1 & 0 & 0 & 1 \end{array}\right) \xrightarrow[\mathrm{III} \to \mathrm{III} - 2\cdot \mathrm{I}]{\mathrm{II} \to \mathrm{II} - 2\cdot \mathrm{I}} \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & -2 & -2 & 1 & 0 \\ 0 & 1 & -3 & -2 & 0 & 1 \end{array}\right) \xrightarrow{\mathrm{III} \to \mathrm{III} - \mathrm{II}} \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & -2 & -2 & 1 & 0 \\ 0 & 0 & -1 & 0 & -1 & 1 \end{array}\right) \xrightarrow{\mathrm{III} \to -\mathrm{III}} \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & -2 & -2 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & -1 \end{array}\right) \xrightarrow[\mathrm{I} \to \mathrm{I} - \mathrm{III}]{\mathrm{II} \to \mathrm{II} + 2\cdot \mathrm{III}} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 1 \\ 0 & 1 & 0 & -2 & 3 & -2 \\ 0 & 0 & 1 & 0 & 1 & -1 \end{array}\right)$$

$$\Rightarrow A^{-1} = \begin{pmatrix} 1 & -1 & 1 \\ -2 & 3 & -2 \\ 0 & 1 & -1 \end{pmatrix}$$

b) Let $T : \mathbb{R}^3 \to \mathbb{R}^4$ be the linear transformation associated to $\begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$

Then $T\left(\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = T\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ so T is not injective.

Since $\operatorname{rank}(T) \leq 3 < 4 = \dim(\mathbb{R}^4)$, T is not surjective.

Since A is not square, T cannot have an inverse.

c) Let $T: \mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation associated to $\begin{pmatrix} 2 & 1 & -1 \\ 3 & 2 & 1 \end{pmatrix}$.

By rank-nullity, $3 = \dim(\operatorname{Ker}(T)) + \dim(\operatorname{Im}(T))$.

Since $\operatorname{Im}(T) \subseteq \mathbb{R}^2$, $\dim(\operatorname{Im}(T)) \leq 2$. Thus $\dim(\operatorname{Ker}(T)) \geq 1$ so T is not injective.

Notice that $\begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ are linearly independent: if $\lambda \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$\left(\begin{array}{cc|c} 1 & -1 & 0 \\ 2 & 1 & 0 \end{array}\right) \xrightarrow{II \to II - 2I} \left(\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 3 & 0 \end{array}\right) \xrightarrow{II \to \frac{1}{3} II} \left(\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 1 & 0 \end{array}\right) \xrightarrow{I \to I + II} \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right)$$

only solution is $\lambda = \mu = 0$.

It follows that $\dim(\operatorname{Im}(T)) \geq 2$, hence since the codomain is $\mathbb{R}^2$, T is surjective.

d) We find an inverse. This will show that T is injective and surjective.

$$\left(\begin{array}{cccc|cccc} 1 & 2 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 2 & 2 & 4 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right) \xrightarrow[III \to -III]{II \to \frac{1}{2} II} \left(\begin{array}{cccc|cccc} 1 & 2 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 2 & 0 & 1/2 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{I \to I - 2II} \left(\begin{array}{cccc|cccc} 1 & 0 & -3 & -4 & 1 & -1 & 0 & 0 \\ 0 & 1 & 1 & 2 & 0 & 1/2 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right)$$

$$\xrightarrow[II \to II - III]{I \to I + 3 \cdot III} \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & -1 & 1 & -1 & -3 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1/2 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right)$$

$$\xrightarrow[III \to III - IV]{I \to I + IV,\ II \to II - IV} \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & \\ 0 & 1 & 0 & 0 & \\ 0 & 0 & 1 & 0 & \\ 0 & 0 & 0 & 1 & \end{array}\right. \left.\boxed{\begin{array}{cccc} 1 & -1 & -3 & 1 \\ 0 & 1/2 & 1 & -1 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 1 \end{array}}\right)$$

We obtained the identity matrix, so this is the inverse.

2. 2. (20 points) Find the values of $\lambda, \mu \in \mathbb{R}$ such that the matrices $A = \begin{pmatrix} 1 & \lambda \\ \lambda - 1 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ commute. (Two matrices commute if $AB = BA$.)

$$AB = \begin{pmatrix} 1 & 2+\lambda \\ \lambda-1 & 2\lambda-2+1 \end{pmatrix} \qquad BA = \begin{pmatrix} 1+2\lambda-2 & \lambda+2 \\ \lambda-1 & 1 \end{pmatrix}$$

$$AB = BA \iff 2\lambda - 2 + 1 = 1 \iff \lambda = 1.$$

3. 3. Let S_1 and S_2 be subspaces of $\mathbb{R}^n$, and let

$$S_1 \cap S_2 = \{v \in \mathbb{R}^n : v \in S_1 \text{ and } v \in S_2\}$$

Prove or disprove: $S_1 \cap S_2$ is a subspace of $\mathbb{R}^n$. Let

$$S_1 \cup S_2 = \{v \in \mathbb{R}^n : v \in S_1 \text{ or } v \in S_2\}$$

Prove or disprove: $S_1 \cup S_2$ is a linear subspace of $\mathbb{R}^n$.

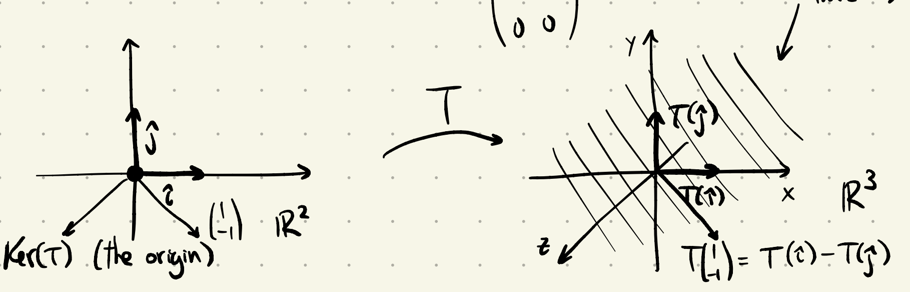
$S_1 \cap S_2$ is a subspace of $\mathbb{R}^n$: take $v_1, v_2 \in S_1 \cap S_2$. Then $v_1 + v_2 \in S_1$ since $v_1, v_2 \in S_1$ and S_1 is a subspace of $\mathbb{R}^n$. Similarly, $v_1 + v_2 \in S_2$. Therefore $v_1 + v_2 \in S_1 \cap S_2$.
Next, take $v \in S_1 \cap S_2$ and $\lambda \in \mathbb{R}$. Then $\lambda v \in S_1$ since $v \in S_1$ and S_1 is a subspace. Similarly, $\lambda v \in S_2$. Therefore $\lambda v \in S_1 \cap S_2$. We conclude that $S_1 \cap S_2$ is a subspace.

$S_1 \cup S_2$ is <u>not</u> a subspace of $\mathbb{R}^n$: let $S_1 = \text{Span}\left(\begin{pmatrix}1\\0\end{pmatrix}\right) \subseteq \mathbb{R}^2$, $S_2 = \text{Span}\left(\begin{pmatrix}0\\1\end{pmatrix}\right) \subseteq \mathbb{R}^2$. Then $\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix} + \begin{pmatrix}0\\1\end{pmatrix}$ is not in $S_1 \cup S_2$, despite the fact that $\begin{pmatrix}1\\0\end{pmatrix}$ and $\begin{pmatrix}0\\1\end{pmatrix}$ are in $S_1 \cup S_2$. It follows that $S_1 \cup S_2$ is <u>not</u> a subspace of $\mathbb{R}^n$.

4. (20 points) Give an example of an injective linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$. Draw your linear transformation in the way we have done it in the course, identifying in your picture: where the basis vectors go, the kernel and the image of the transformation. Explain also how to obtain the image of the vector $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ in terms of your drawing.

Consider $T: \mathbb{R}^2 \to \mathbb{R}^3$ with matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$

Picture:



5.

(a) A linear transformation $T : \mathbb{R}^4 \to \mathbb{R}^3$ may have rank 4.

(b) A linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^5$ of rank 3 must be injective.

(c) If AB is the identity matrix, then A and B must be square matrices.

(d) If S and T are subspaces of $\mathbb{R}^n$ and S is contained in T, then $\dim(S) \leq \dim(T)$.

(e) If $x_0 \in \mathbb{R}^n$ is a solution of the equation $Ax = b$ where A is an $m \times n$ matrix, and $x_1 \in \mathbb{R}^n$ is in the kernel of A, then $x_0 + x_1$ is another solution to the equation $Ax = b$.

(a) False: $\operatorname{rank}(T) \leq \dim(\text{codomain}) = 3 < 4$.

(b) True: by rank-nullity, $3 = \dim(\operatorname{Ker}(T)) + \underbrace{\dim(\operatorname{Im}(T))}_{= 3}$

So $\dim(\operatorname{Ker}(T)) = 0$ and so $\operatorname{Ker}(T) = \{0\}$, so T is injective.

(c) False. Counterexample: $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

(d) True: $\dim(S) =$ maximum number of l.i. vectors in S

$\underset{\text{since } S \subseteq T}{\leq}$ maximum number of l.i. vectors in T

$= \dim(T)$.

(e) True: Let T be the linear transformation assoc. to T. Then,

$$\begin{aligned} A(x_0 + x_1) &= T(x_0 + x_1) \\ &\underset{T \text{ linear}}{=} T(x_0) + \underbrace{T(x_1)}_{0 \text{ since } x_1 \in \operatorname{Ker}(T)} \\ &= T(x_0) \\ &= Ax_0 \\ &\underset{\text{by assumption}}{=} b \end{aligned}$$

6. I omit the details, but here is a sketch:

Let $E_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $E_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$. Since $DE_1 = E_1 D$, we get $a = b$

$DE_2 = E_2 D$, we get $b = c$.

Therefore $a = b = c$ as desired.