Linear Algebra Proposed Extra Credit Projects

Cite your sources! The examples must constitute original work. No form of plagiarism will be permitted. You may work in groups of 2-3. You may choose a project out of the following, or feel free to do your own project (but you must get approval for it first). Creativity is encouraged.

Application to biology: Leslie matrices. Explain how the Leslie matrix is constructed and why, and compute three explicit examples where the eigenvalue λ with maximum absolute value satisfies respectively $0 < \lambda < 1$, $\lambda = 1$ and $\lambda > 1$. Taking powers of these matrices (you may use software or do it by hand), relate the value of λ to the asymptotic growth of the population.

Application to economics: Leontief model. Describe the input-output matrix and explain why it is defined as it is. Set up your own example with three industries, and let the resulting matrix be A. Choose values for the entries of A and fix an "external demand" vector b. Then find what the production values of each industry should be to meet the demand b. What happens if $I_3 - A$ is not invertible?

Application to number theory: Fibonacci numbers. Use a matrix to describe the recursion in the Fibonacci sequence. Diagonalize the matrix and use this to give an explicit formula for F_n (one that does not involve previous terms). Use the formula to find the limit of F_{n+1}/F_n as $n \to \infty$.

Application to probability: Markov chains. Explain intuitively the concept of a Markov chain, describe the transition matrix and explain why it is defined as it is. Set up your own example with 3 nodes, explain what the powers of the transition matrix represent, compute some explicit probabilities after 3 or 4 steps, and compute these probabilities in the limit as the number of steps goes to infinity.

Application to physics: Lorentz transformations. Derive the Lorentz transformation matrix for two inertial frames S, S' where S' is moving away from S only in the x direction (you may assume the speed of light is 1). Give some explicit examples of the features of special relativity (e.g. time dilation, length contraction) using your matrix. (You may assume the basics of special relativity).

Application to quantum computing: quantum logic gates. briefly explain the set-up for quantum circuits (you may assume the basics of quantum mechanics) in terms of linear transformations. Then, describe two quantum logic gates of your choice. Give an example computation where you compose the two gates and apply it to an input of your choice.

Application to abstract algebra: field extensions. Define field extensions, finite field extensions, and the degree of a field extension (you may assume the definition of a field). Then prove that if L/K and K/F are finite field extensions, then L/F is a finite field extension of degree $[L : F] = [L : K] \cdot [K/F]$. Give an explicit example with $[L : K] = 2$ and $[K : F] = 3$.

Application to computer graphics: triangle on a screen. Derive the 3×3 matrices for rotations about the x, y and z axis. Use this to derive the 3×3 matrix of an arbitrary rotation in 3D space. Derive also the 3×3 matrix for the projection matrix onto the (x, y) plane (this is how computers "project onto the screen"). Take the triangle with vertices $(1, 1, 1)$, $(1, 1, 3)$, $(1, 2, 2)$. Show that its projection on the screen has zero area, "the triangle is not visible". Find which rotations make the triangle visible and which don't.

Application to machine learning: neural networks. Explain neural networks as compositions of linear transformations, with biases and linear activation functions in between. Do not include the use of a sigmoid function. Define a neural network with 2 inputs, 1 output, and a single hidden layer with 2 nodes. Describe the two weight matrices: from the first layer to the second, and from the second layer to the third. Finally choose values for the weights and biases so that input $(1, 0)$ and $(0, 1)$ activate the final output, but input $(1, 1)$ does not. Prove that this could not have been achieved with a single 2×1 matrix directly from the input to the output.