Linear Algebra HW Week 5

1. Let S be the subspace of $\mathbb{R}^3$ spanned by the vectors $\begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$. Describe:

   (a) The basis of S obtained by the Gram-Schmidt algorithm on $\begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$.

   (b) The QR factorization of $\begin{pmatrix} 3 & 1 \\ 1 & -1 \\ 1 & 1 \end{pmatrix}$.

   (c) The matrix of proj_S.

   (d) The vectors $v^{||}$ and $v^{\perp}$ given that $v = \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$.

2. Compute the QR factorization of the matrix $A = \begin{pmatrix} 1 & -1 & 0 \\ -2 & 3 & 1 \\ -2 & 1 & -2 \end{pmatrix}$. Use it to solve the system $Ax = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ quickly.

3. Determine whether the following matrices are orthogonally diagonalizable and find their orthogonal diagonalizations if possible.

   (a) $A = \begin{pmatrix} 4 & \sqrt{6} & \sqrt{2} \\ \sqrt{6} & 3 & -\sqrt{3} \\ \sqrt{2} & -\sqrt{3} & 5 \end{pmatrix}$

   (b) $A = \begin{pmatrix} 4 & 0 & \sqrt{2} \\ \sqrt{6} & 3 & -\sqrt{3} \\ \sqrt{2} & -\sqrt{3} & 5 \end{pmatrix}$

4. (20 points) Fix a unit vector $u \in \mathbb{R}^n$. Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be the transformation such that $T(v) = v - 2(v \cdot u)u$ for all $v \in \mathbb{R}^n$.

   (a) Prove that T is a linear transformation.

   (b) Prove that T is orthogonal (Hint: show that it preserves the dot product, in the sense from the lectures).

   (c) Take your favorite unit vector in $\mathbb{R}^3$ and describe the linear transformation T geometrically in your case.

5. Determine if the following statements are true or false. If they are true, provide a proof. If they are false, provide a counterexample.

   (a) If Q_1 and Q_2 are both orthogonal $n \times n$ matrices, then $Q_1 Q_2$ is orthogonal.

   (b) If A and B are both symmetric $n \times n$ matrices, then AB is symmetric.

   (c) If A is an invertible $n \times n$ matrix, then A^T is invertible and $(A^T)^{-1} = (A^{-1})^T$.

   (d) If A and B are square $n \times n$ matrices, and they are both symmetric, then $A+B$ is symmetric.

   (e) (Harder) If A is an invertible symmetric $n \times n$ matrix, then A^{-1} is symmetric.