Linear Algebra HW Week 1

1. Determine whether the following matrices have inverses and compute them when they do.

   (a) $\begin{pmatrix} 1 & 1 & 2 \\ -1 & 1 & 3 \\ 2 & -1 & 5 \end{pmatrix}$

   (b) $\begin{pmatrix} 1 & 1 & 2 \\ -1 & 1 & 3 \\ 2 & -1 & -7/2 \end{pmatrix}$

   (c) $\begin{pmatrix} 1 & 1 & 2 \\ -1 & 1 & 3 \end{pmatrix}$

   (d) $\begin{pmatrix} 1 & 1 \\ -1 & 1 \\ 2 & -1 \end{pmatrix}$

2. Consider the linear transformations $T_1 : \mathbb{R}^2 \to \mathbb{R}^2$ and $T_2 : \mathbb{R}^2 \to \mathbb{R}^2$ associated to the matrices $A = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

   (a) Interpret these linear transformations geometrically.

   (b) Compute A^2, B^2, AB and show that $(AB)^3 = I_2$.

   (c) Using the previous part or otherwise, show that A and B don't commute.

3. Let $T : \mathbb{R}^n \to \mathbb{R}^m$, and let $v_1, ..., v_k$ in $\mathbb{R}^n$. For each of the following statements, decide if they are true or false. If the given statement is true, prove it. If it is false, give a counterexample.

   (a) Suppose $v_1, ..., v_k$ are linearly independent and T is injective. Are $T(v_1), ..., T(v_k)$ necessarily linearly independent?

   (b) Suppose $v_1, ..., v_k$ are linearly independent and T is surjective. Are $T(v_1), ..., T(v_k)$ necessarily linearly independent?

   (c) Suppose $v_1, ..., v_k$ span $\mathbb{R}^n$ and T is injective. Do $T(v_1), ..., T(v_k)$ necessarily span $\mathbb{R}^n$?

   (d) Suppose $v_1, ..., v_k$ span $\mathbb{R}^n$ and T is surjective. Do $T(v_1), ..., T(v_k)$ necessarily span $\mathbb{R}^n$?

4. Determine the kernel and image of the following linear transformations.

   (a) The linear transformation associated to the matrix $\begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{pmatrix}$

(b) The linear transformation associated to the matrix $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

(c) The linear transformation associated to the matrix $\begin{pmatrix} 1 & 4 & 5 \\ 3 & 5 & 1 \\ 2 & -1 & -8 \end{pmatrix}$

(d) The linear transformation associated to the matrix $\begin{pmatrix} -4 & 2 & 0 \\ 2 & -1 & 0 \\ 6 & -3 & 0 \end{pmatrix}$

5. Find a basis for each kernel and image in the previous exercise, indicating the dimension of each subspace. Verify that the rank-nullity theorem holds in each case.