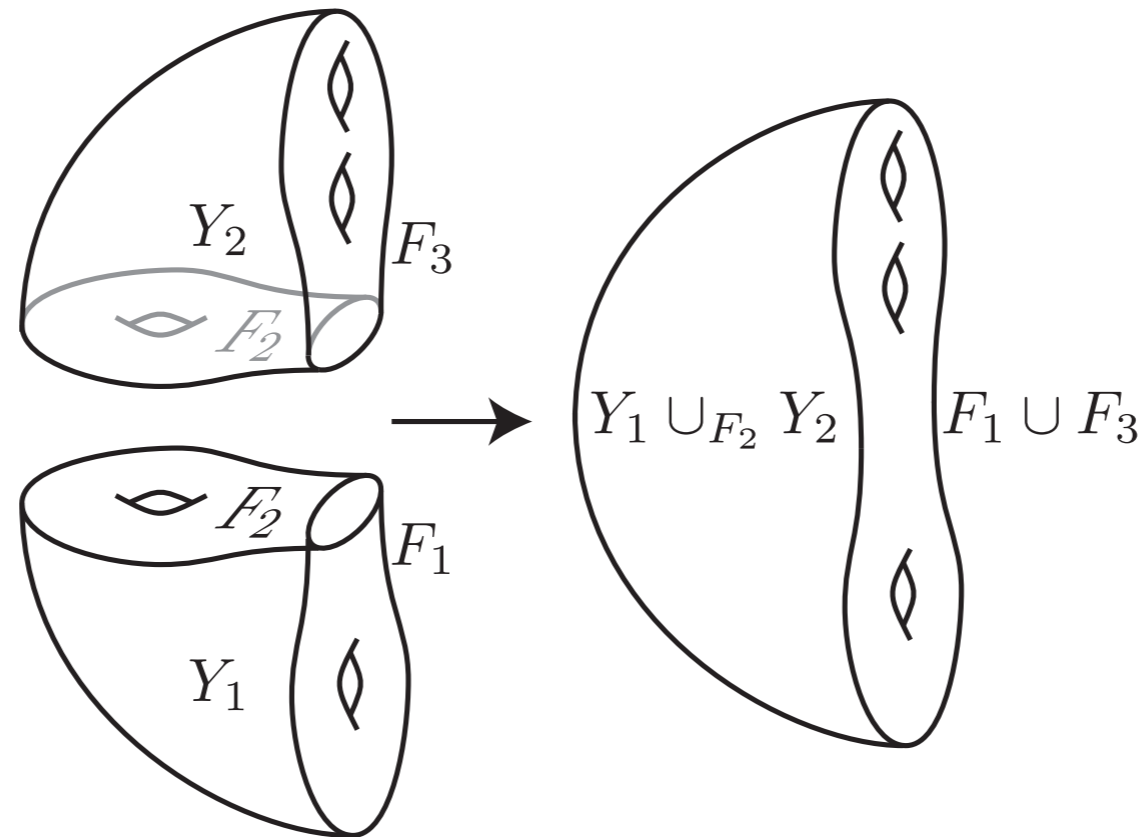


Cornered Floer Homology

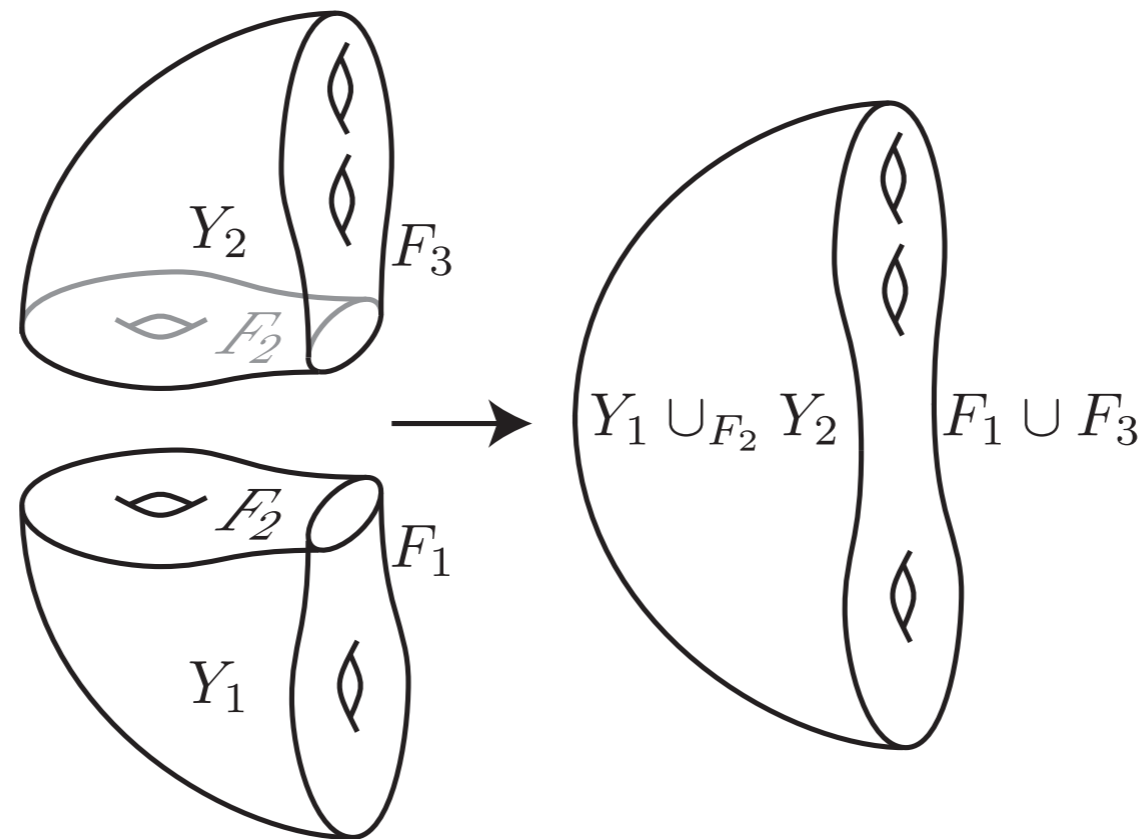
Robert Lipshitz
May 20, 2013

Joint with Chris Douglas and Ciprian Manolescu

Gluing 3-manifolds with corners and Heegaard Floer homology



Gluing 3-manifolds with corners and Heegaard Floer homology



1. Background: bordered Floer and Heegaard Floer.
2. Cornered gluing from bordered Floer, via trimodules.
3. Cornered Floer homology.

Heegaard Floer Homology and the Four-Ball Genus
On the Heegaard Floer Homology of Seifert Surfaces
On the Heegaard Floer Homology of Knots and Links
A Cylindrical Reformulation of Heegaard Floer Homology
On the Floer Homology of Two-Bridge Knots
On the Floer Homology of Plumbed Three-Manifolds
Holomorphic Triangle Invariants and the Topology of Symplectic Four-Manifolds
Absolutely Graded Floer Homologies and Intersection Forms for Four-Manifolds with Boundary
Holomorphic Triangles and Invariants for Smooth Four-Manifolds
Holomorphic Disks and Three-Manifold Invariants: Properties and Applications
Holomorphic Disks and Topological Invariants for Closed Three-Manifolds

Heegaard Floer and Bordered Floer

Heegaard Floer Homology

(Ozsváth-Szabó)

- Y^3 based 3-manifold
→ $\widehat{HF}(Y)$ abelian group.

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→ $\widehat{F}_W : \widehat{HF}(Y_1) \rightarrow \widehat{HF}(Y_2)$.

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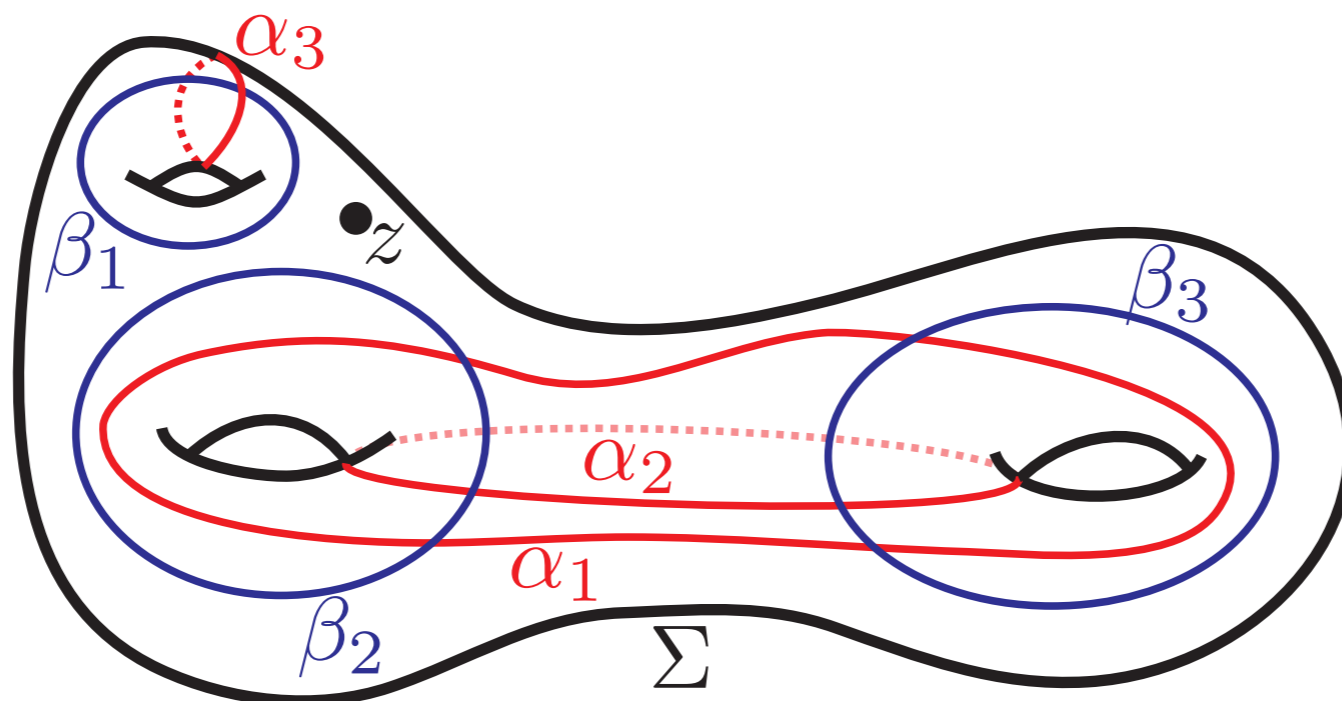
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- Composition → composition.
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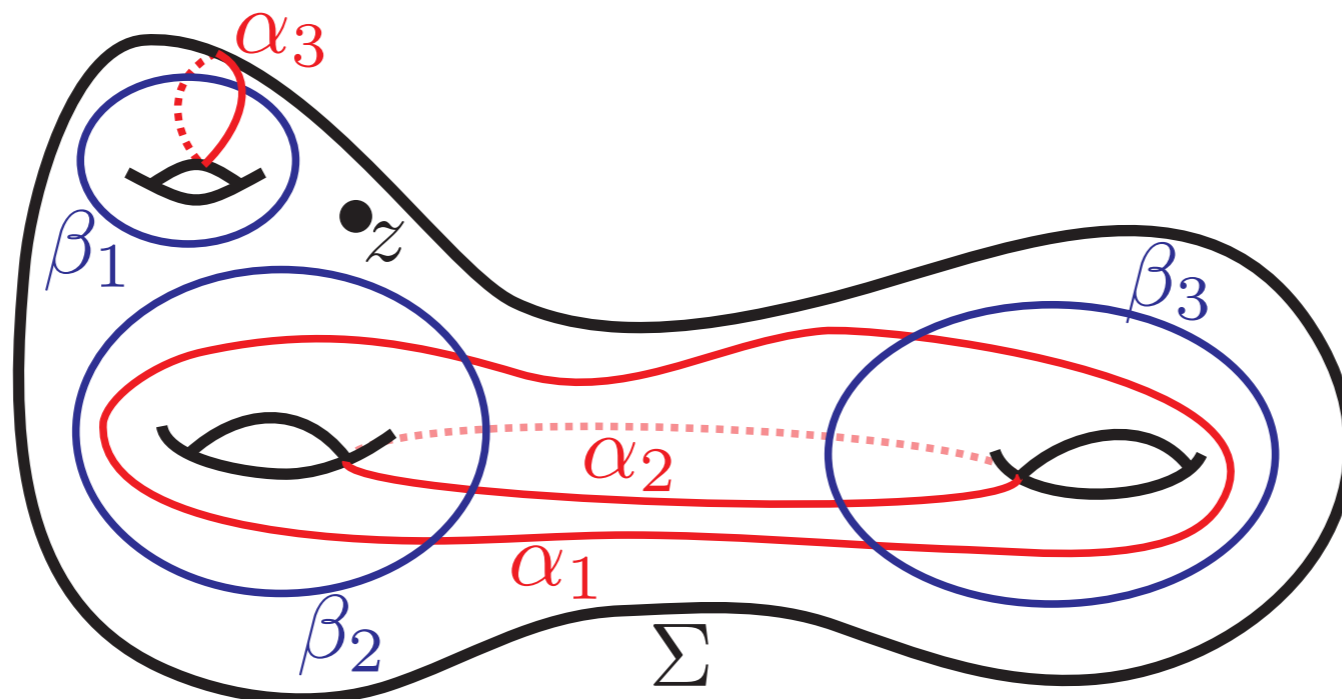
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- Also comes in HF^- , HF^+ flavors.



Bordered Floer Homology

(L-Ozsváth-Thurston)

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Bordered Floer Homology

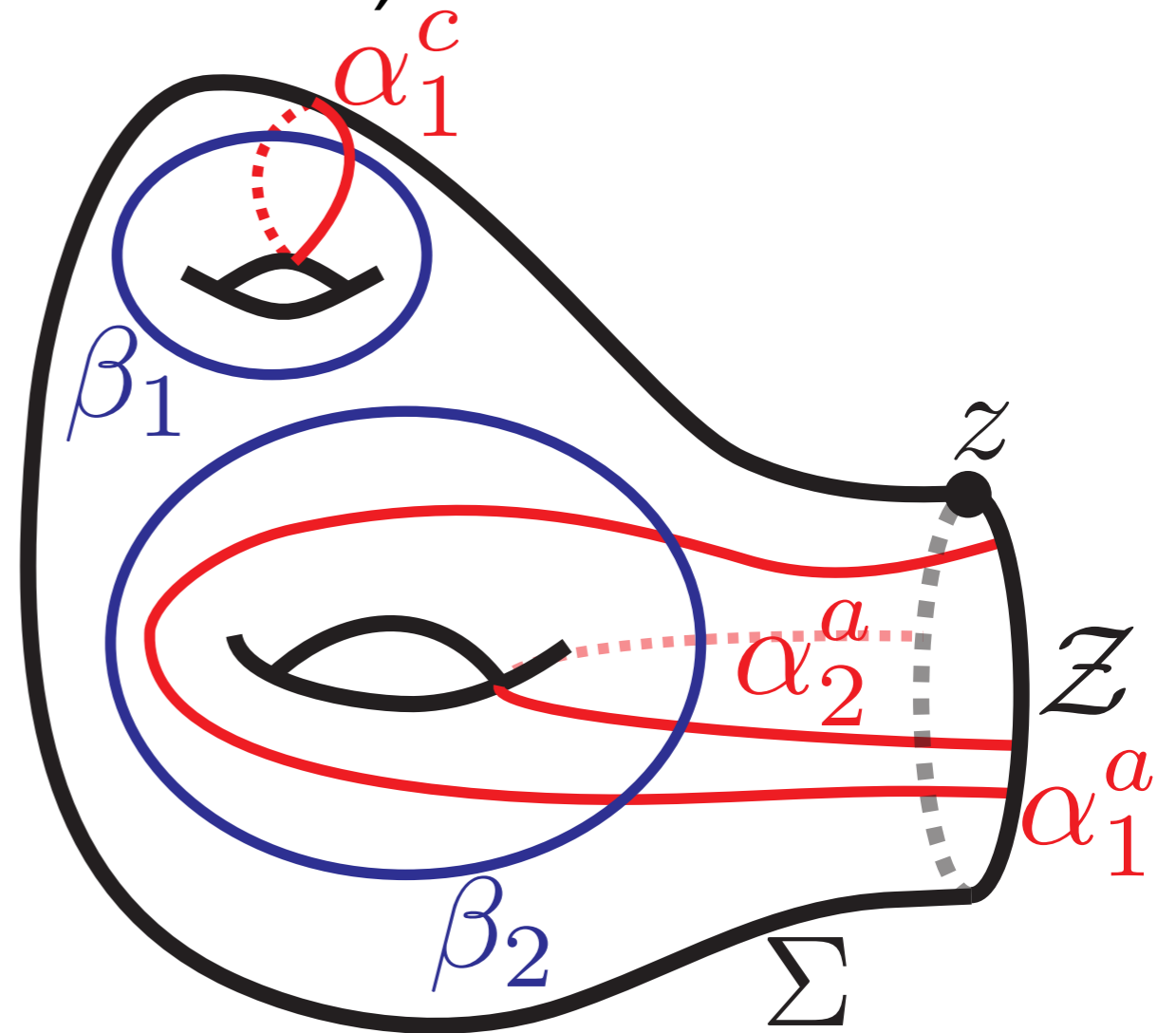
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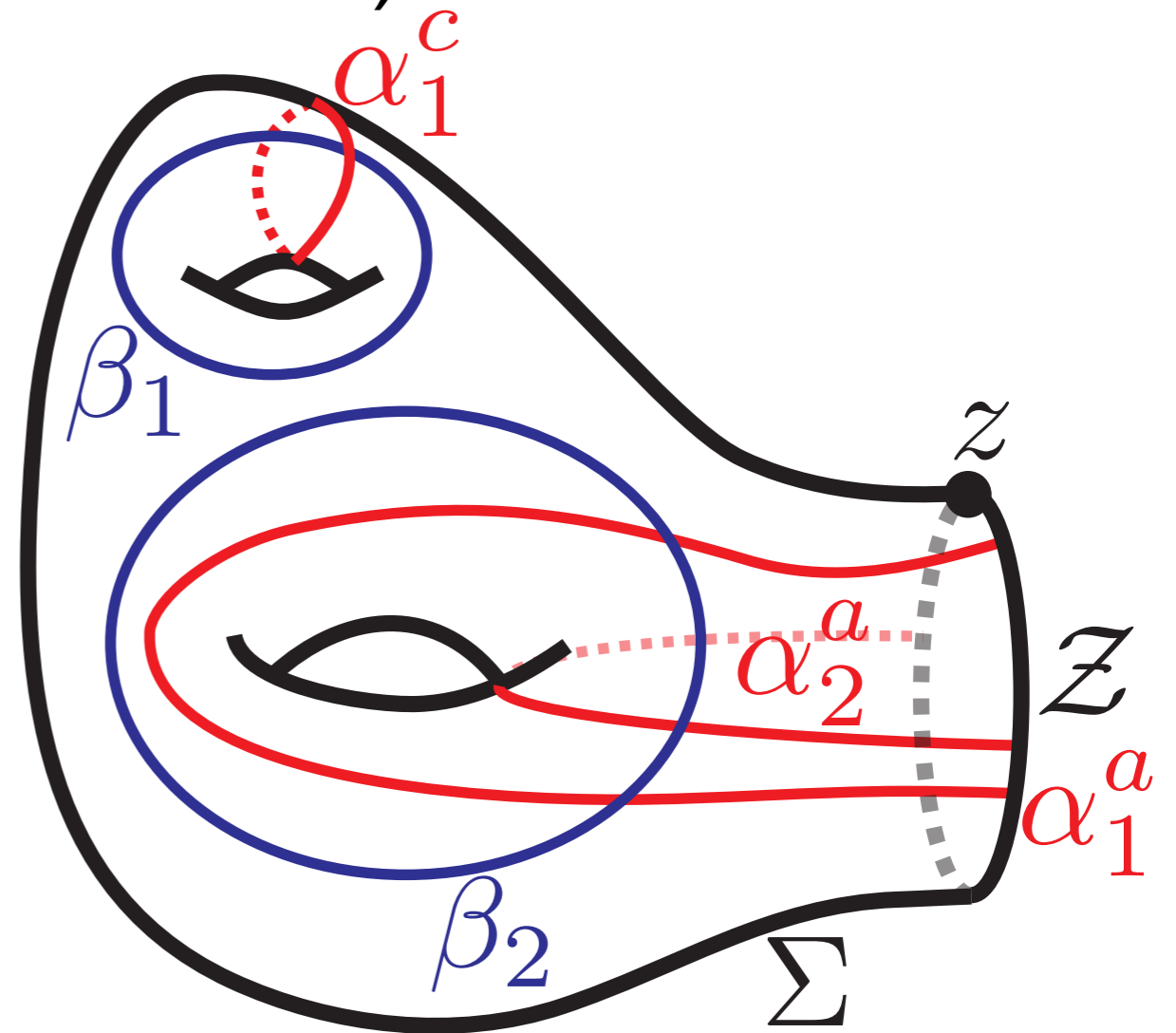
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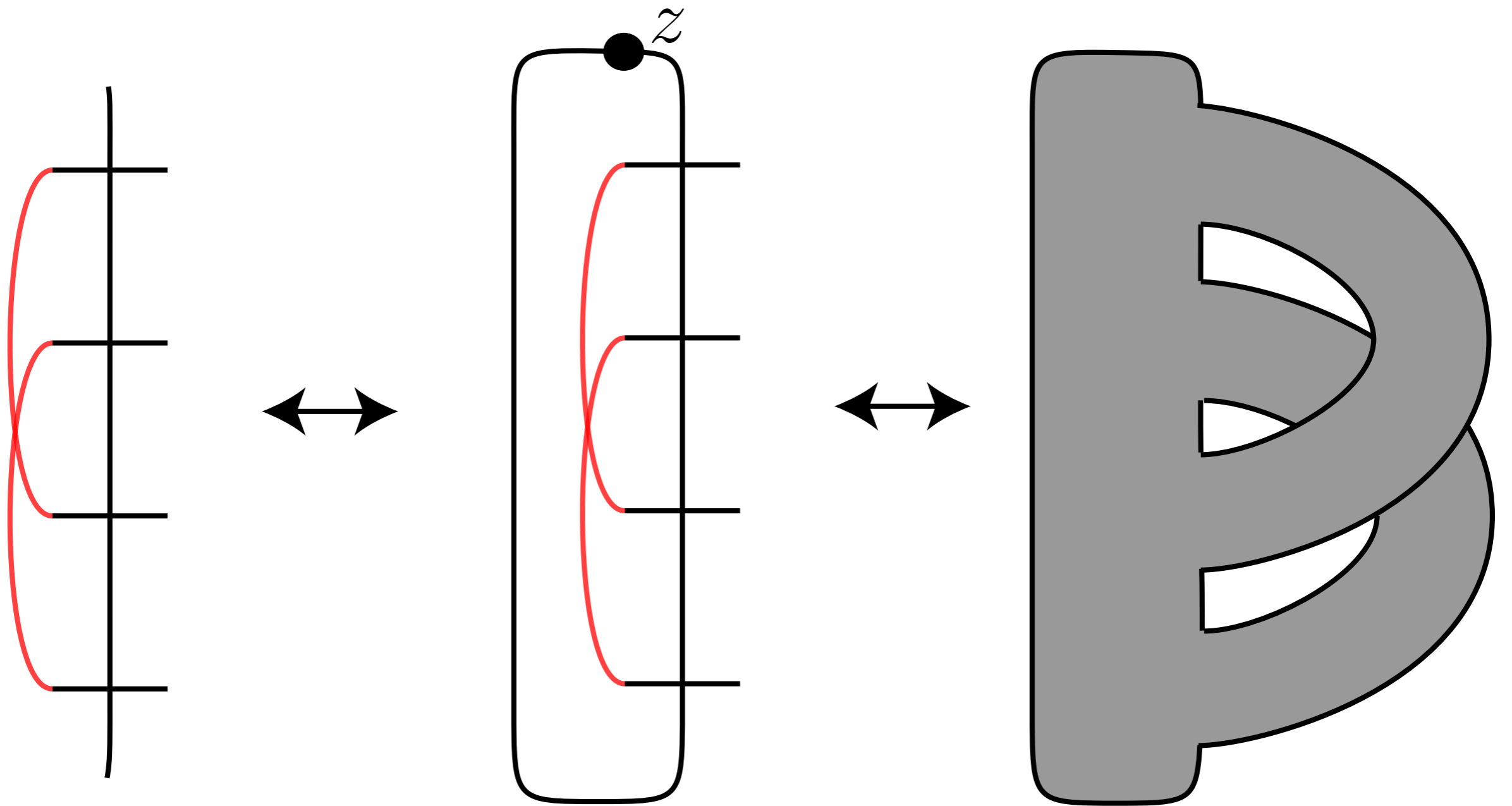
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- Cobordisms \longrightarrow Bimodules

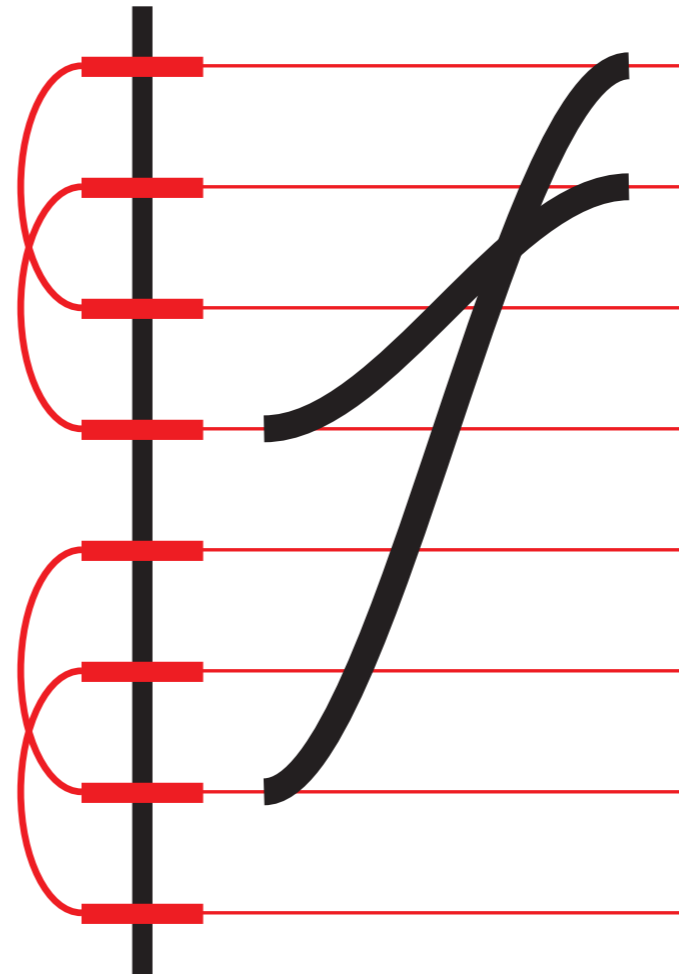


Pointed matched circles



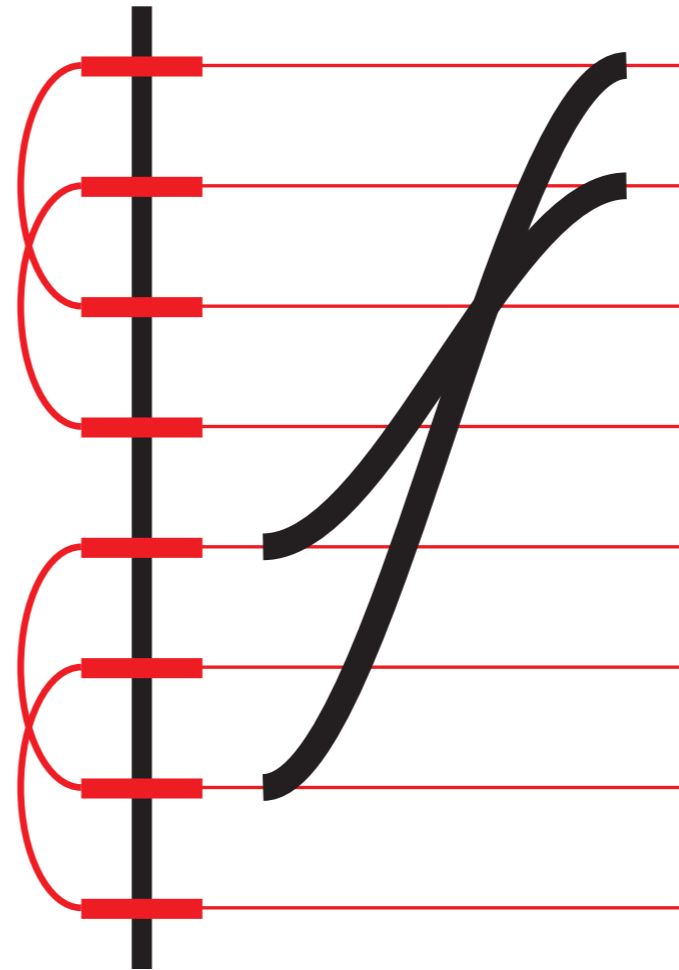
The Bordered Algebras

- Upward-veering strand diagrams.



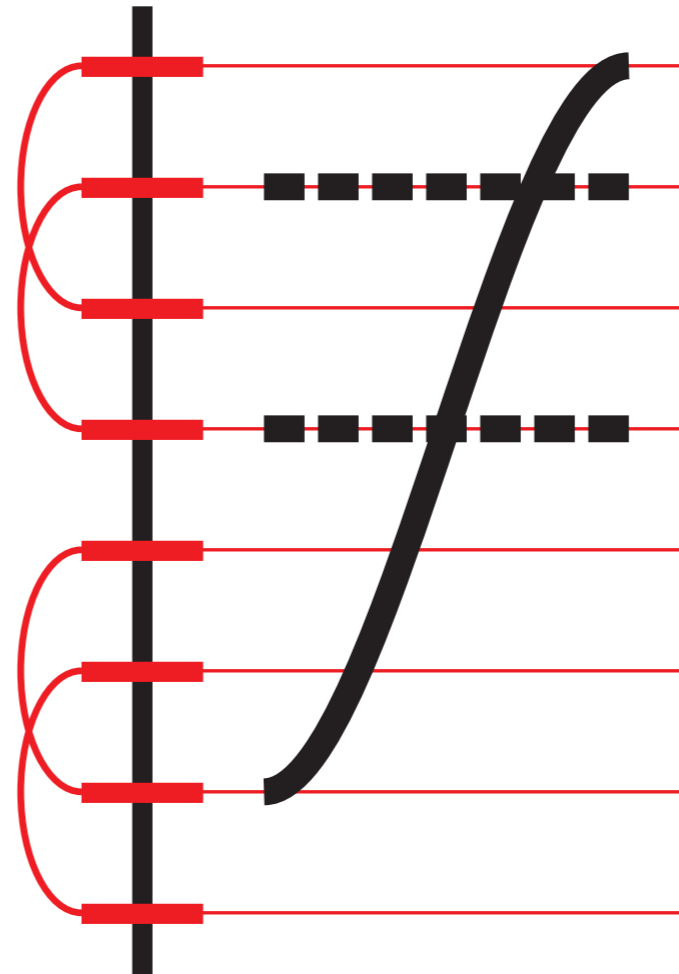
The Bordered Algebras

- Upward-veering strand diagrams.
- No initial (resp. terminal) endpoints matched.



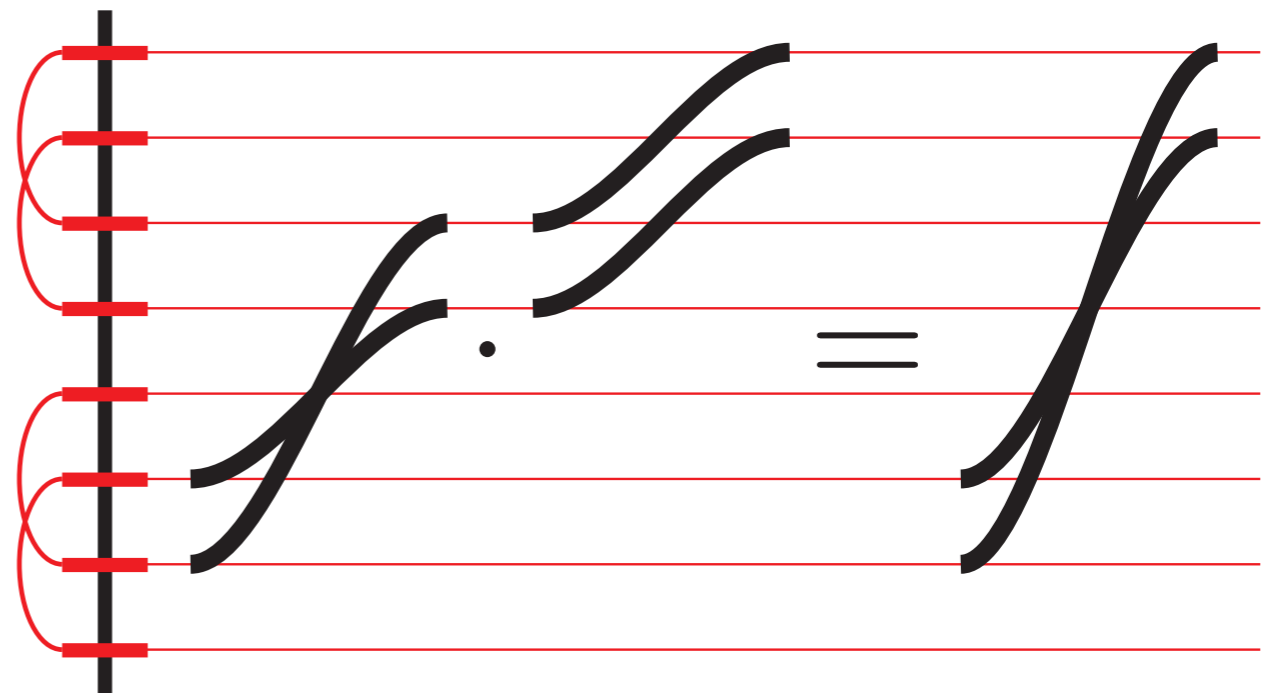
The Bordered Algebras

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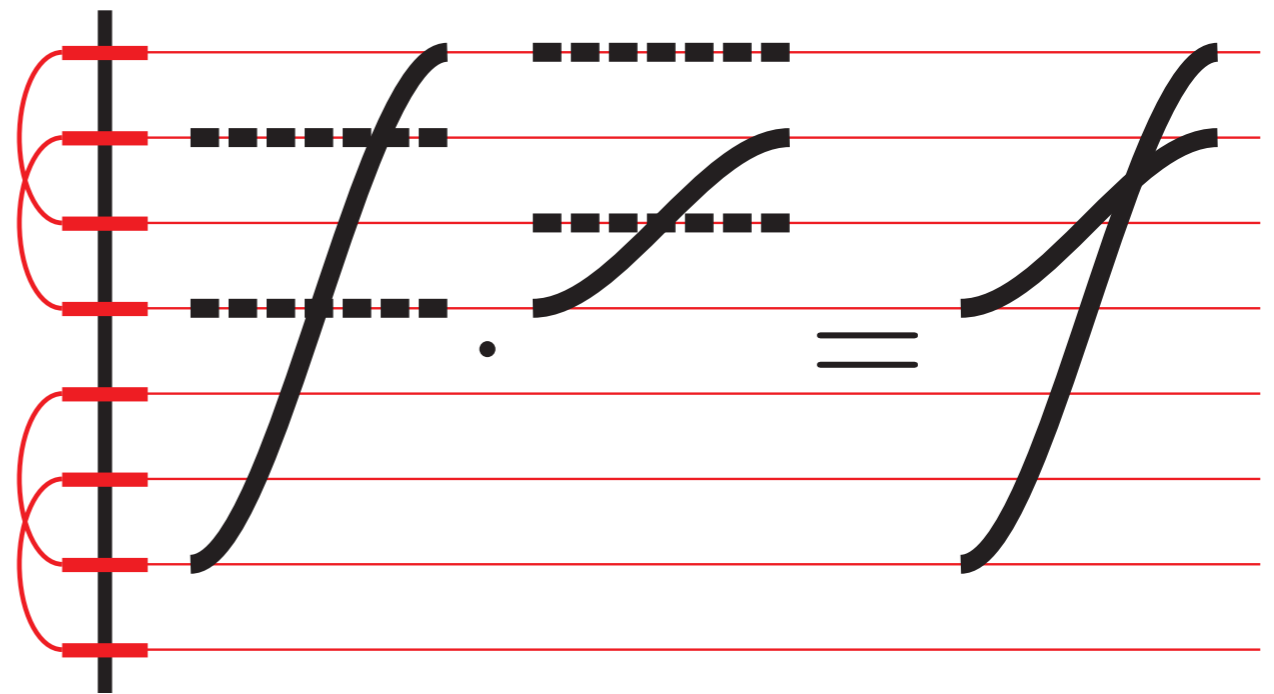
The Bordered Algebras

- Upward-veering strand diagrams.
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- Smeared horizontal lines
- Multiplication is concatenation.



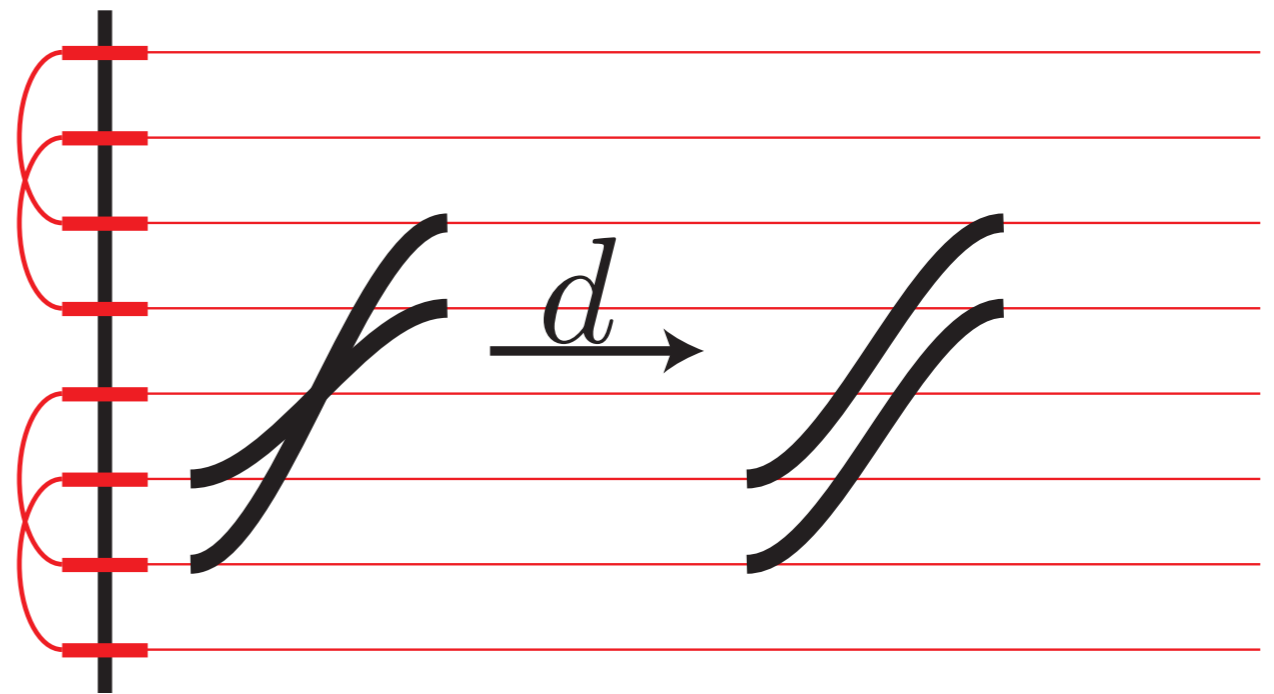
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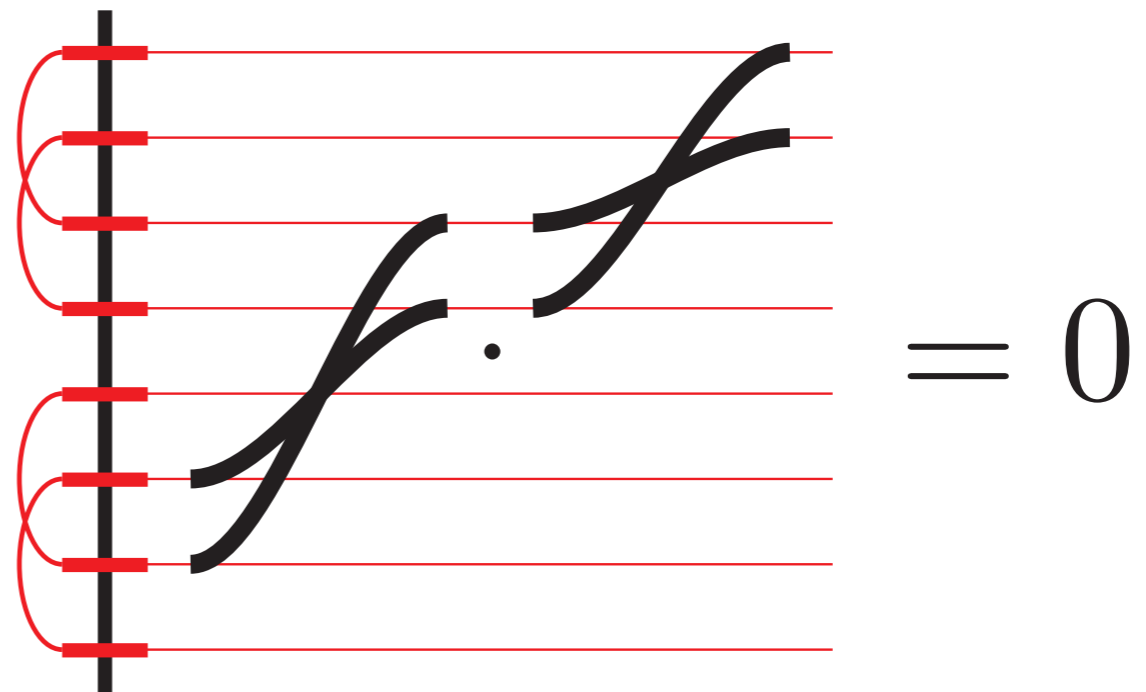
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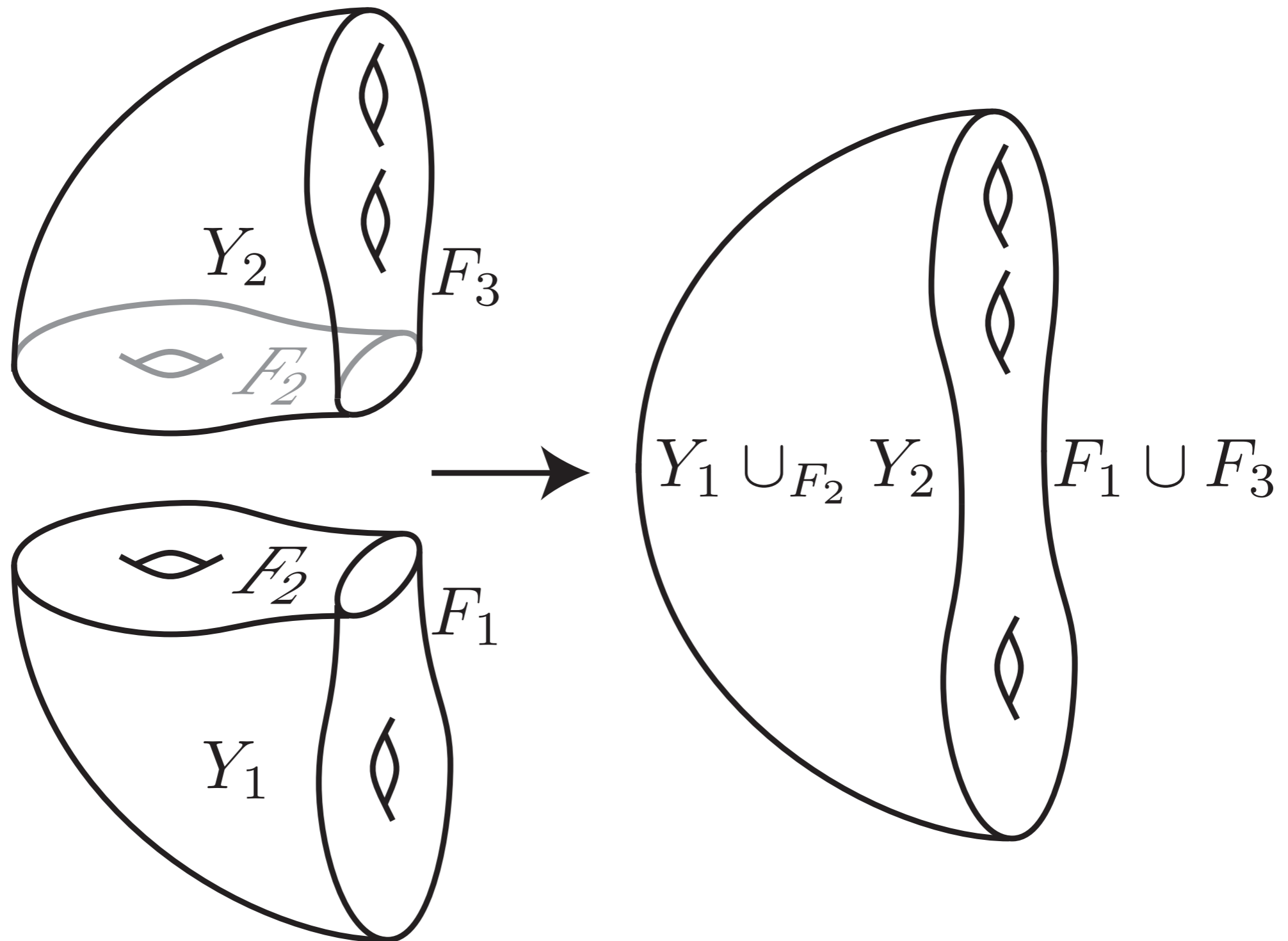
- Double crossings = 0.



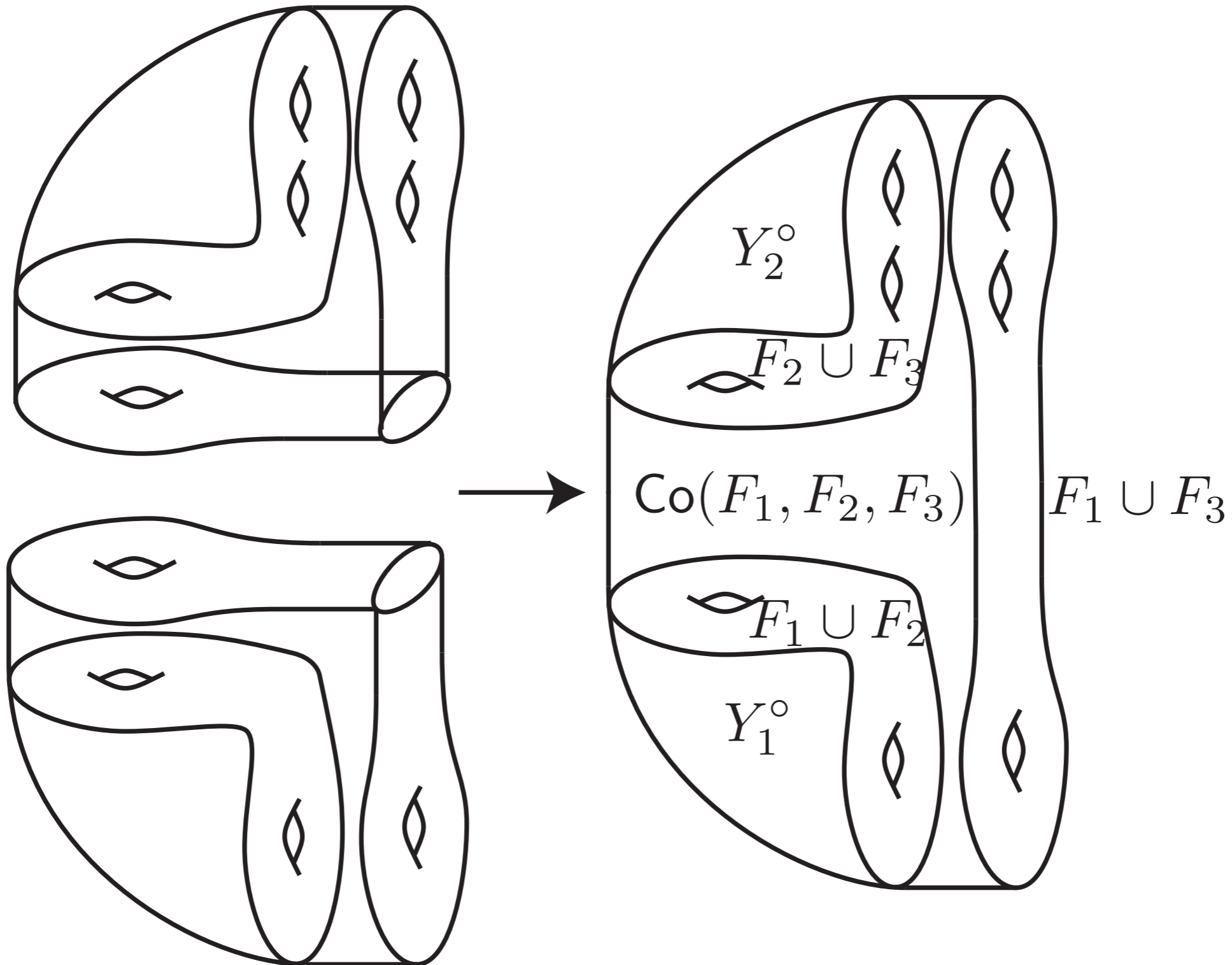


**Cornered Gluing in
Bordered Floer**

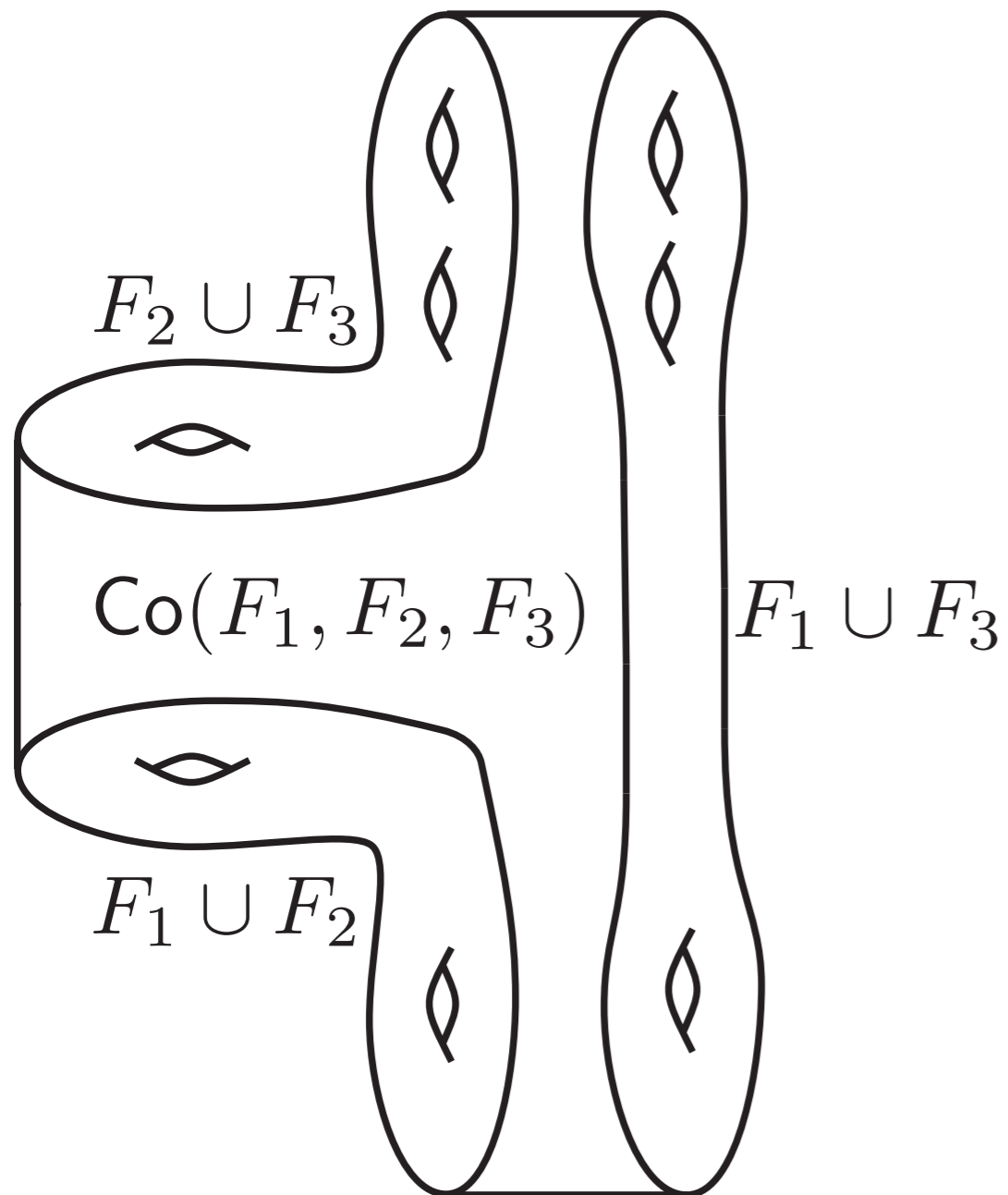
Cornered Gluing...



...via Trimodules



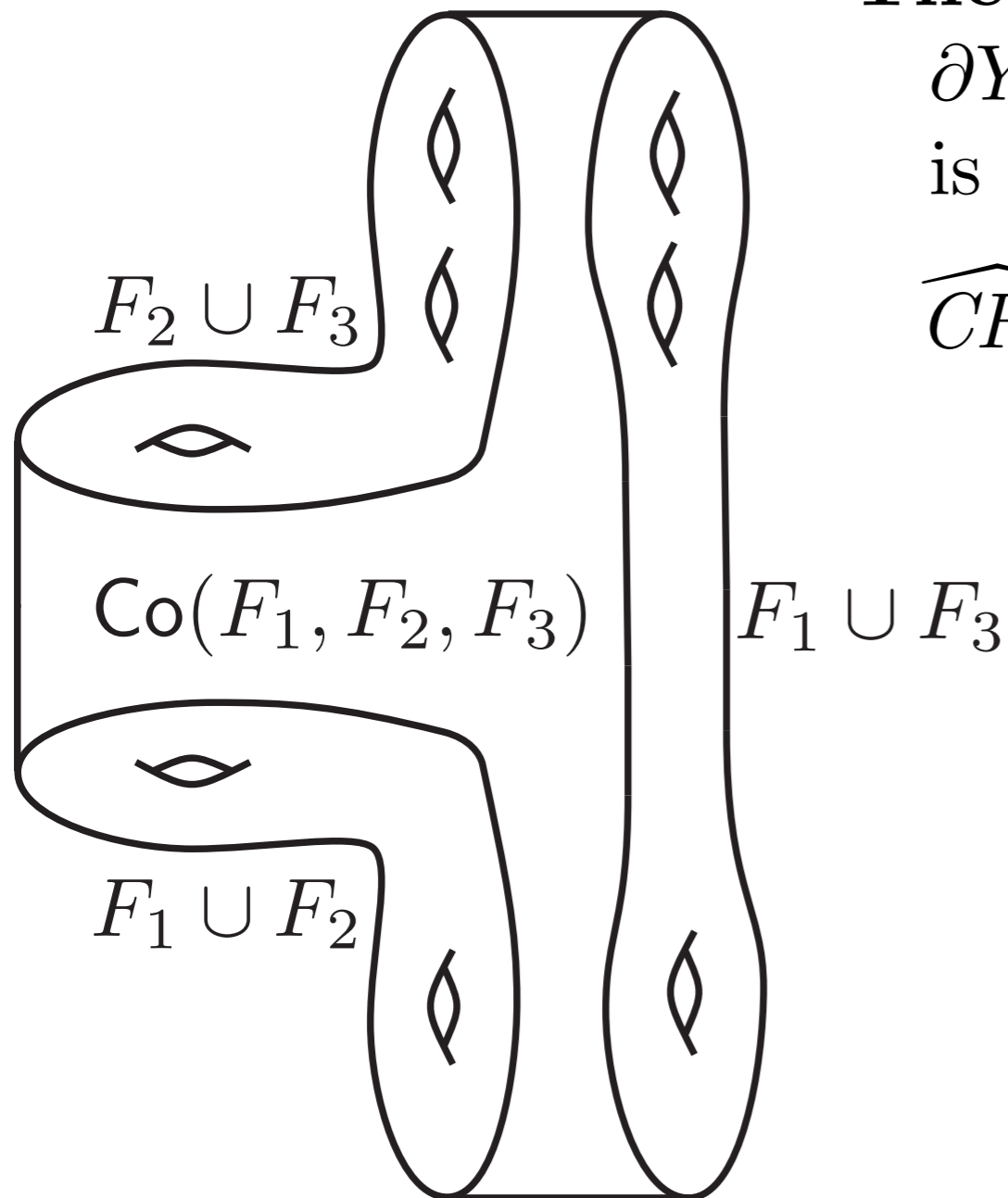
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Theorem. If $\partial Y_1 = F_1 \cup F_2$ and $\partial Y_2 = F_2 \cup F_3$ then $\widehat{CFA}(Y_1 \cup_{F_2} Y_2)$ is quasi-isomorphic to

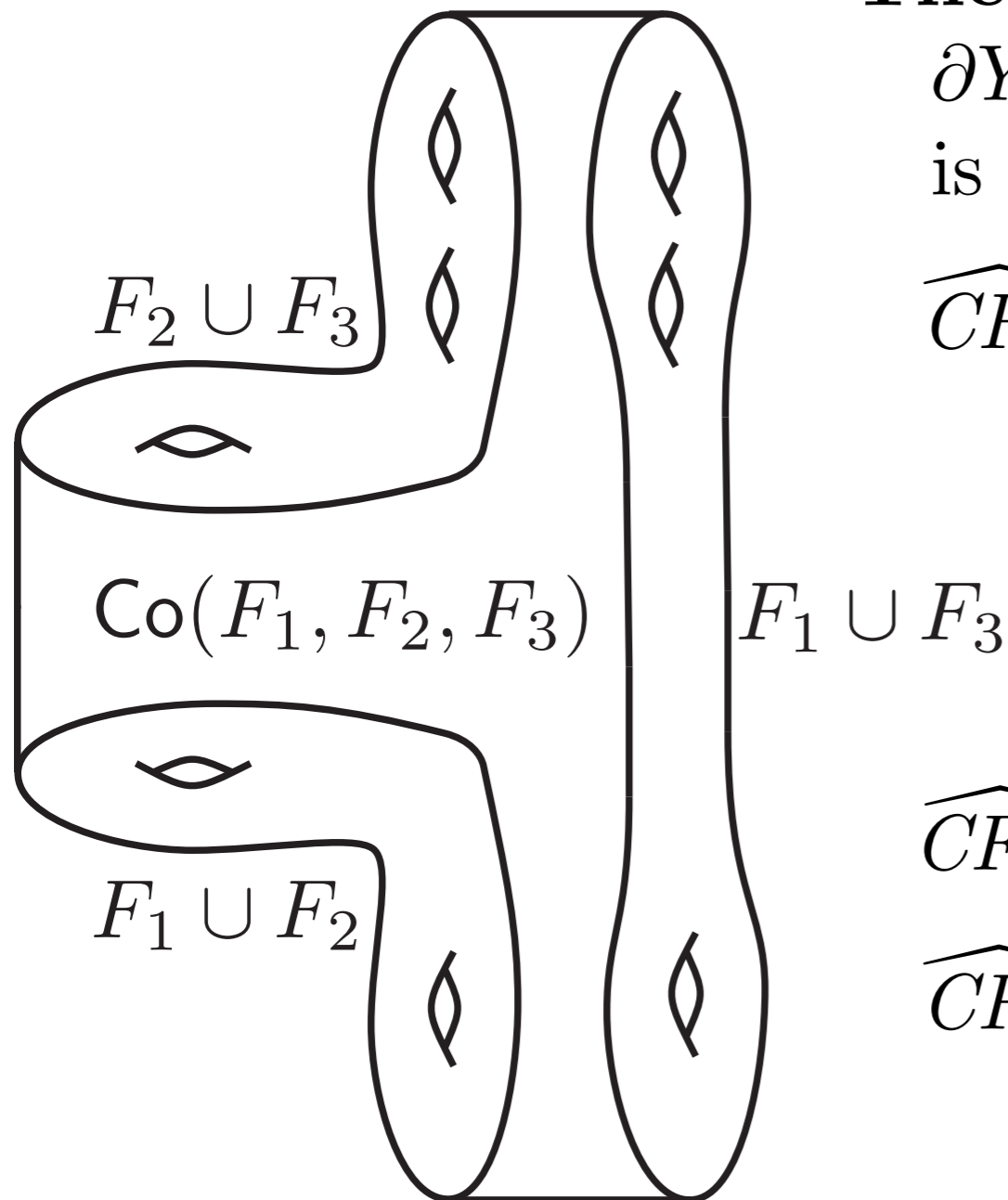
$$\widehat{CFA}(Y_1) \otimes_{\mathcal{A}(F_1 \cup F_2)} \widehat{CFA}(Y_2) \otimes_{\mathcal{A}(F_2 \cup F_3)} \widehat{CFDDA}(\text{Co}(F_1, F_2, F_3))$$



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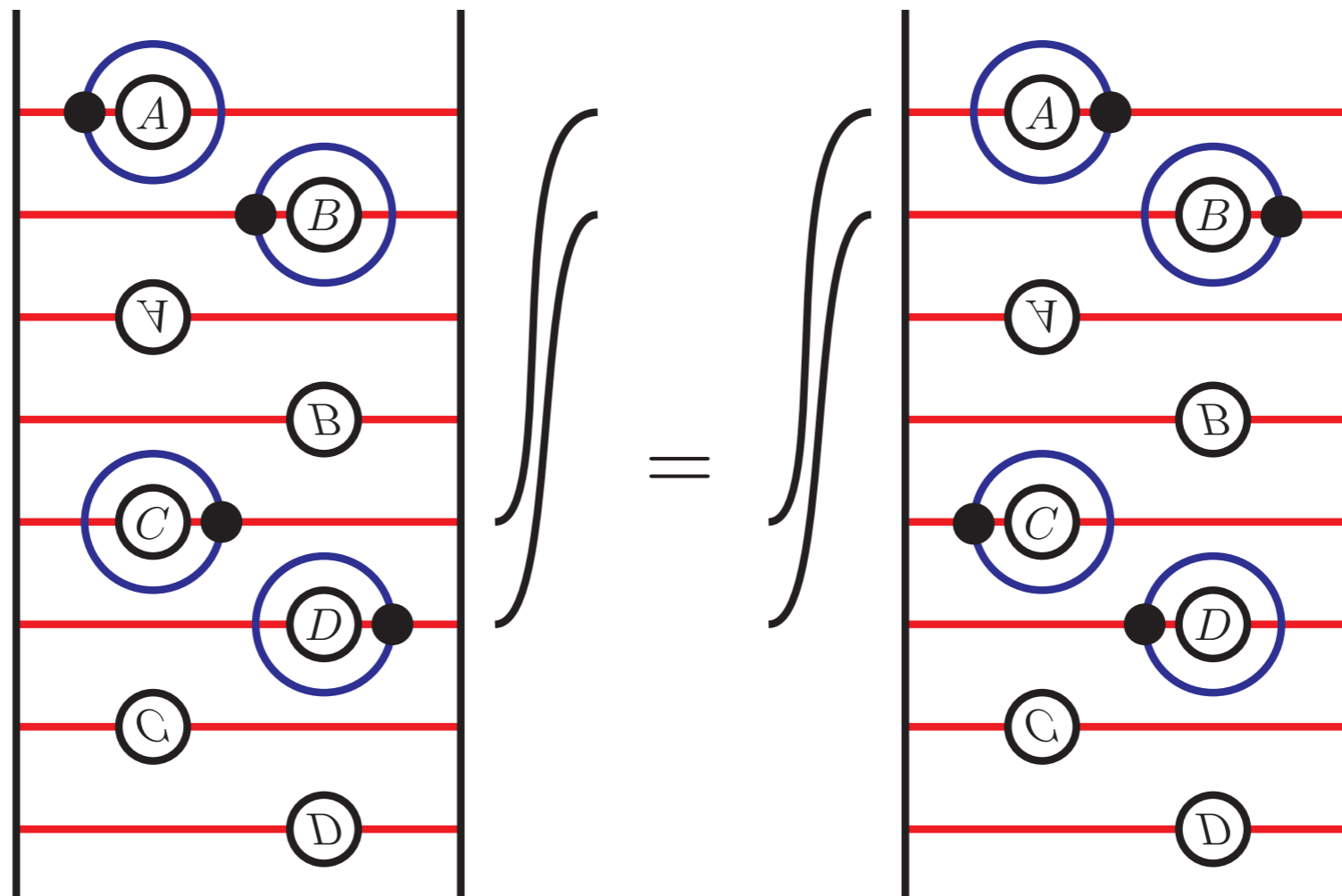
$\widehat{CFD}(Y_1 \cup_{F_2} Y_2)$ is quasi-isomorphic to

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The DA Identity Bimodule

$$\widehat{CFDA}([0, 1] \times F)$$

- $\mathcal{A}(F)$ as an $\mathcal{A}(F)$ -bimodule

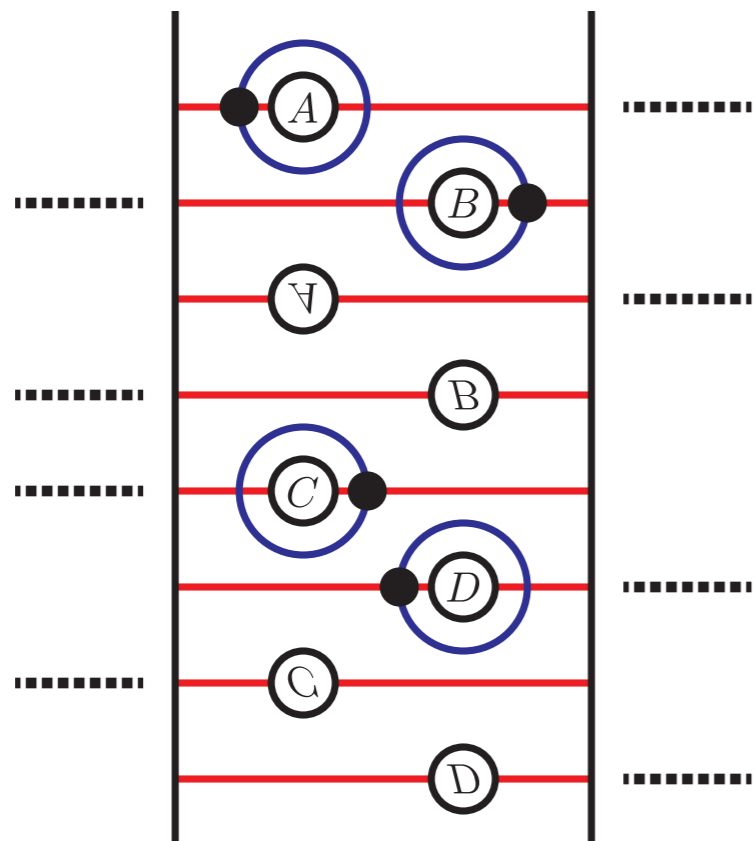


(LOT, "Bimodules in Bordered Floer Homology")

The DD Identity Bimodule

$$\widehat{CFDD}([0, 1] \times F)$$

- Free (projective) module over $\mathcal{A}(F) \otimes \mathcal{A}(F)$ generated by *complementary idempotents* $I \otimes J$.

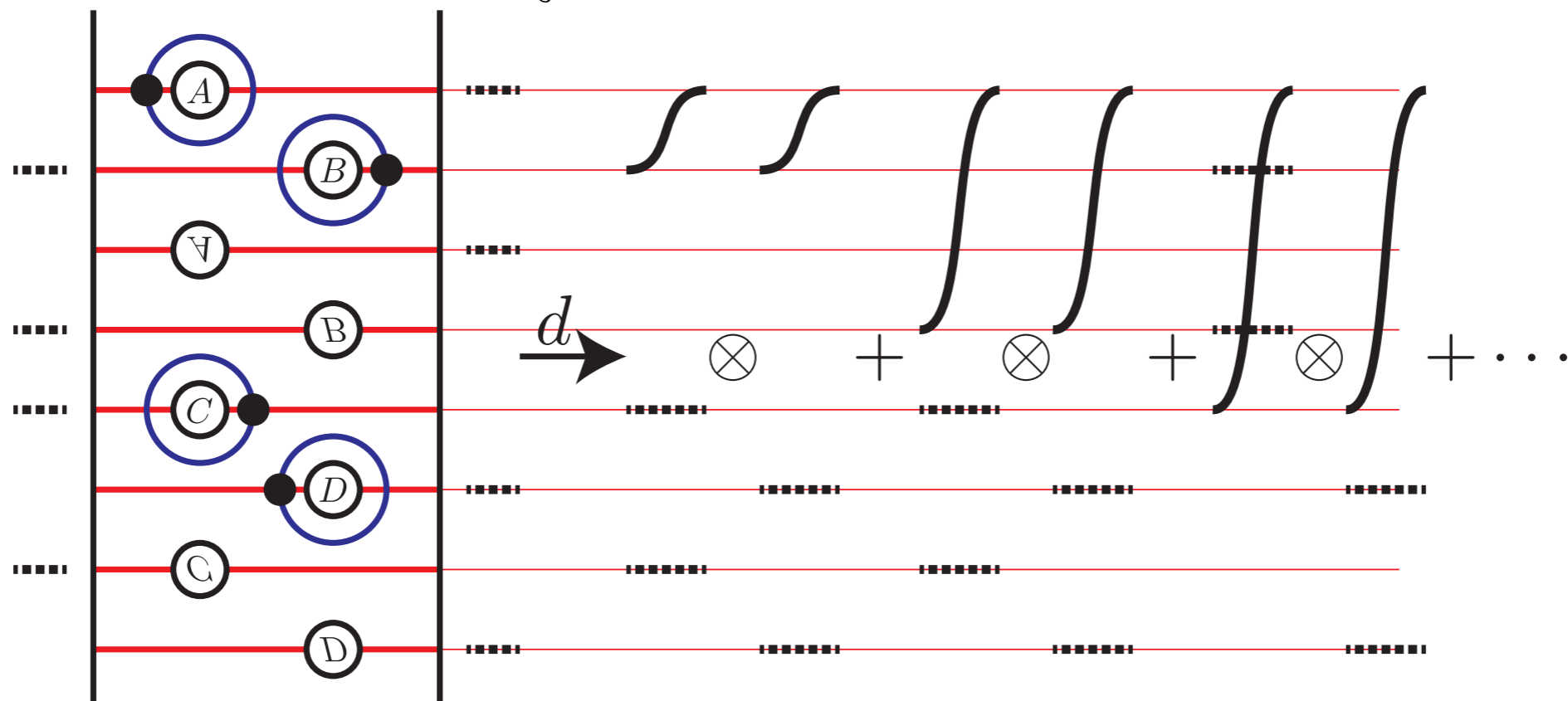


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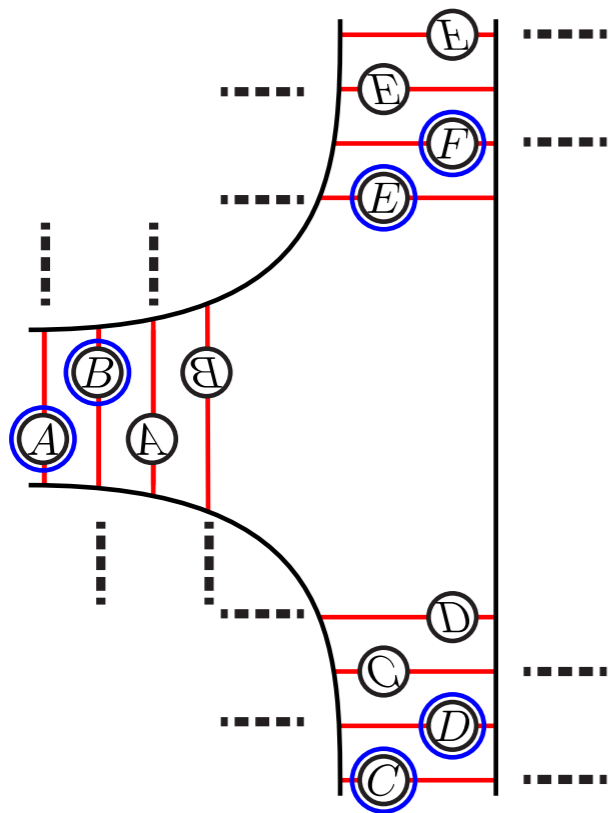
- $$d(I \otimes J) = \sum_{\text{chords } \xi} I \cdot a(\xi) \otimes a(\xi) \cdot J$$



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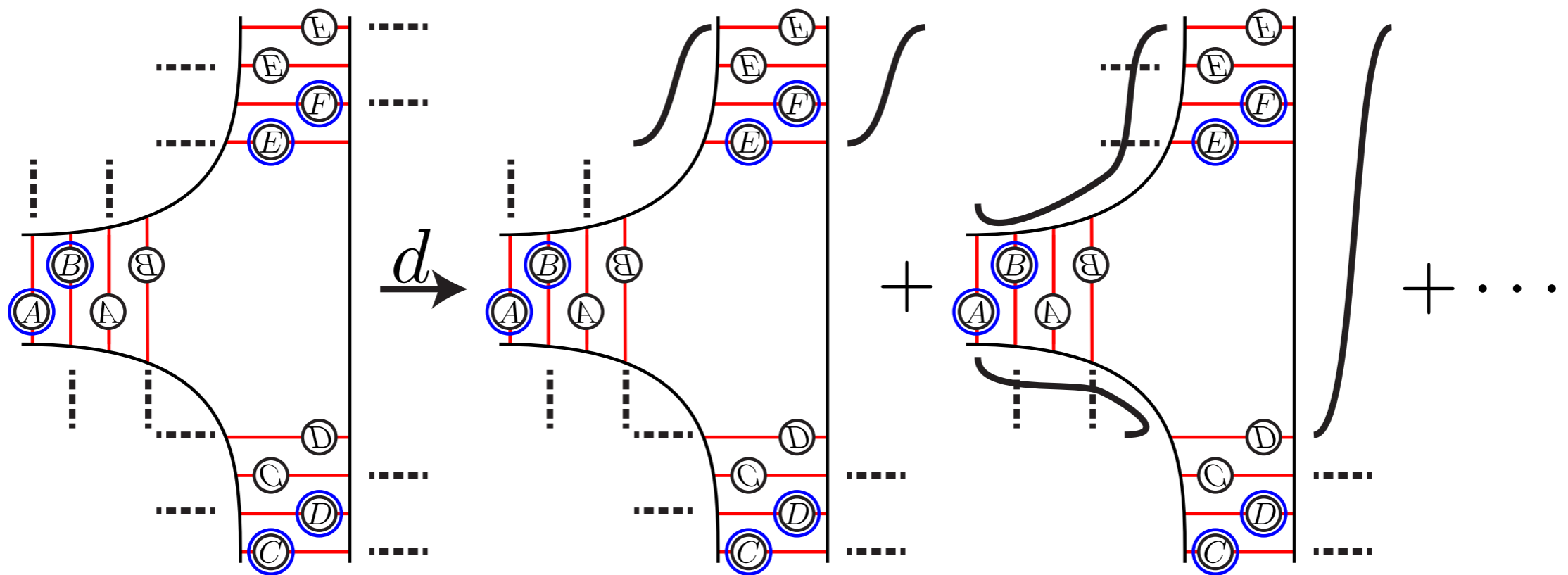
The DDD Cornering Trimodule

- Like the DD identity module.
- Generated by complementary idempotent triples.
- Differential is a sum over chords.



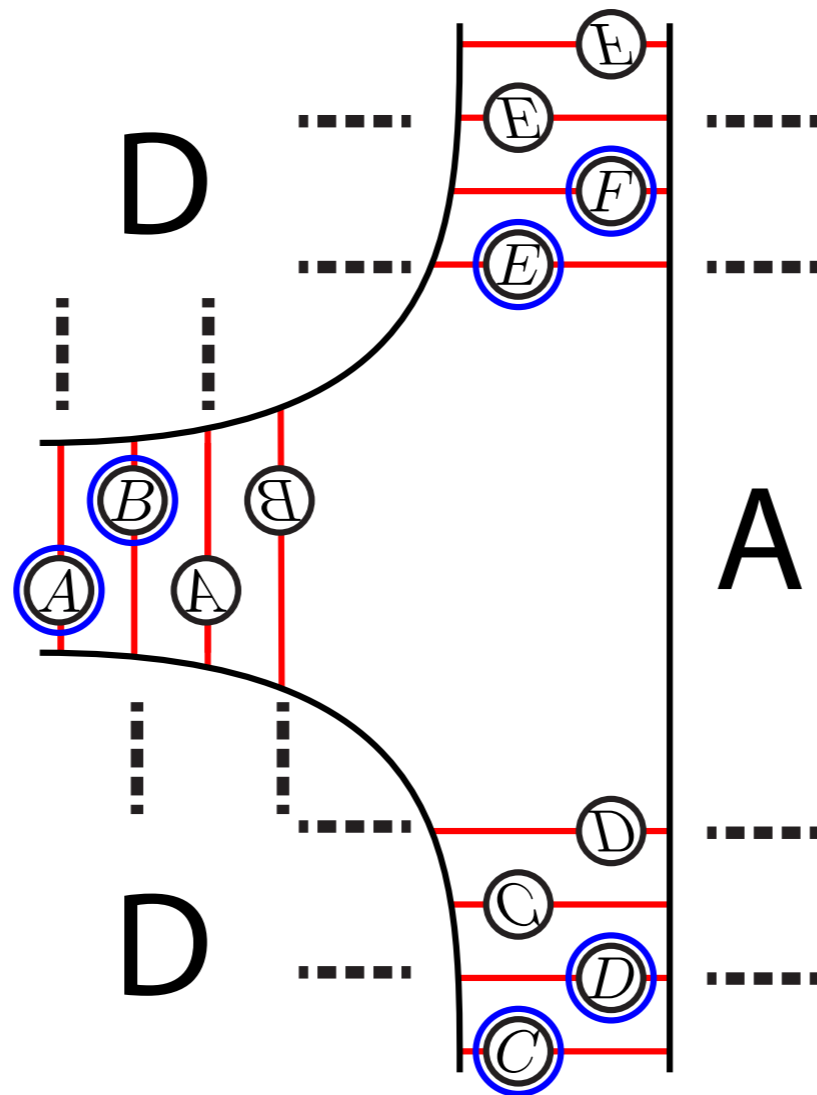
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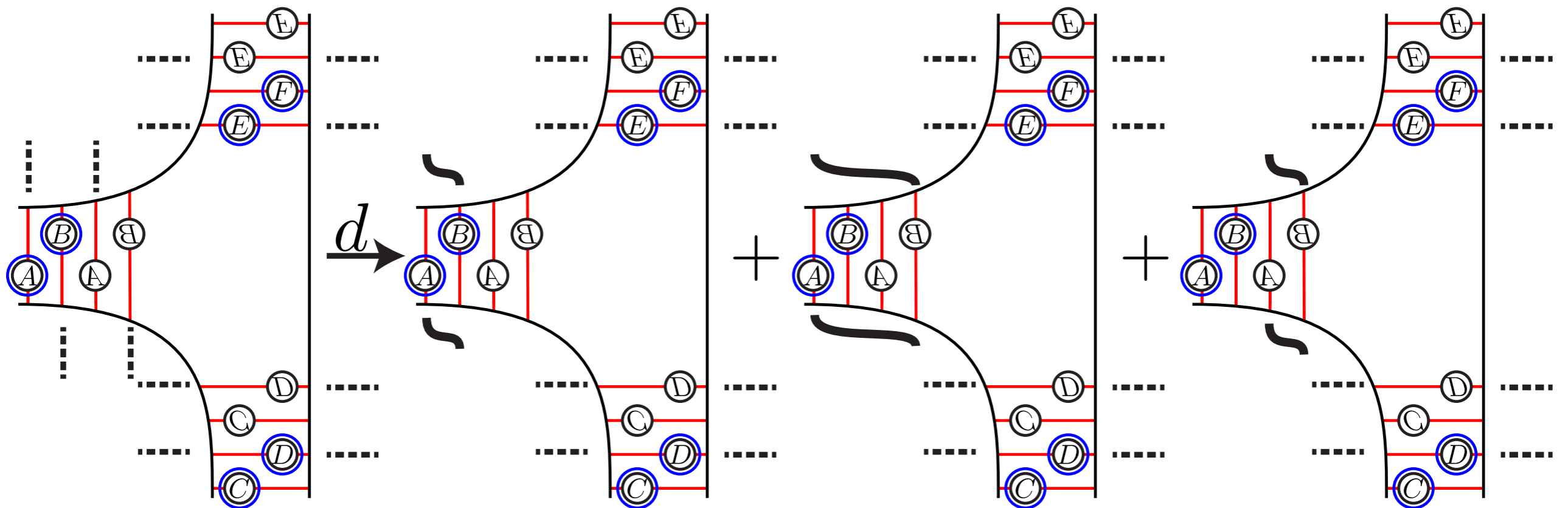
The *DDA* Cornering Trimodule

- A hybrid between the *DD* and *DA* identity modules.



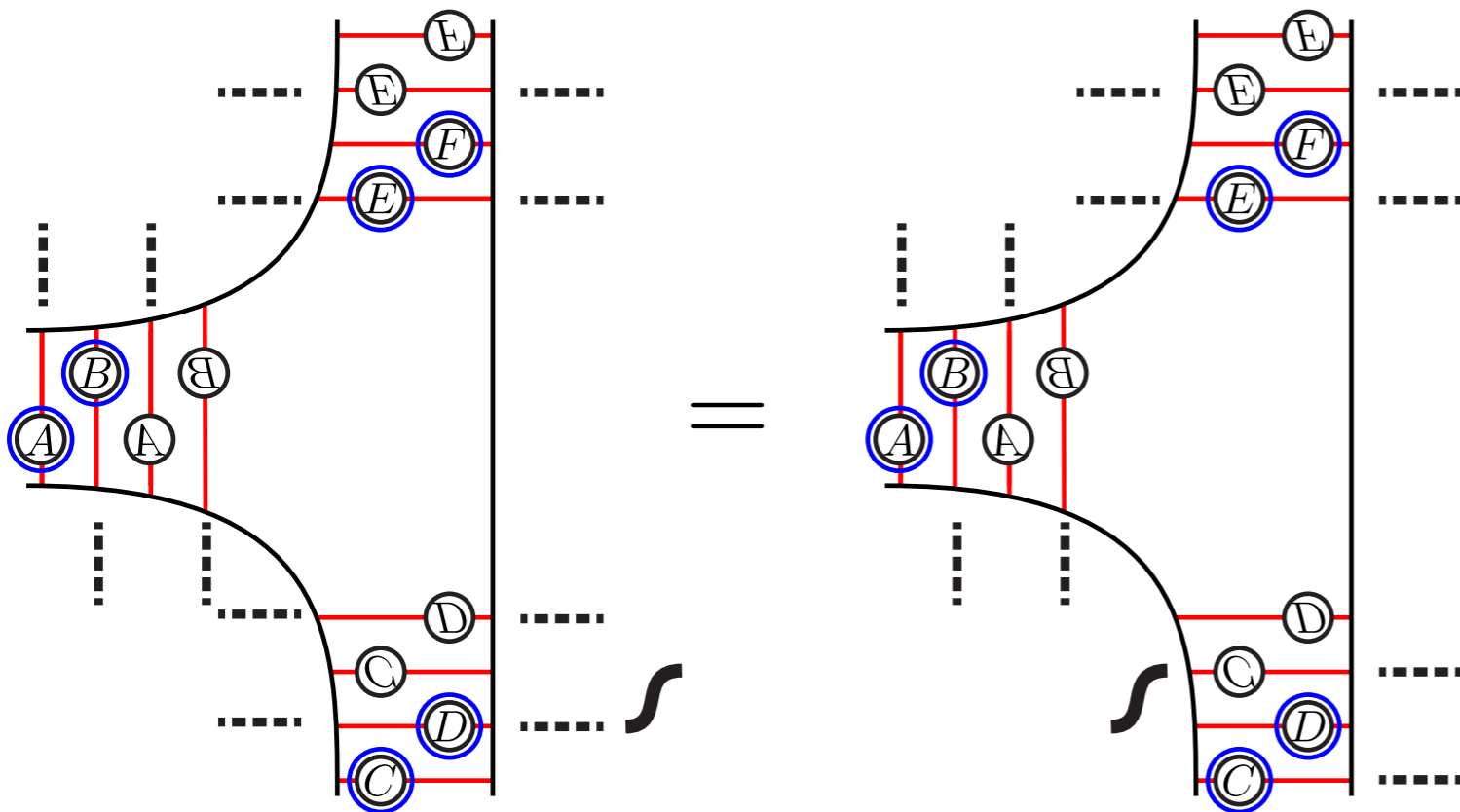
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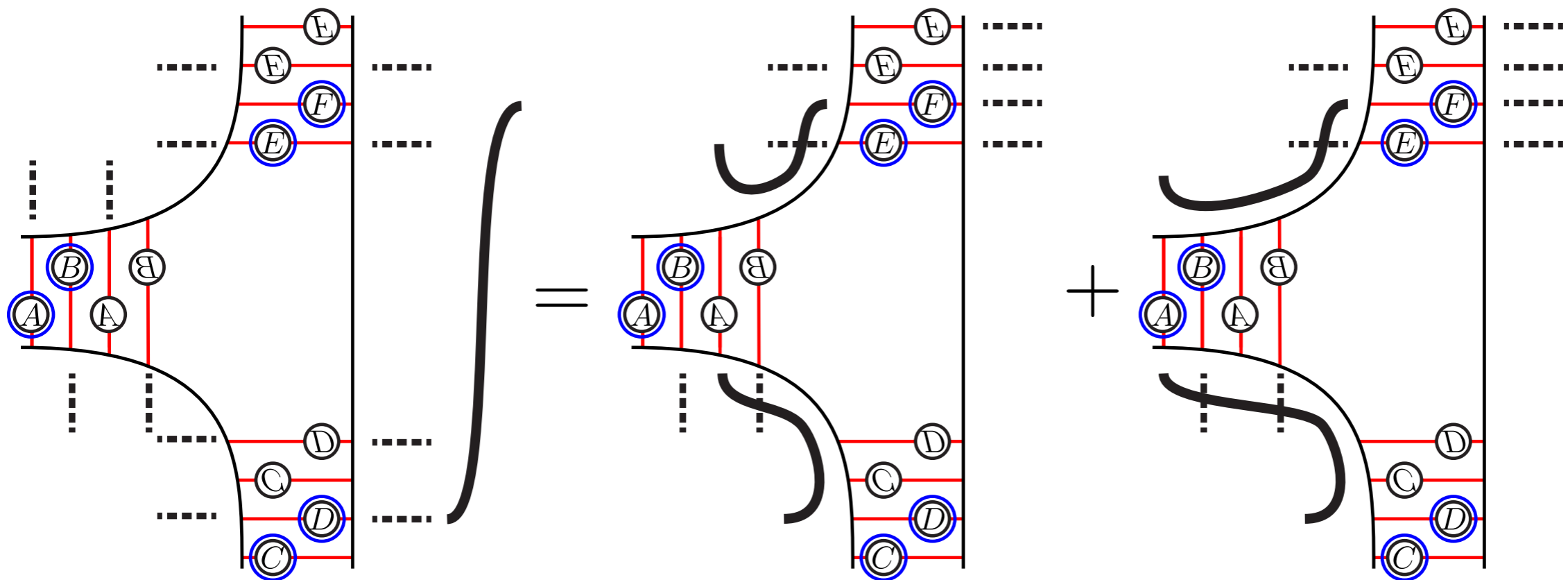
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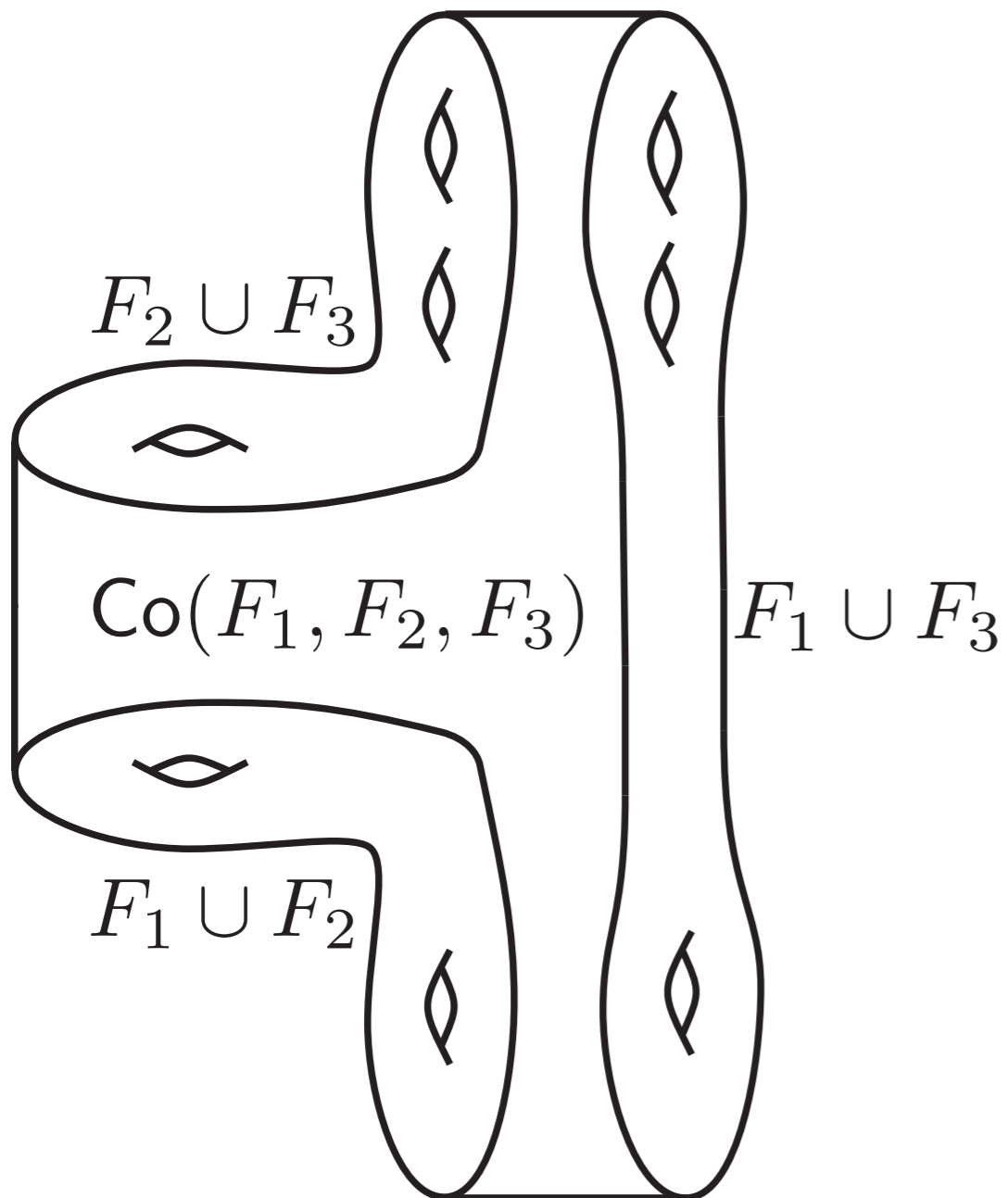


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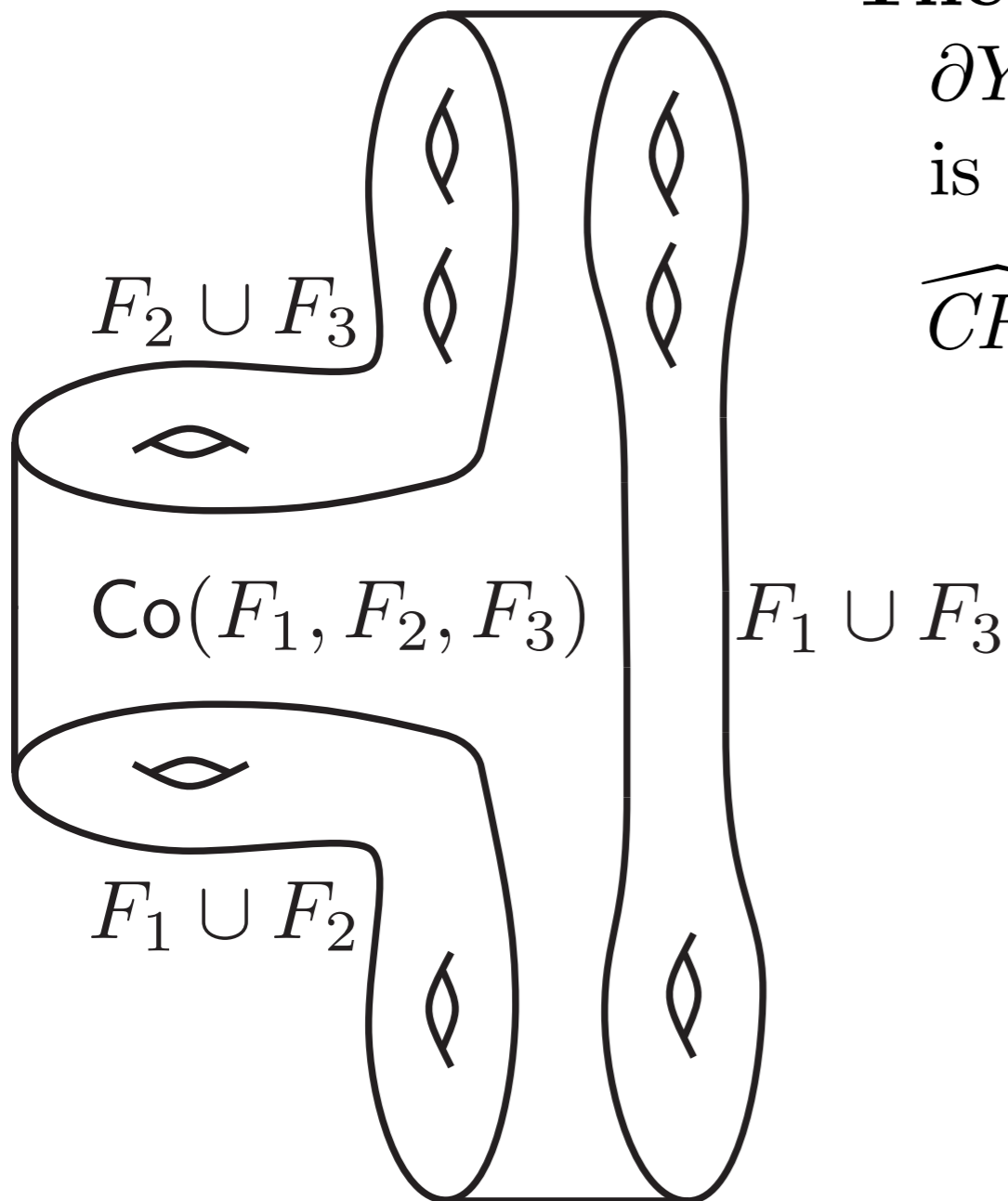
Recap:



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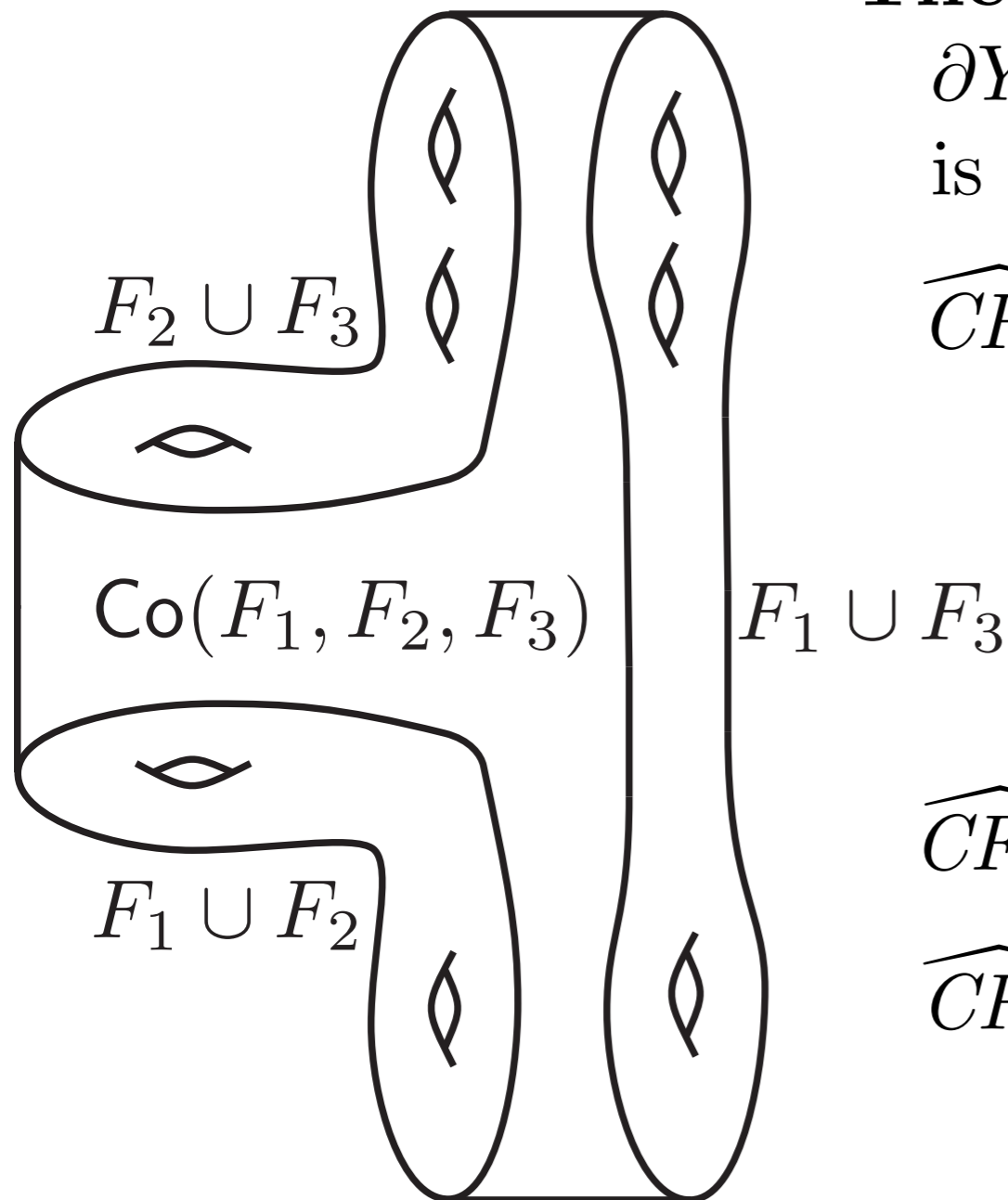
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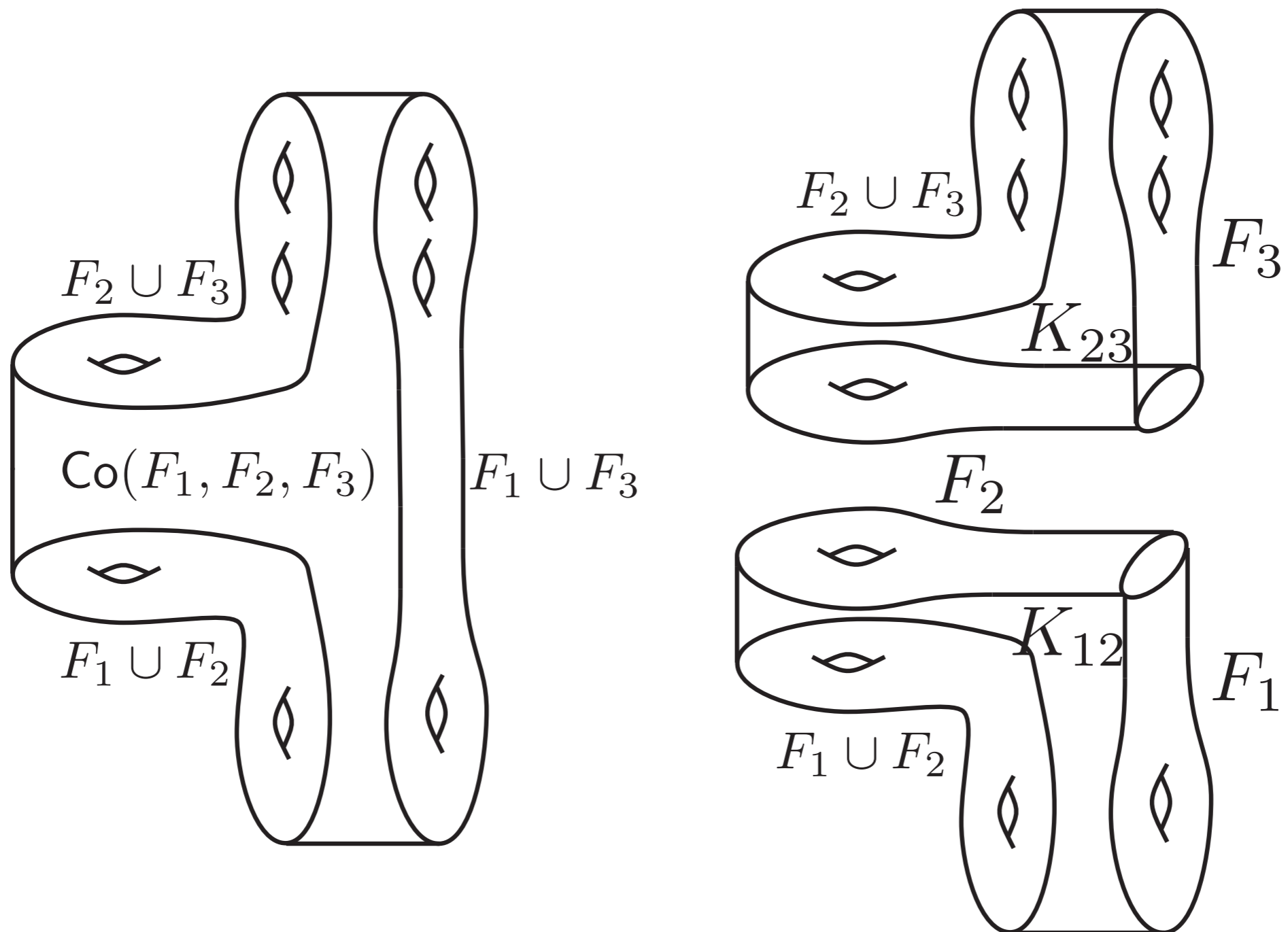
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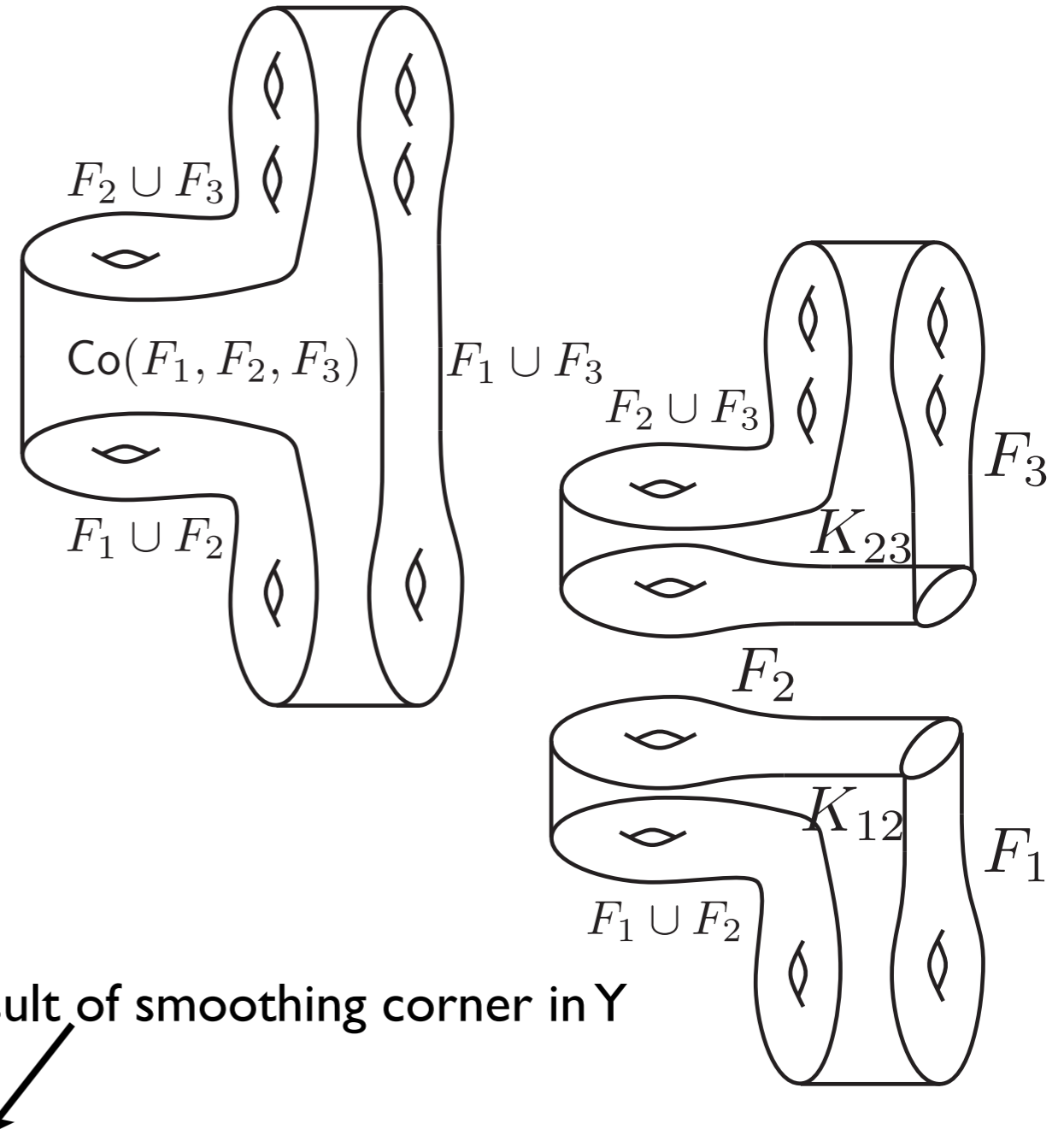
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Towards a cornered invariant



Towards a cornered invariant

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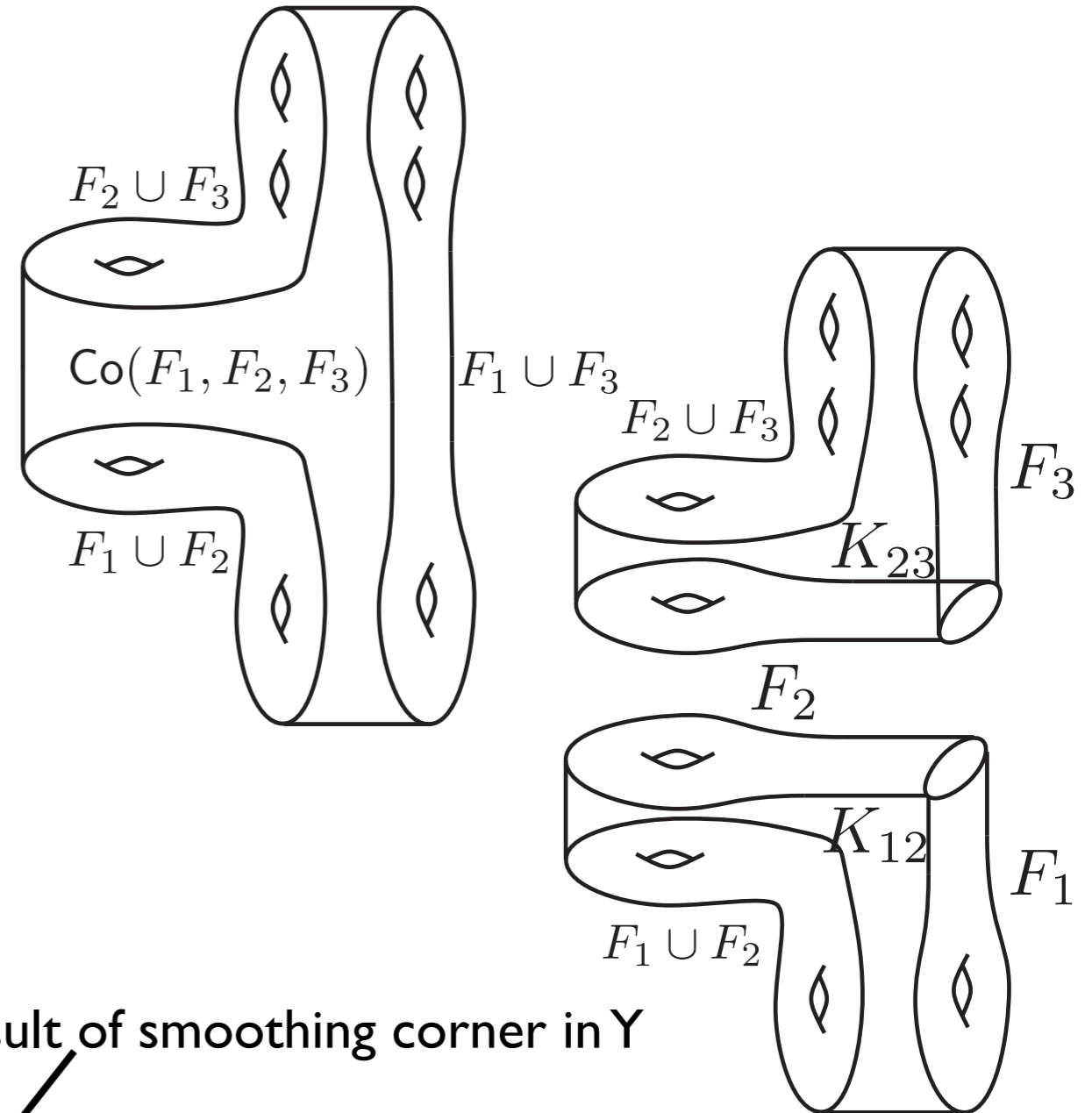


Towards a cornered invariant

Need:

- Algebraic framework with

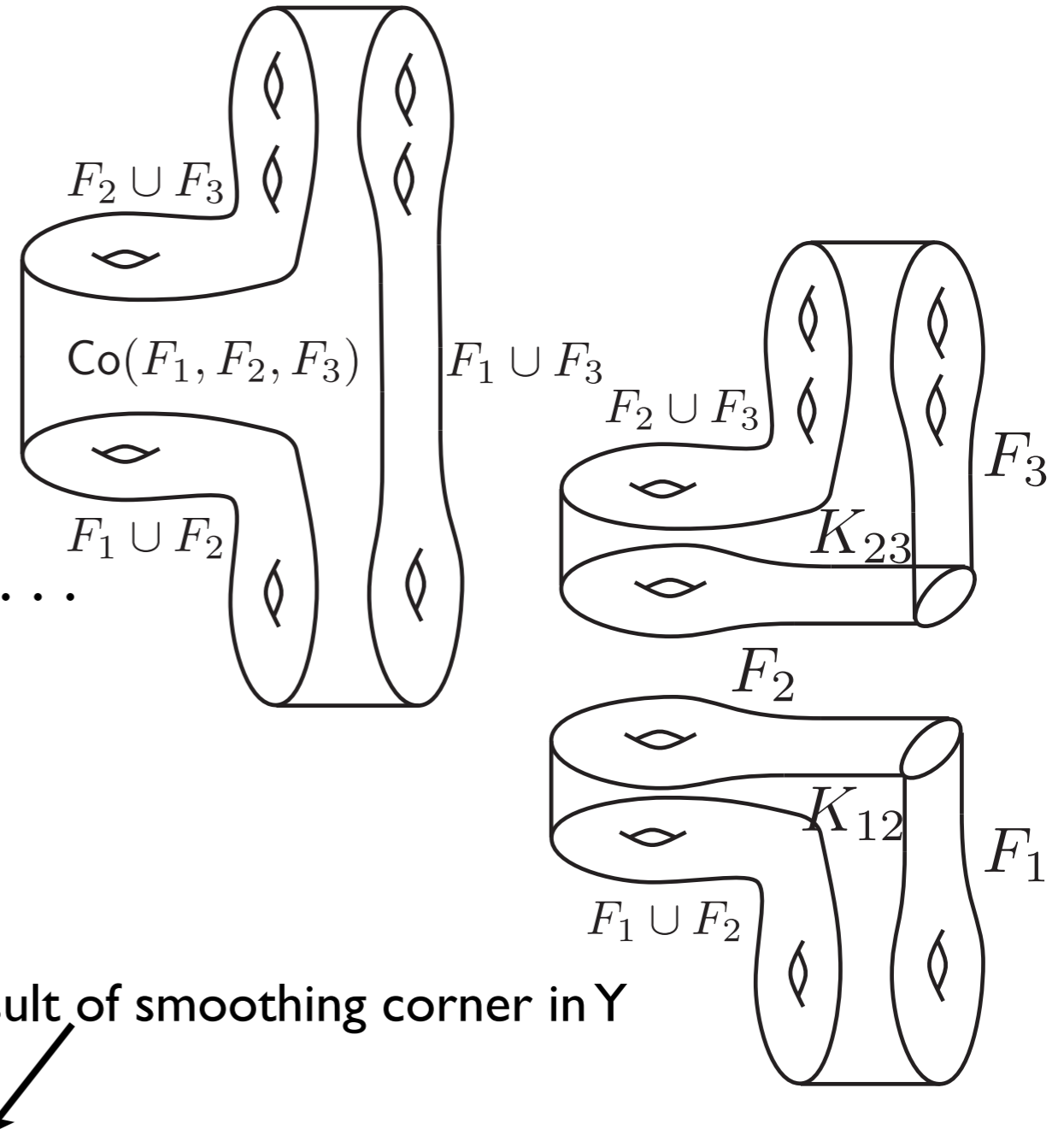
$$\mathcal{A}(F) = \mathcal{B}(F_3) \blacksquare \mathcal{T}(F_1)$$



Towards a cornered invariant

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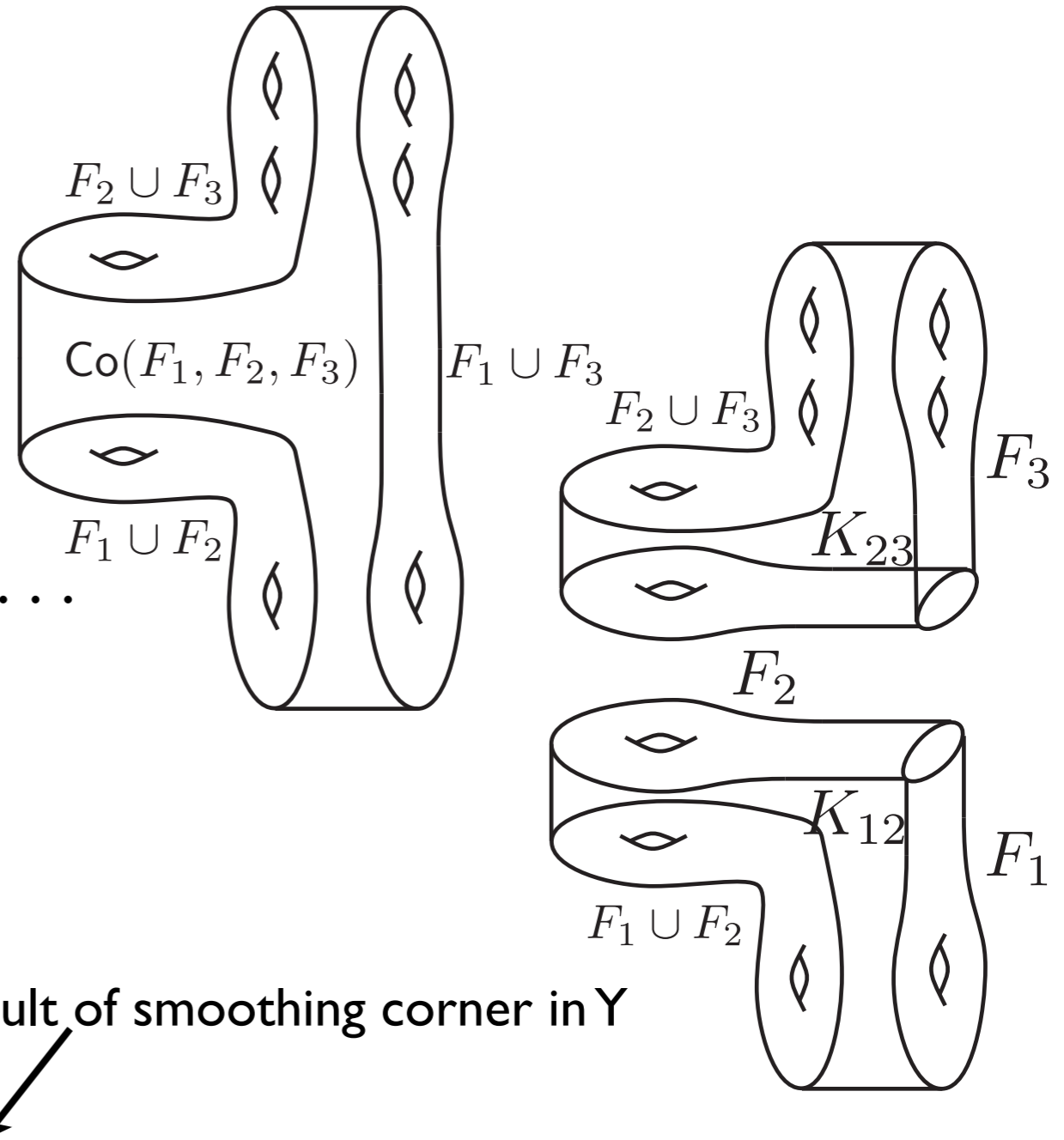
- Algebraic framework with $\mathcal{A}(F) = \mathcal{B}(F_3) \blacksquare \mathcal{T}(F_1)$
- Invariants $C_{D\{AA\}}(K_{12})$ and $C_{D\{DA\}}(K_{23})$ such that...



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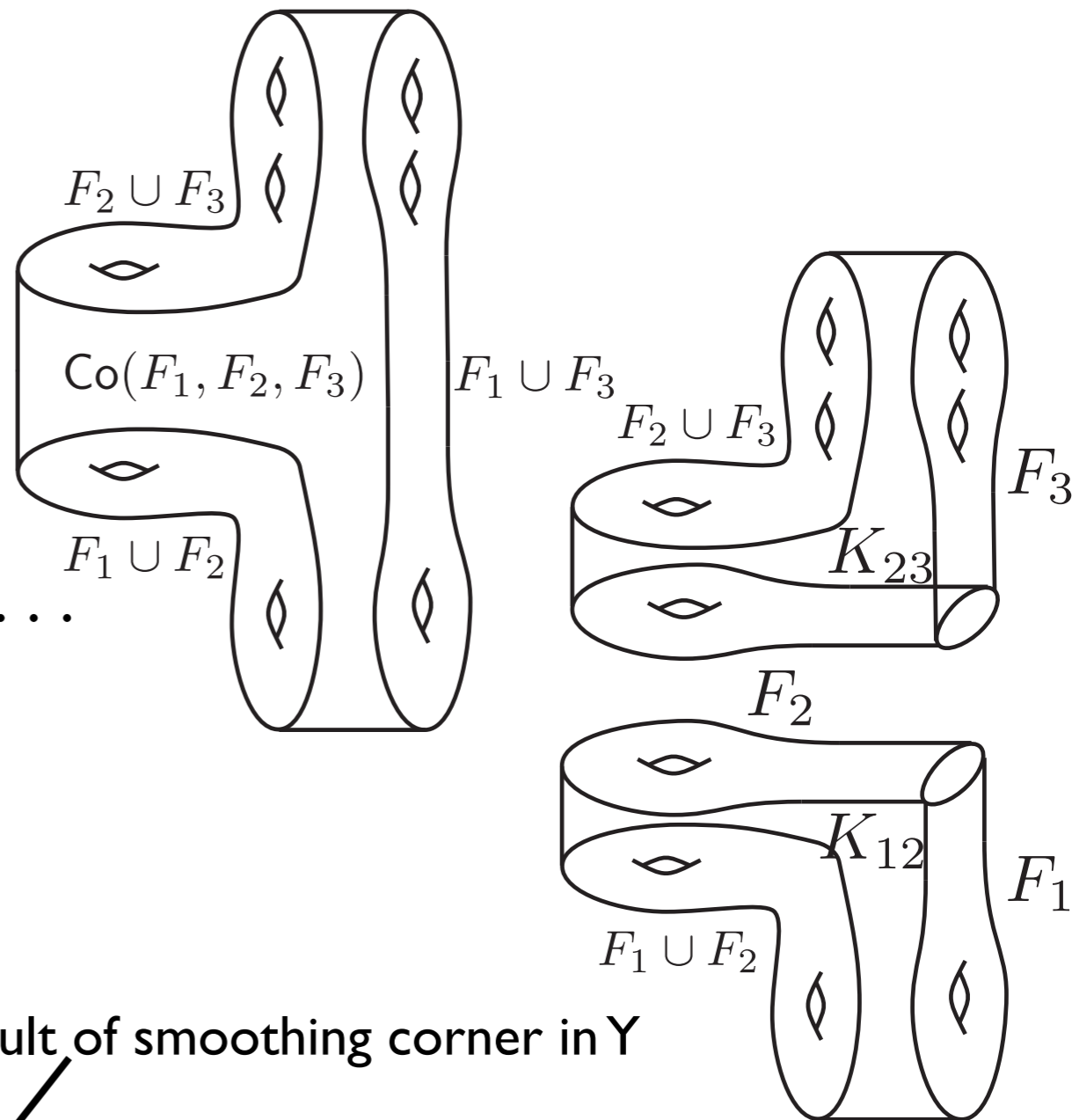
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Towards a cornered invariant

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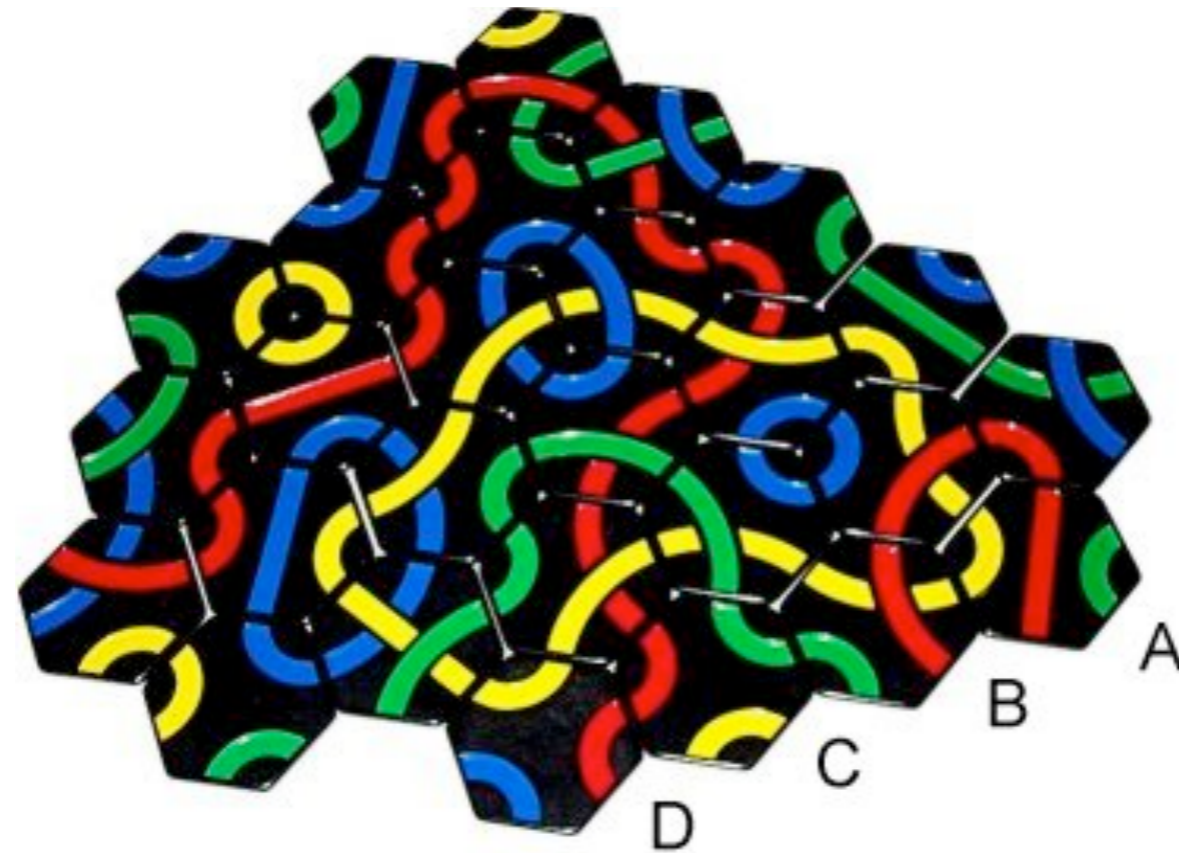
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Then define:

$$CF\{AA\}(Y) = \widehat{CFA}(Y^\circ) \otimes_{\mathcal{A}(F_1 \cup F_2)} C_{D\{AA\}}(K_{12})$$

$$CF\{DA\}(Y) = \widehat{CFA}(Y^\circ) \otimes_{\mathcal{A}(F_2 \cup F_3)} C_{D\{DA\}}(K_{23})$$



(Tantrix. Picture from thegamesjournal.com)

Cornered Floer Homology

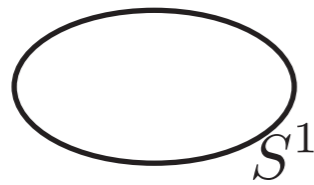
Cornered Floer Homology

(Douglas-Manolescu, L-Douglas-Manolescu)

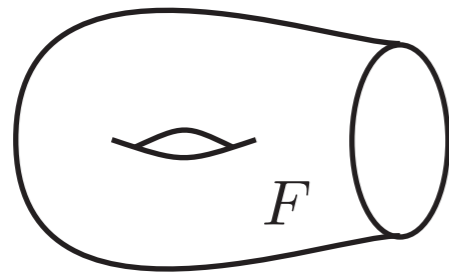
$$\text{S}^1 \longrightarrow \text{nilCoxeter 2-algebra } \mathcal{N} \text{ or } \mathcal{D}.$$

Cornered Floer Homology

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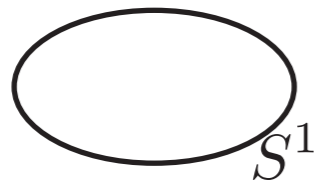
nilCoxeter 2-algebra \mathcal{N} or \mathcal{D} .



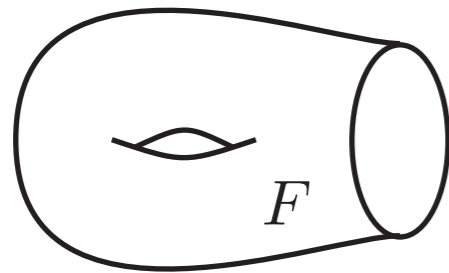
algebra-module $\mathcal{R}(F)$, $\mathcal{T}(F)$,
 $\mathcal{L}(F)$ or $\mathcal{B}(F)$.

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(Douglas-Manolescu, L-Douglas-Manolescu)



→ nilCoxeter 2-algebra \mathcal{N} or \mathcal{D} .

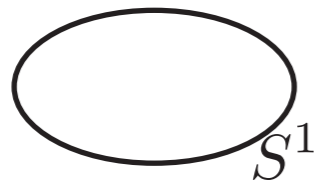


→ algebra-module $\mathcal{R}(F)$, $\mathcal{T}(F)$,
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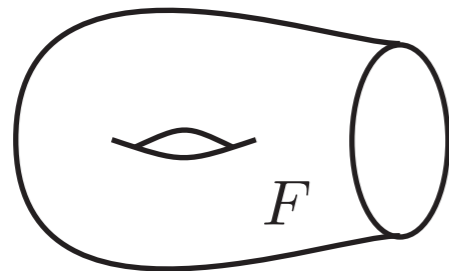
Theorem. $\mathcal{A}(F_1 \cup_{S^1} F_2) \cong \mathcal{T}(F_1) \otimes_{\mathcal{D}} \mathcal{B}(F_2) \cong \mathcal{R}(F_1) \otimes_{\mathcal{D}} \mathcal{L}(F_2)$.

Cornered Floer Homology

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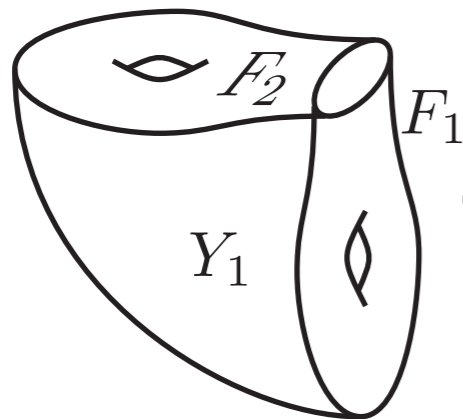


→ nilCoxeter 2-algebra \mathcal{N} or \mathcal{D} .



→ algebra-module $\mathcal{R}(F)$, $\mathcal{T}(F)$, $\mathcal{L}(F)$ or $\mathcal{B}(F)$.

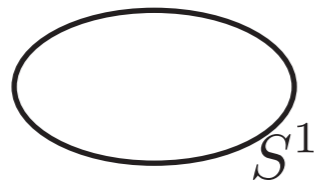
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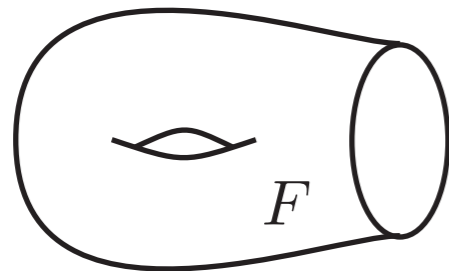
→ 2-module $\widehat{CF\{AA\}}(Y)$, $\widehat{CF\{DA\}}(Y)$, $\widehat{CF\{AD\}}(Y)$ or $\widehat{CF\{DD\}}(Y)$.

Cornered Floer Homology

(Douglas-Manolescu, L-Douglas-Manolescu)

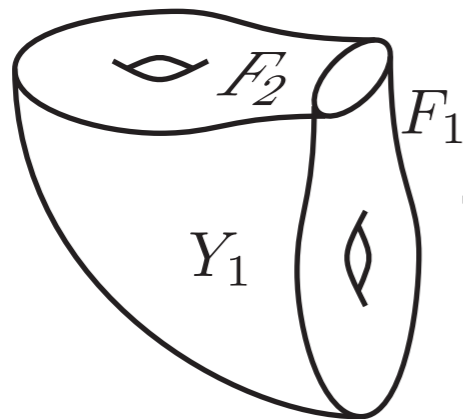


→ nilCoxeter 2-algebra \mathcal{N} or \mathcal{D} .



→ algebra-module $\mathcal{R}(F)$, $\mathcal{T}(F)$, $\mathcal{L}(F)$ or $\mathcal{B}(F)$.

Theorem. $\mathcal{A}(F_1 \cup_{S^1} F_2) \cong \mathcal{T}(F_1) \otimes_{\mathcal{D}} \mathcal{B}(F_2) \cong \mathcal{R}(F_1) \otimes_{\mathcal{D}} \mathcal{L}(F_2)$.

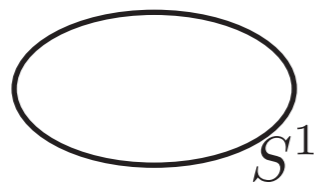


→ 2-module $\widehat{CF\{AA\}}(Y)$, $\widehat{CF\{DA\}}(Y)$, $\widehat{CF\{AD\}}(Y)$ or $\widehat{CF\{DD\}}(Y)$.

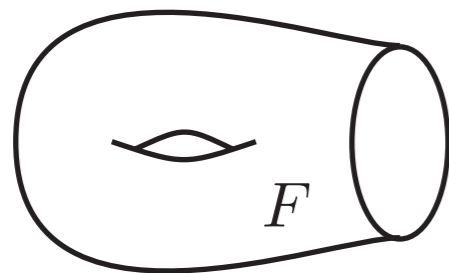
Theorem. $\widehat{CFA}(Y_1 \cup_F Y_2) \simeq \widehat{CF\{AA\}}(Y_1) \otimes_{\mathcal{R}(F)} \widehat{CF\{DA\}}(Y_2)$.

Cornered Floer Homology

(Douglas-Manolescu, L-Douglas-Manolescu)

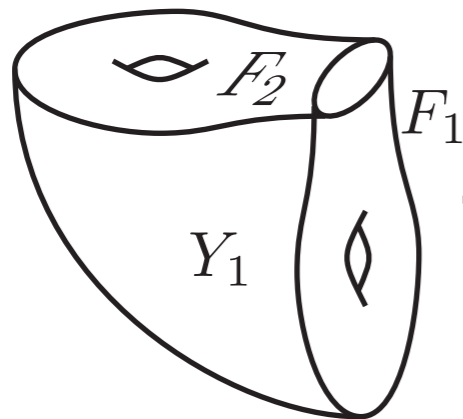


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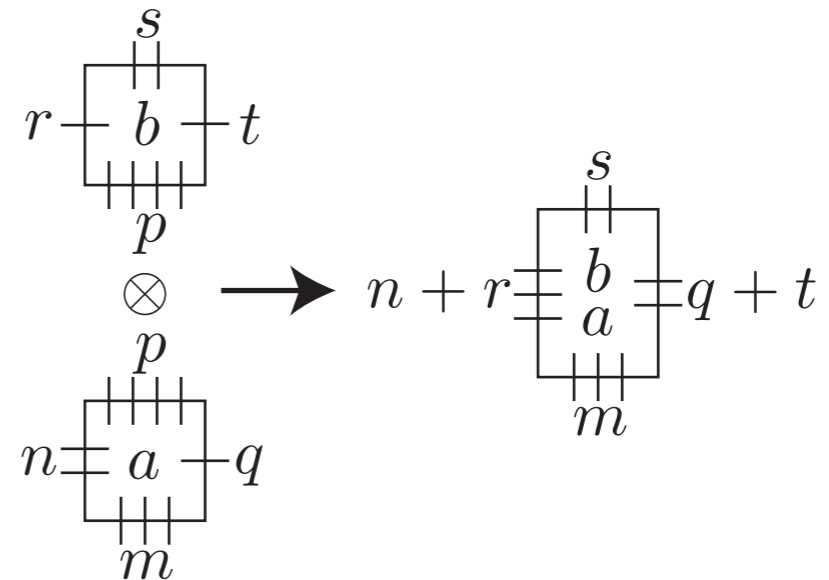
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Two versions: sequential and planar.

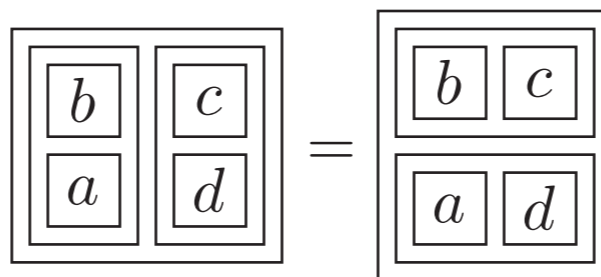
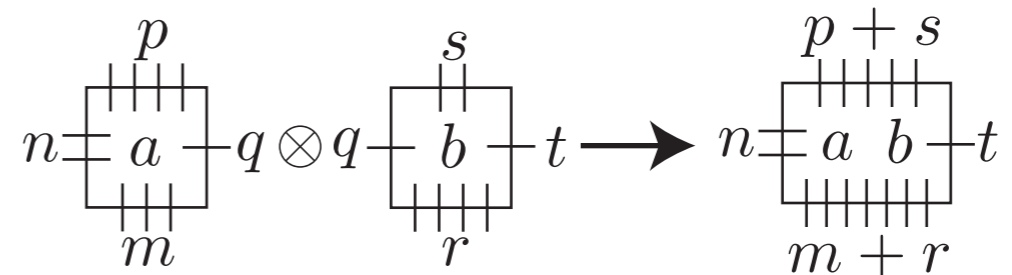
Abstract 2-algebra

2-Algebra: Chain complexes ${}_n\mathcal{D}_q^p$ and maps

$$\begin{array}{c} s \\ r\mathcal{D}_t \\ p \end{array} \otimes \begin{array}{c} p \\ n\mathcal{D}_q \\ m \end{array} \longrightarrow \begin{array}{c} s \\ n+r\mathcal{D}_{q+t} \\ m \end{array}$$

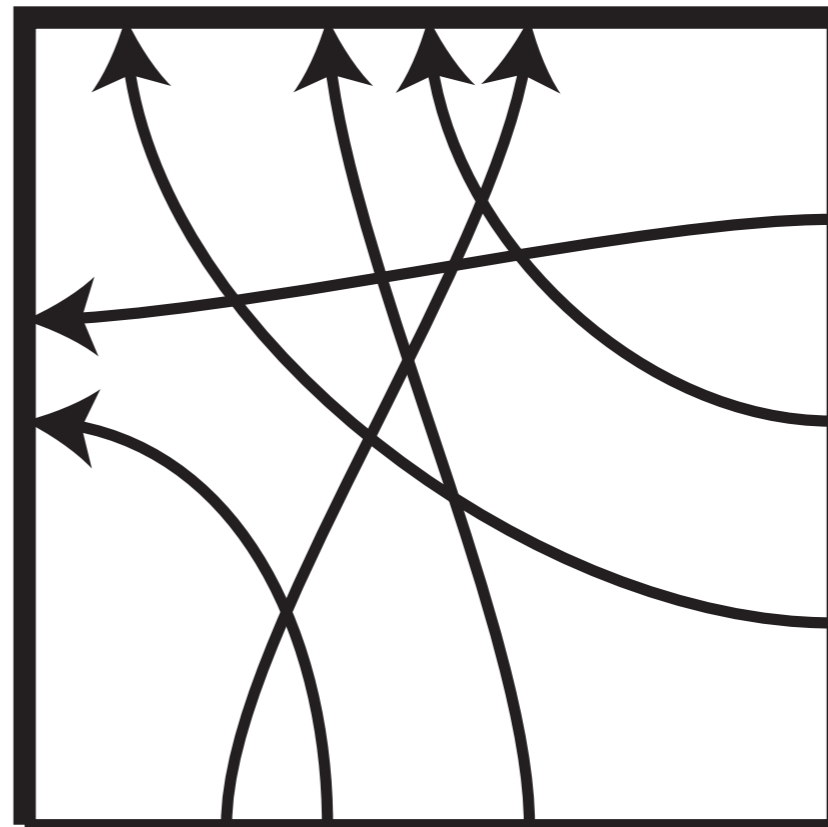


$$\begin{array}{c} p \\ n\mathcal{D}_q \\ m \end{array} \otimes \begin{array}{c} s \\ q\mathcal{D}_t \\ r \end{array} \longrightarrow \begin{array}{c} p+s \\ n\mathcal{D}_{t} \\ m+r \end{array}$$



The nilCoxeter 2-algebra

- Multiplication is concatenation.
- Differential smooths crossings.
- Double crossings = 0.
- No closed components.



More abstract 2-algebra

Right algebra-module: Chain complexes \mathcal{R}_m^p and maps

$$\begin{array}{c} \mathcal{R} \\ \hline \mathcal{R} \end{array} \begin{array}{c} q \\ n \\ m \end{array} \begin{array}{c} r \\ p \end{array} \longrightarrow \begin{array}{c} \mathcal{R} \\ \hline \mathcal{R} \end{array} \begin{array}{c} q \\ m \end{array} \begin{array}{c} p+r \end{array} \quad \begin{array}{c} \mathcal{R} \\ \hline \mathcal{R} \end{array} \begin{array}{c} n \\ m \end{array} \begin{array}{c} p \end{array} \begin{array}{c} \mathcal{D} \\ \hline \mathcal{D} \end{array} \begin{array}{c} s \\ r \end{array} \begin{array}{c} t \end{array} \longrightarrow \begin{array}{c} \mathcal{R} \\ \hline \mathcal{R} \end{array} \begin{array}{c} n+s \\ m+r \end{array} \begin{array}{c} t \end{array}$$

\mathcal{R}	\mathcal{D}
\mathcal{R}	\mathcal{D}

More abstract 2-algebra

Right algebra-module: Chain complexes \mathcal{R}_q^p and maps

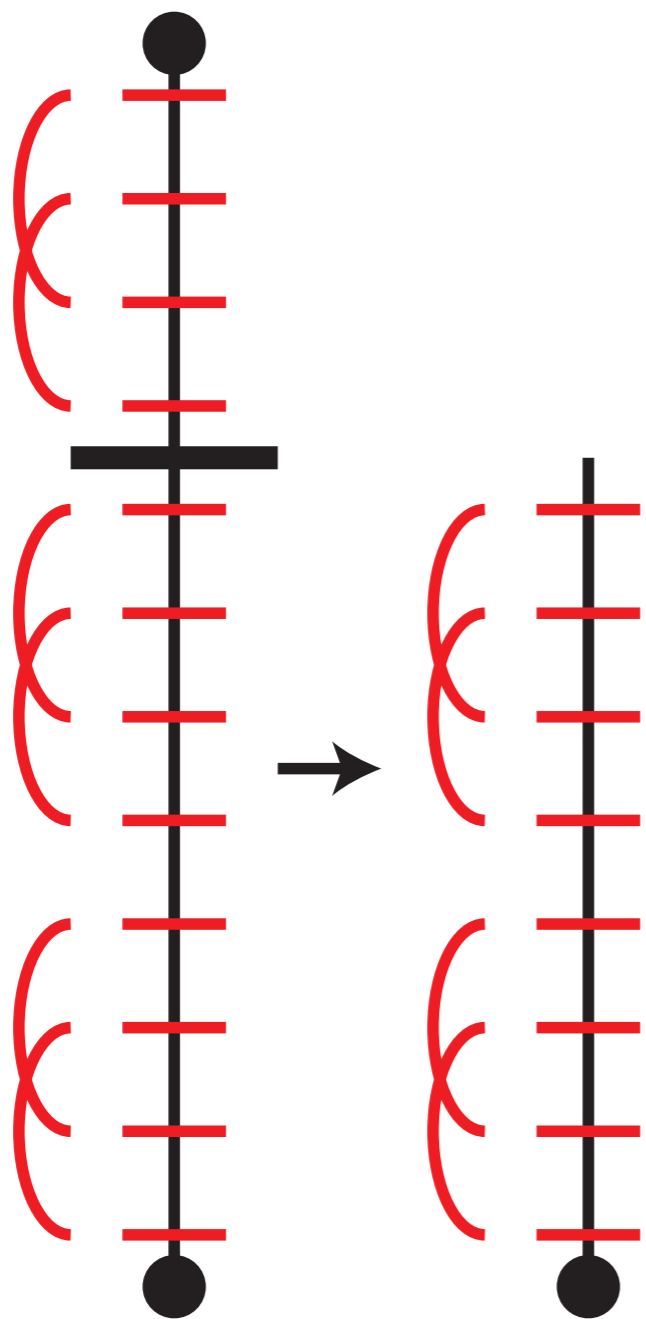
$$\begin{array}{c} \overline{\mathcal{R}}^q \\ \hline \overline{\mathcal{R}}^n \\ \hline \overline{\mathcal{R}}^m \end{array} \begin{array}{l} r \\ \\ p \end{array} \longrightarrow \begin{array}{c} \overline{\mathcal{R}}^q \\ \hline \overline{\mathcal{R}}^m \end{array} \begin{array}{l} p+r \\ \\ \end{array} \quad \begin{array}{c} \overline{\mathcal{R}}^n \\ \hline \overline{\mathcal{R}}^m \end{array} \begin{array}{l} p \\ \\ \end{array} \begin{array}{c} \overline{\mathcal{D}}^s \\ \hline \overline{\mathcal{D}}^r \end{array} \begin{array}{l} t \\ \\ \end{array} \longrightarrow \begin{array}{c} \overline{\mathcal{R}}^{n+s} \\ \hline \overline{\mathcal{R}}^{m+r} \end{array} \begin{array}{l} t \\ \\ \end{array}$$

$$\begin{array}{c} \overline{\mathcal{R}} \quad \overline{\mathcal{D}} \\ \hline \overline{\mathcal{R}} \quad \overline{\mathcal{D}} \end{array}$$

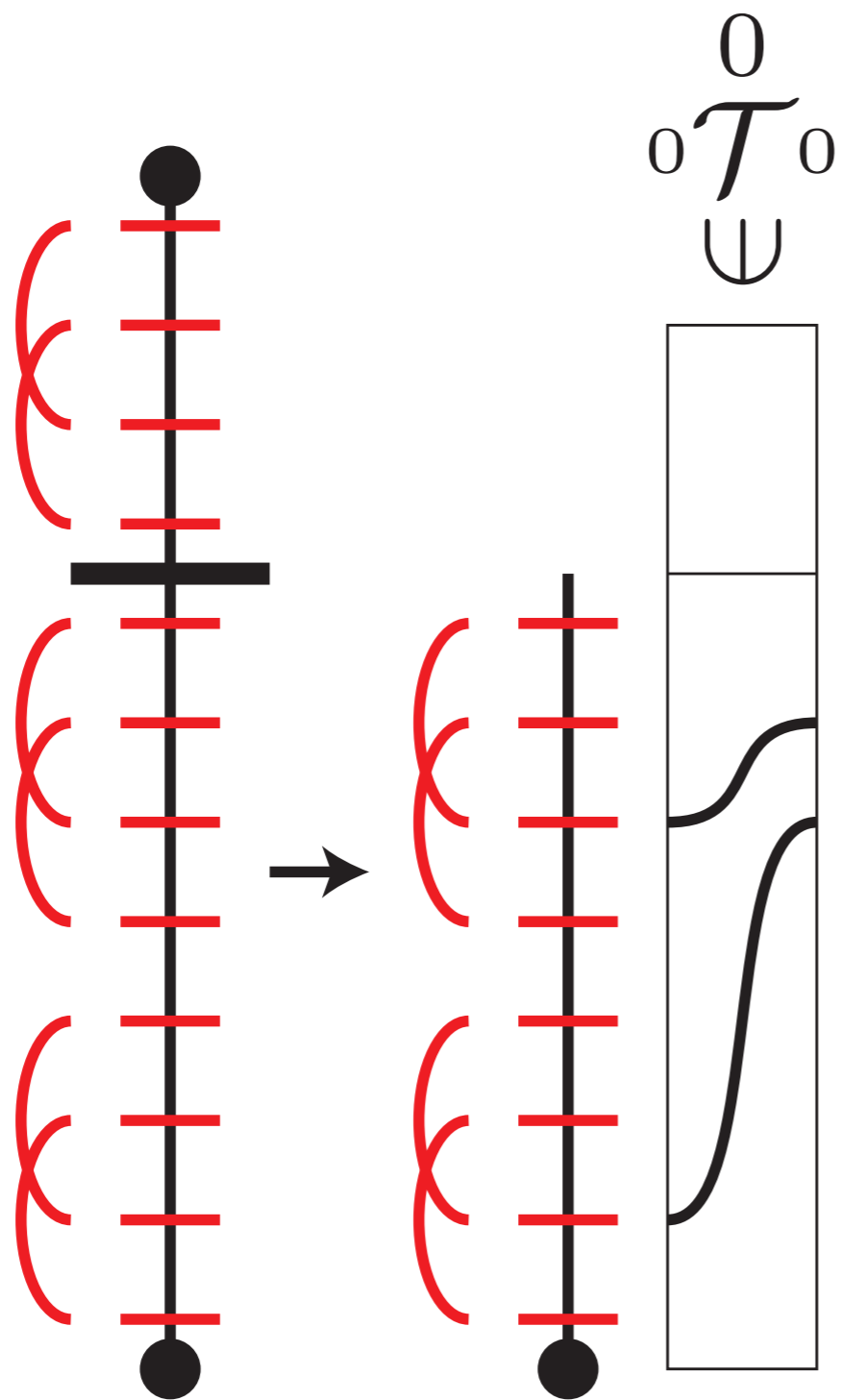
Top-right 2-module: Chain complexes \mathcal{TR}_n^m and maps

$$\begin{array}{c} \overline{\mathcal{R}}^p \\ \hline \overline{\mathcal{R}}^n \end{array} \begin{array}{l} q \\ \\ \end{array} \begin{array}{c} \overline{\mathcal{TR}} \\ \hline \overline{\mathcal{TR}} \end{array} \begin{array}{l} m \\ \\ m+q \end{array} \longrightarrow \begin{array}{c} \overline{\mathcal{TR}} \\ \hline \overline{\mathcal{TR}} \end{array} \begin{array}{l} p \\ \\ m+q \end{array} \quad \begin{array}{c} \overline{\mathcal{TR}}^n \\ \hline \overline{\mathcal{TR}}^m \end{array} \begin{array}{l} p \\ \\ m \end{array} \begin{array}{c} \overline{\mathcal{T}}^p \\ \hline \overline{\mathcal{T}}^q \end{array} \begin{array}{l} q \\ \\ q \end{array} \longrightarrow \begin{array}{c} \overline{\mathcal{TR}}^{n+p} \\ \hline \overline{\mathcal{TR}}^q \end{array} \begin{array}{l} q \\ \\ q \end{array} \quad \begin{array}{c} \overline{\mathcal{R}} \quad \overline{\mathcal{D}} \\ \hline \overline{\mathcal{TR}} \quad \overline{\mathcal{T}} \end{array}$$

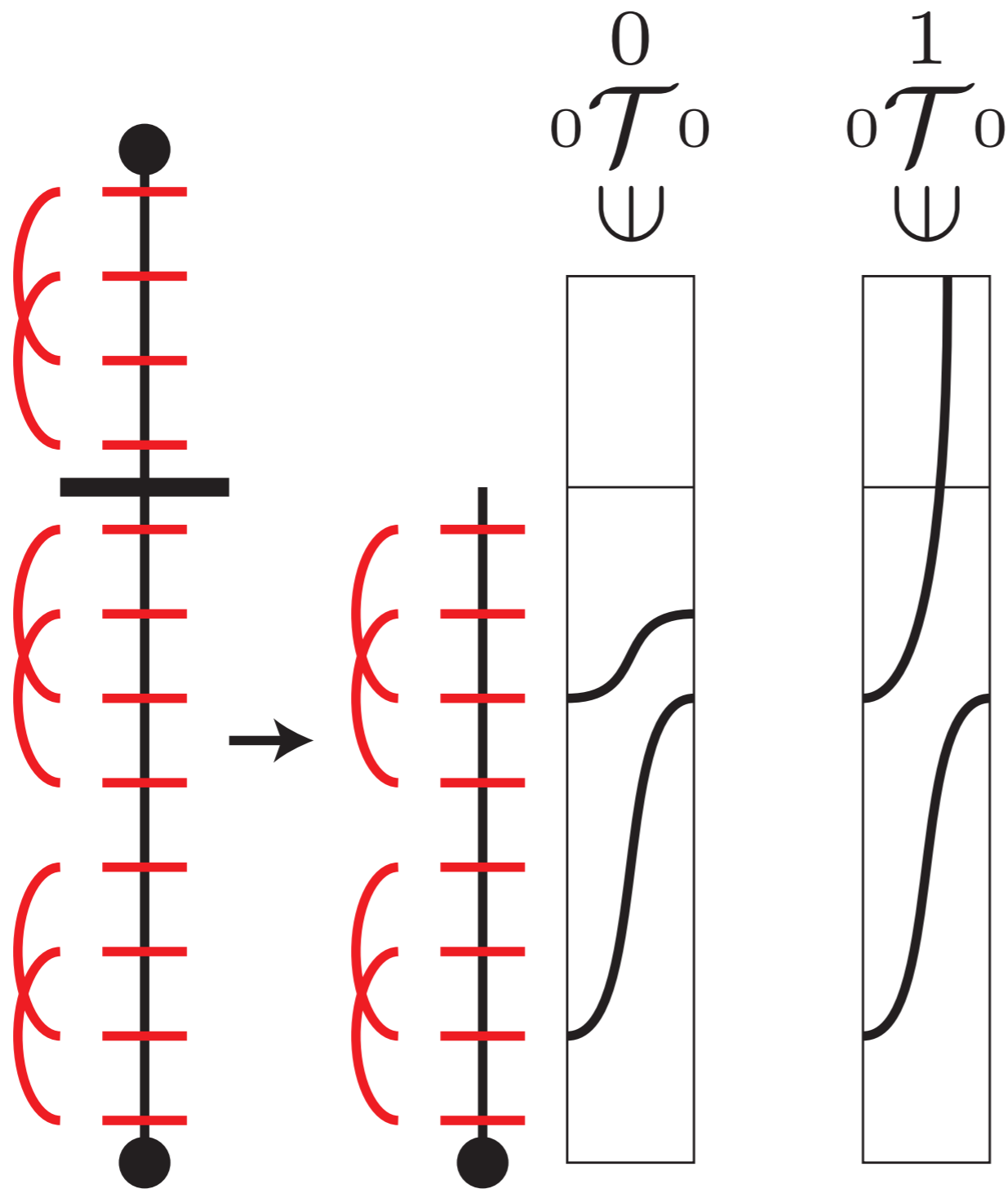
The algebra-modules



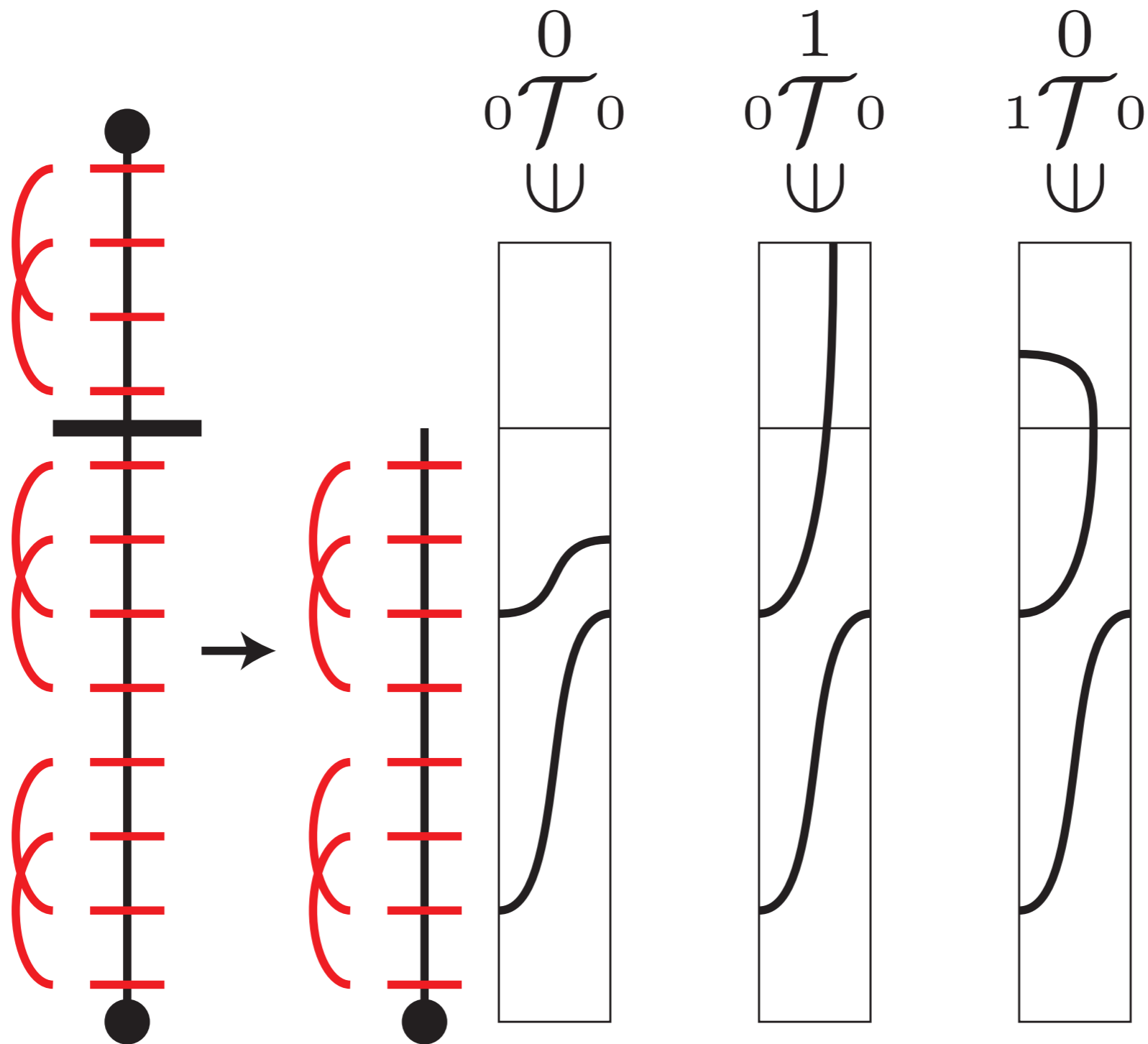
The algebra-modules



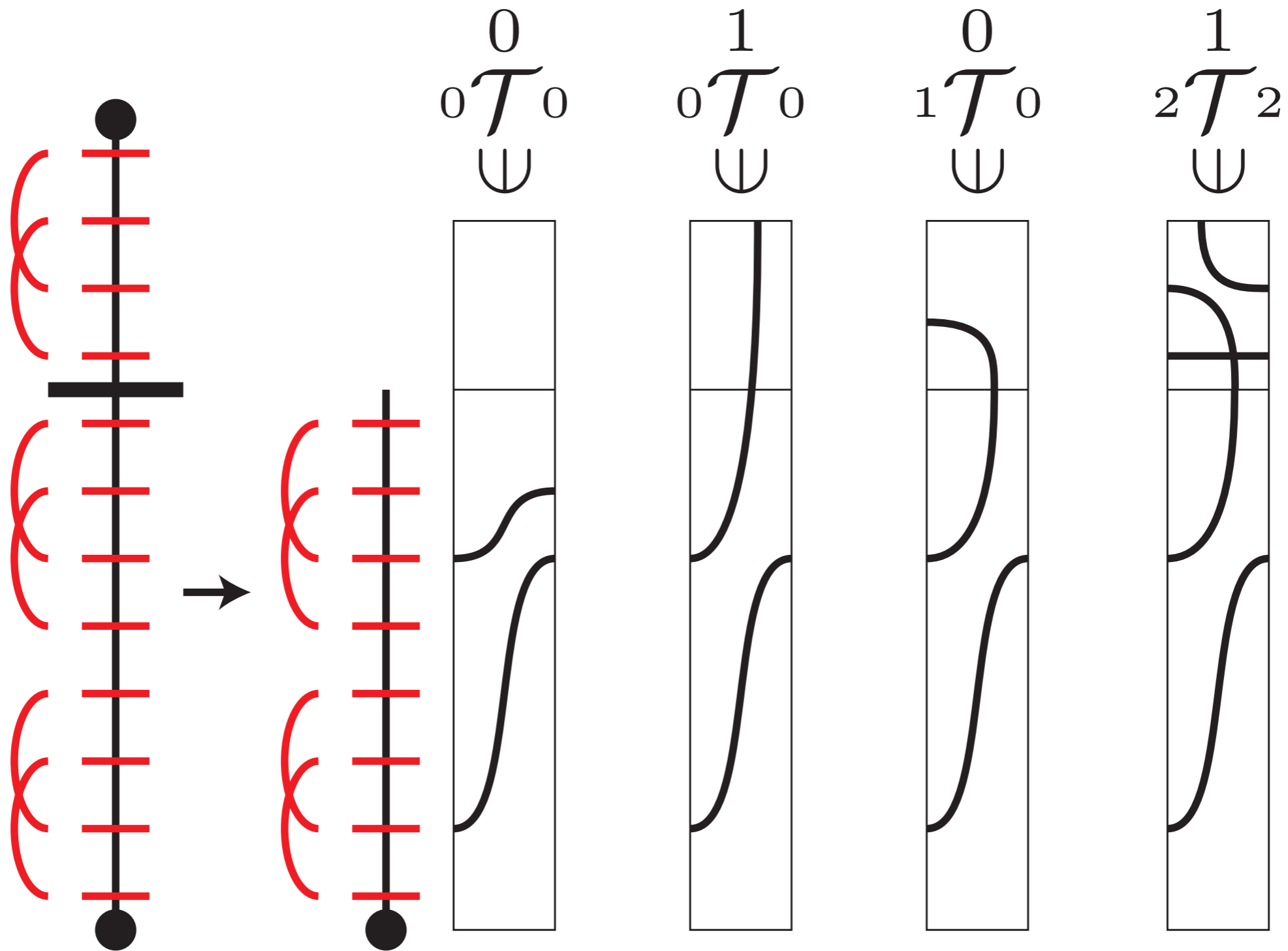
The algebra-modules



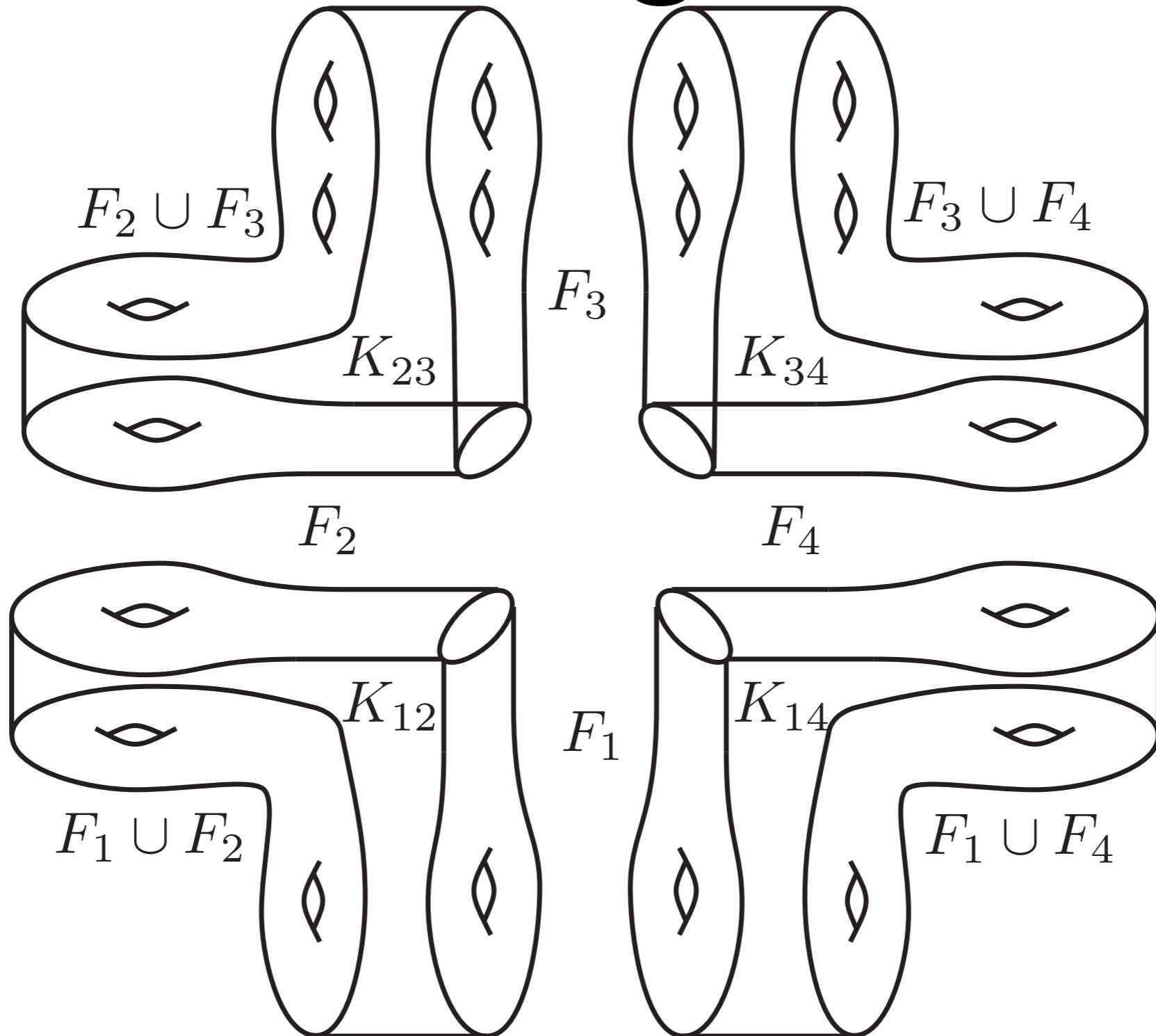
The algebra-modules



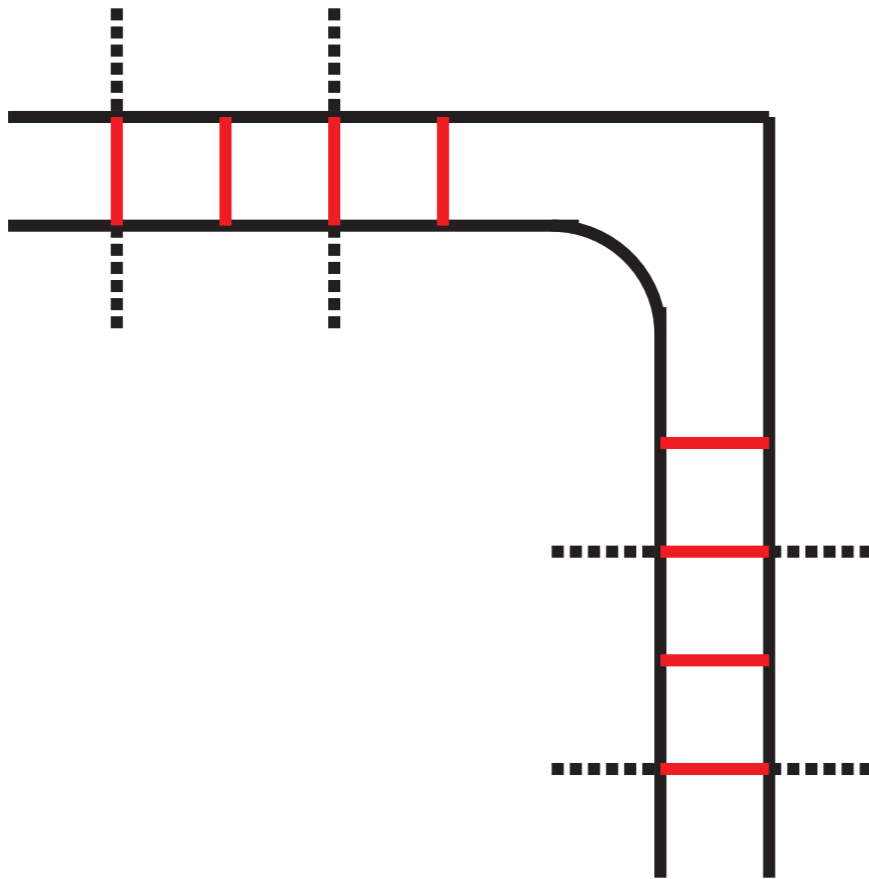
The algebra-modules



Cornering Pieces

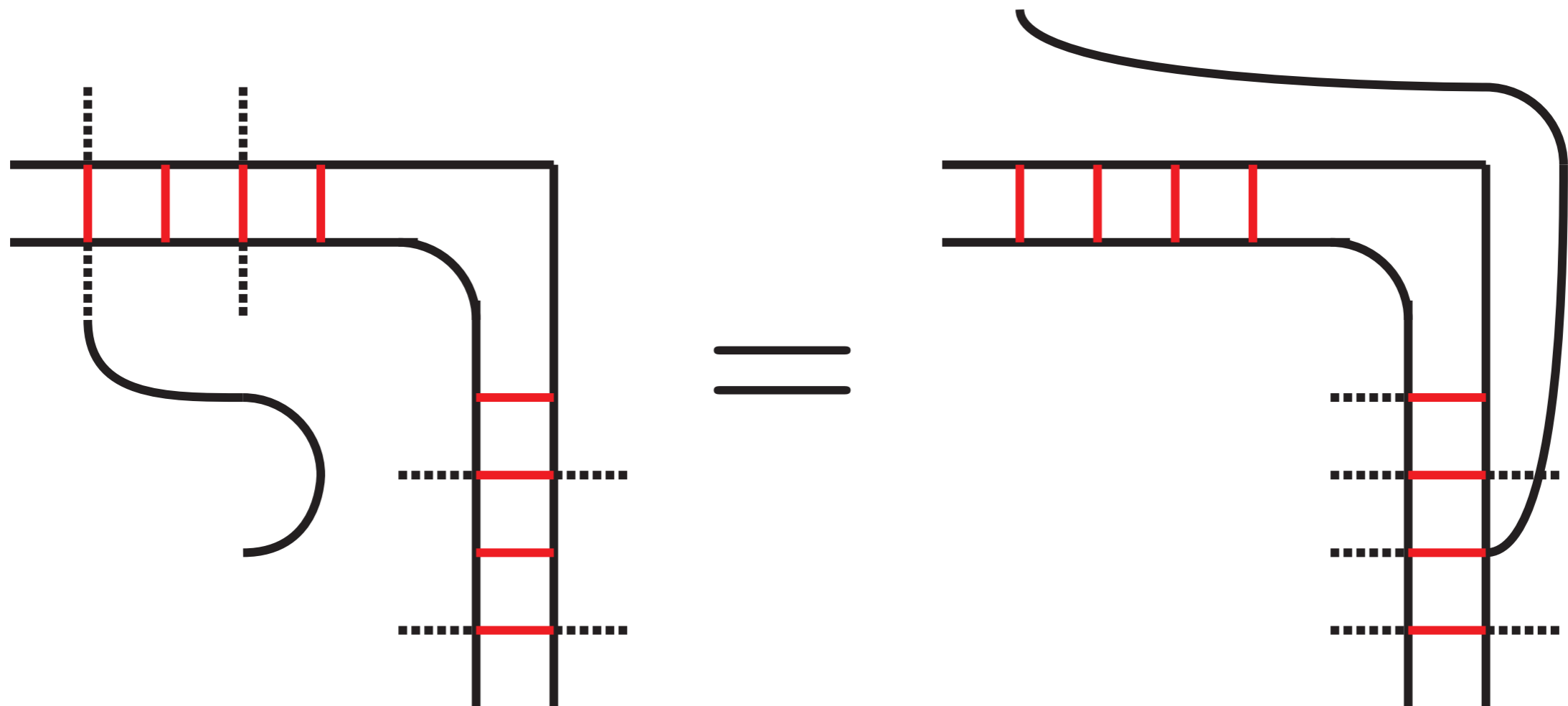


The *DAA* cornering 2-module



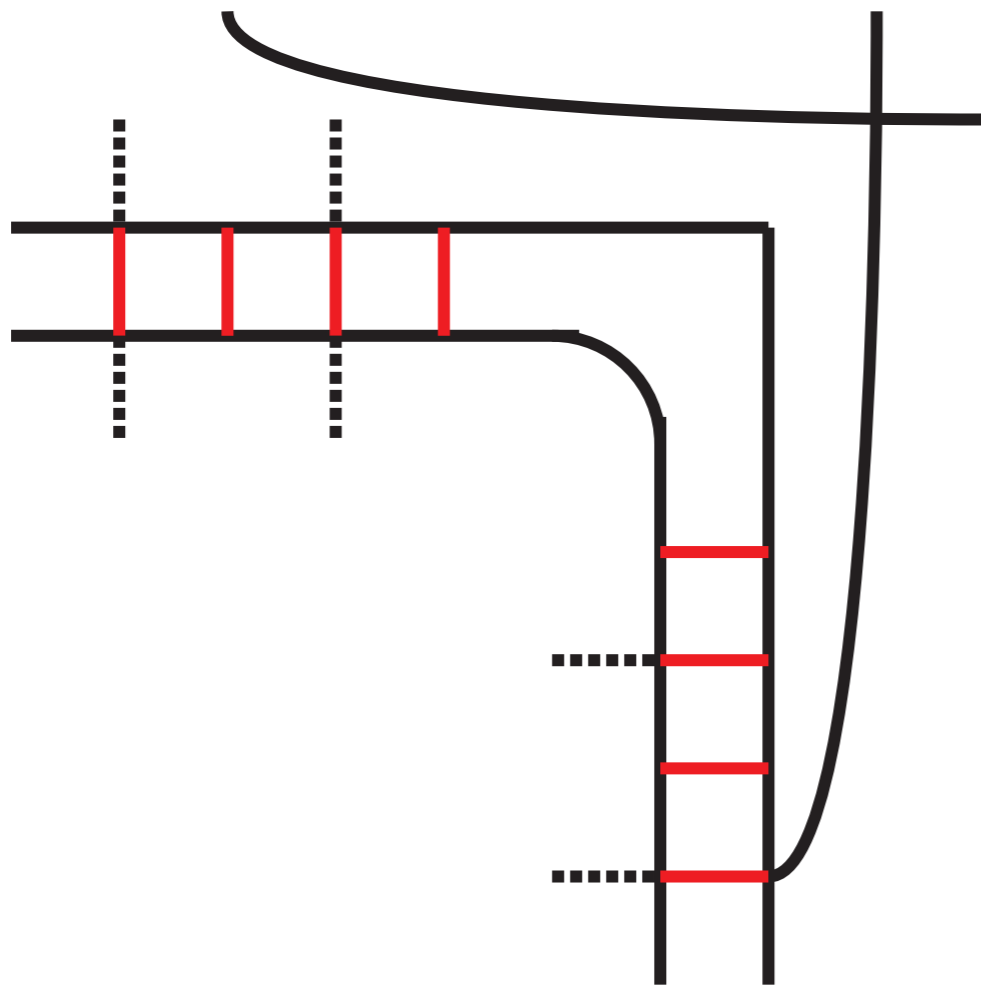
$$C_{D\{AA\}}(K_{12})$$

The *DAA* cornering 2-module



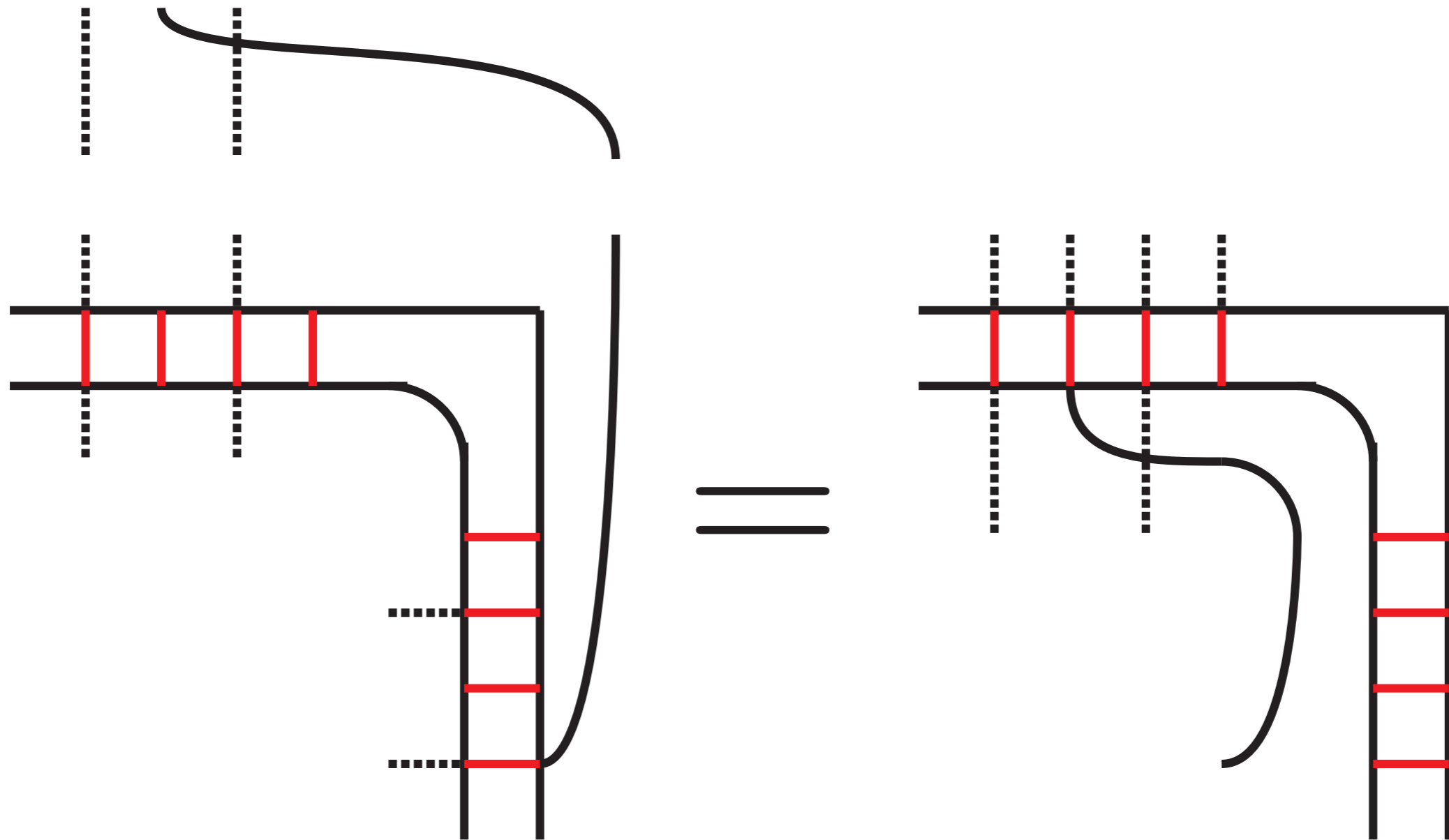
$$C_{D\{AA\}}(K_{12})$$

The *DAA* cornering 2-module



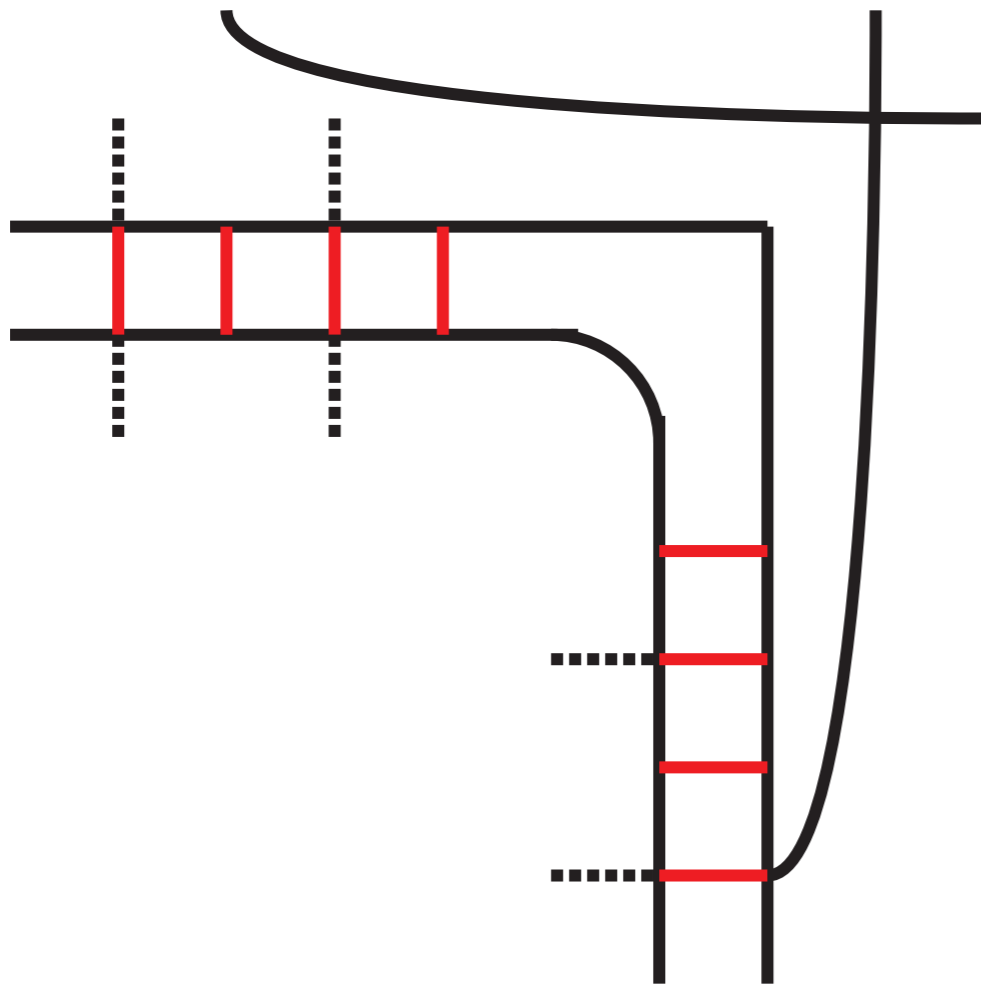
$$C_{D\{AA\}}(K_{12})$$

The *DAA* cornering 2-module



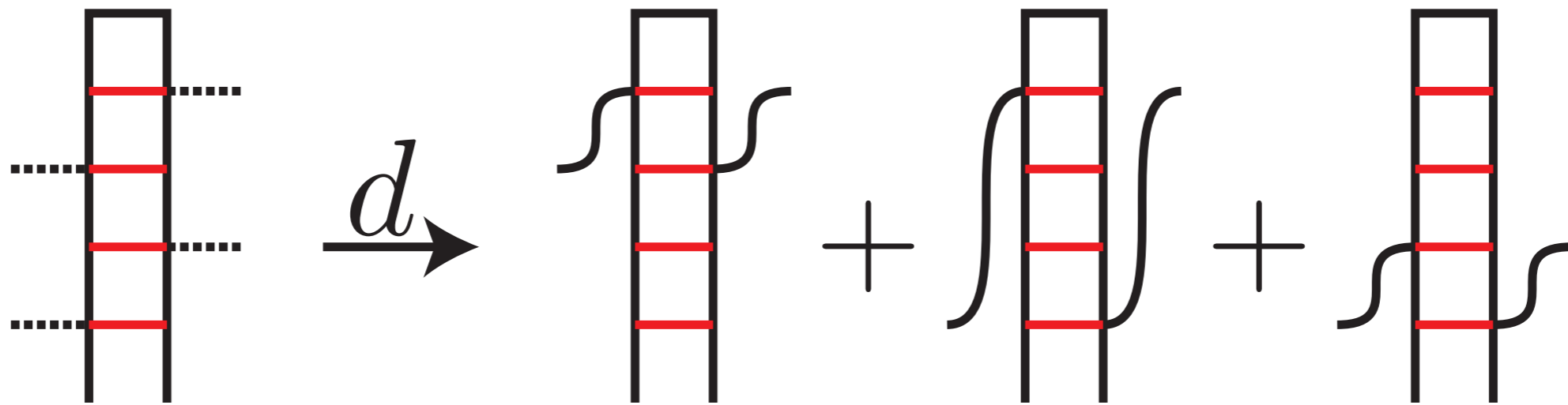
$$C_{D\{AA\}}(K_{12})$$

The DAA cornering 2-module



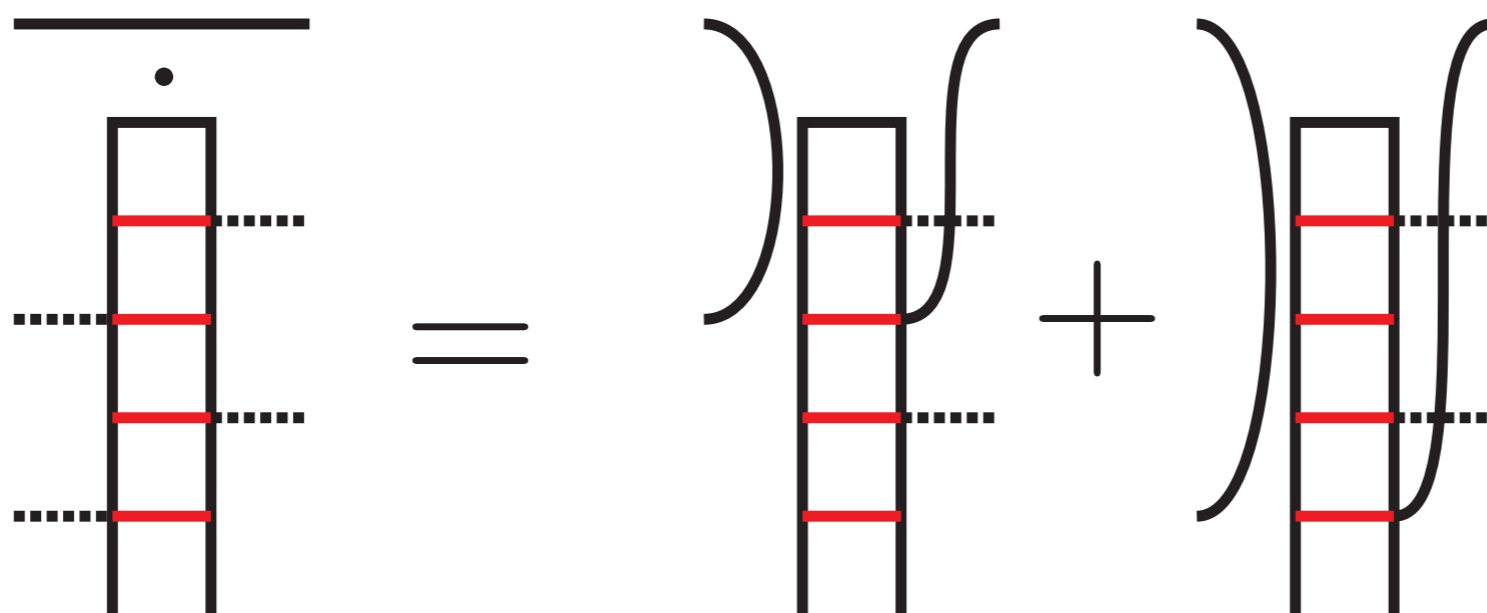
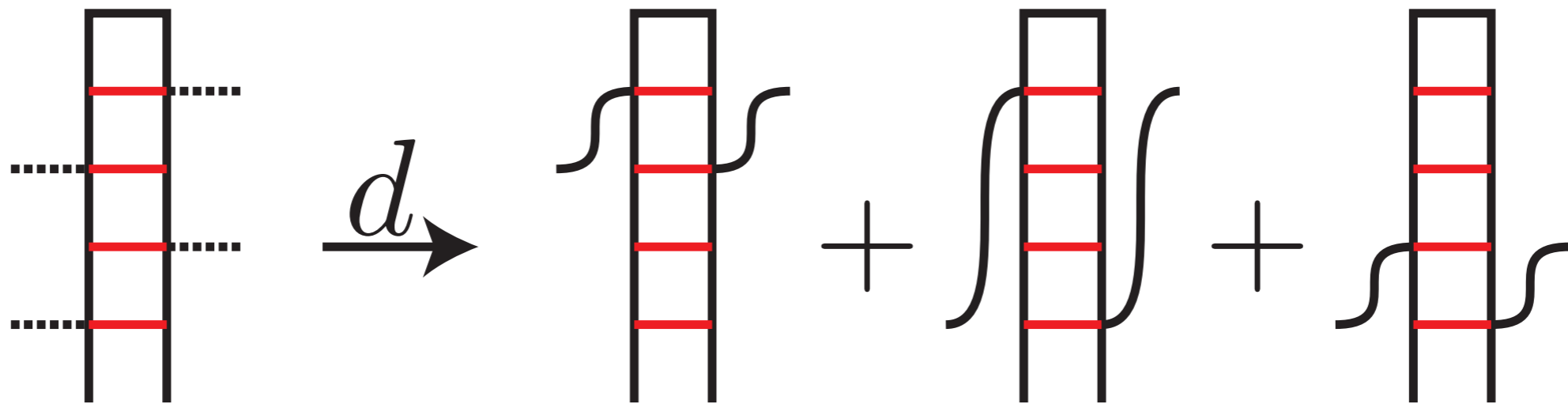
$$C_{D\{AA\}}(K_{12}) = \begin{array}{c} \bullet \\ \mathcal{R} \bullet \\ 0 \end{array} \begin{array}{c} \bullet \\ \cdot \mathcal{D} \cdot \\ 0 \end{array} \begin{array}{c} \bullet \\ \mathcal{D} \bullet \\ 0 \end{array} \begin{array}{c} \bullet \\ \cdot \mathcal{T} \cdot \\ 0 \end{array}$$

The DD Identity Module



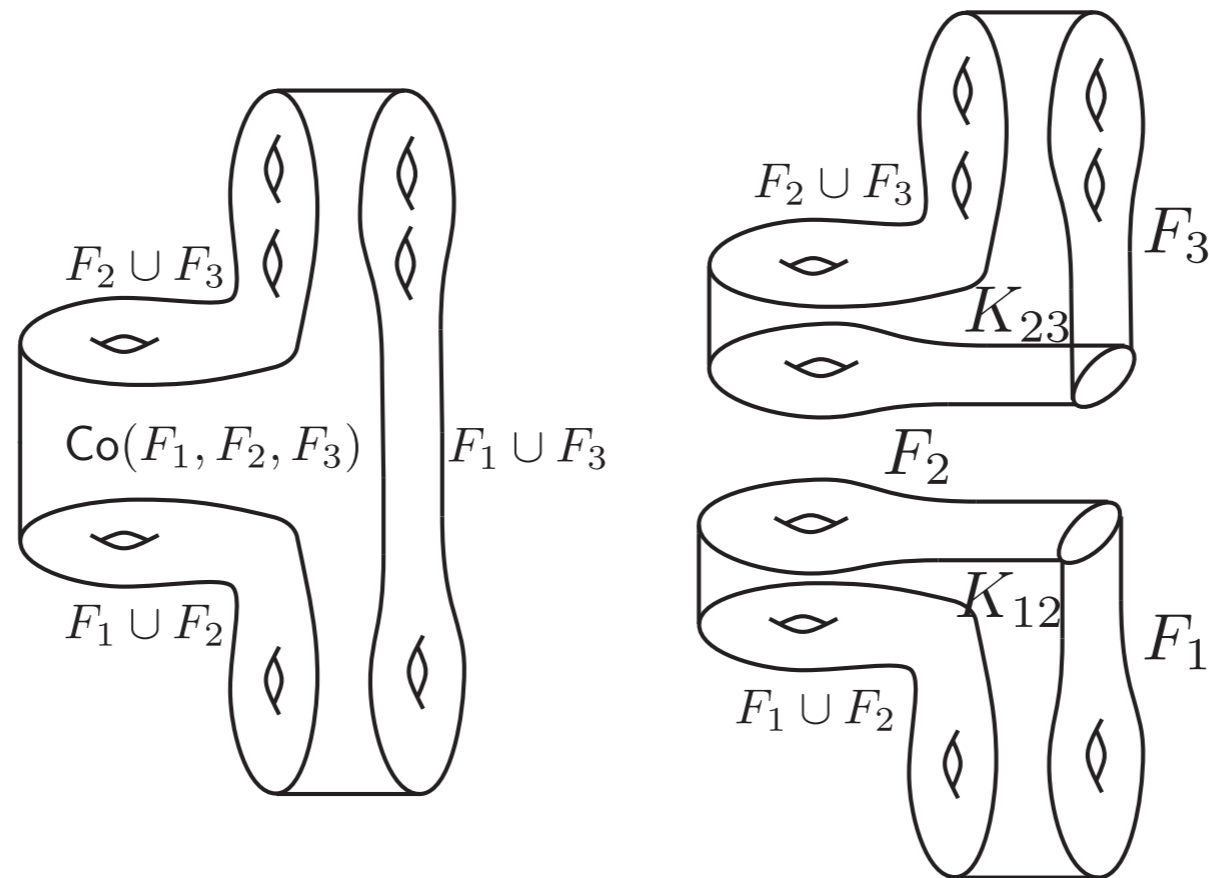
$DD(\mathbb{I}_F)$

The DD Identity Module



$DD(\mathbb{I}_F)$

The other cornering pieces



$$C_{D\{DA\}}(K_{23}) = C_{D\{AA\}}(K_{23}) \otimes_{\mathcal{R}(F_2)} DD(\mathbb{I}_{F_2})$$

$$C_{D\{AD\}}(K_{14}) = C_{D\{AA\}}(K_{14}) \otimes_{\mathcal{T}(F_1)} DD(\mathbb{I}_{F_1})$$

$$C_{D\{DD\}}(K_{34}) = C_{D\{AA\}}(K_{34}) \otimes_{\mathcal{T}(F_3)} DD(\mathbb{I}_{F_3}) \otimes_{\mathcal{R}(F_4)} DD(\mathbb{I}_{F_4})$$

The 2-modules

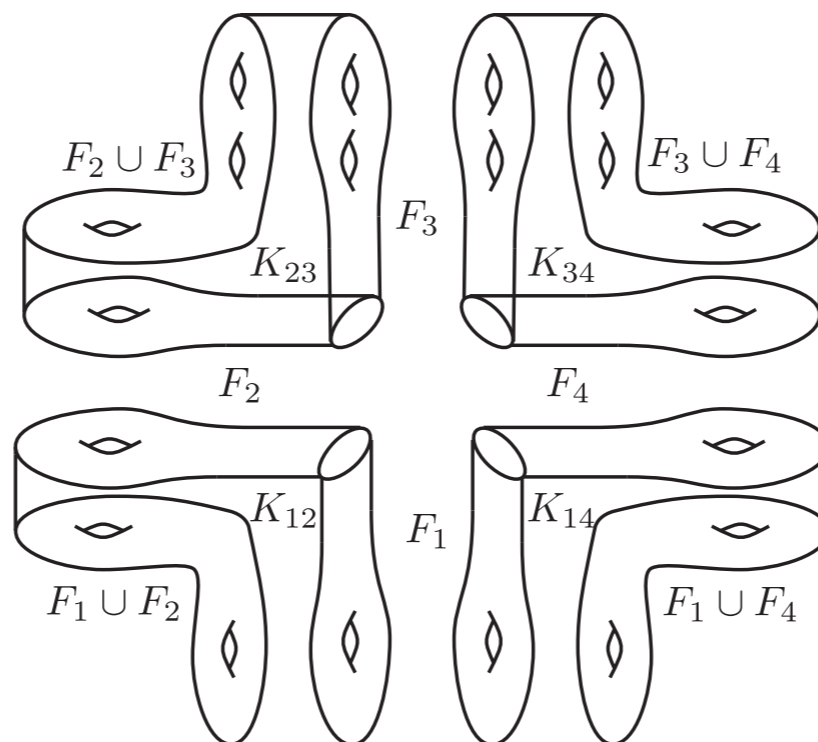
Define:

$$CF\{AA\}(Y) = \widehat{CFA}(Y^\circ) \otimes_{\mathcal{A}(F_1 \cup F_2)} C_D\{AA\}(K_{12})$$

$$CF\{DA\}(Y) = \widehat{CFA}(Y^\circ) \otimes_{\mathcal{A}(F_2 \cup F_3)} C_D\{DA\}(K_{23})$$

$$CF\{AD\}(Y) = \widehat{CFA}(Y^\circ) \otimes_{\mathcal{A}(F_1 \cup F_4)} C_D\{AD\}(K_{14})$$

$$CF\{DD\}(Y) = \widehat{CFA}(Y^\circ) \otimes_{\mathcal{A}(F_3 \cup F_4)} C_D\{DD\}(K_{34})$$



Invariance and pairing

Theorem. Up to quasi-isomorphism, $\widehat{CF}\{AA\}(Y)$, $\widehat{CF}\{DA\}(Y)$, $\widehat{CF}\{AD\}(Y)$ and $\widehat{CF}\{DD\}(Y)$ are invariants of Y .

Invariance and pairing

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Theorem.

$$C_{D\{AA\}}(K_{23}) \otimes_{\mathcal{R}(F_2)} C_{D\{DA\}}(K_{12}) \simeq \widehat{CFDDA}(\text{Co}(F_1, F_2, F_3))$$

$$C_{D\{AD\}}(K_{23}) \otimes_{\mathcal{R}(F_2)} C_{D\{DD\}}(K_{12}) \simeq \widehat{CFDDD}(\text{Co}(F_1, F_2, F_3))$$

Invariance and pairing

Theorem. Up to quasi-isomorphism, $\widehat{CF}\{AA\}(Y)$, $\widehat{CF}\{DA\}(Y)$, $\widehat{CF}\{AD\}(Y)$ and $\widehat{CF}\{DD\}(Y)$ are invariants of Y .

Theorem.

$$C_{D\{AA\}}(K_{23}) \otimes_{\mathcal{R}(F_2)} C_{D\{DA\}}(K_{12}) \simeq \widehat{CFDDA}(\text{Co}(F_1, F_2, F_3))$$

$$C_{D\{AD\}}(K_{23}) \otimes_{\mathcal{R}(F_2)} C_{D\{DD\}}(K_{12}) \simeq \widehat{CFDDD}(\text{Co}(F_1, F_2, F_3))$$

Corollary. $\widehat{CFA}(Y_1 \cup_F Y_2) \simeq \widehat{CF}\{AA\}(Y_1) \otimes_{\mathcal{R}(F)} \widehat{CF}\{DA\}(Y_2)$

$$\widehat{CFD}(Y_1 \cup_F Y_2) \simeq \widehat{CF}\{AD\}(Y_1) \otimes_{\mathcal{R}(F)} \widehat{CF}\{DD\}(Y_2)$$