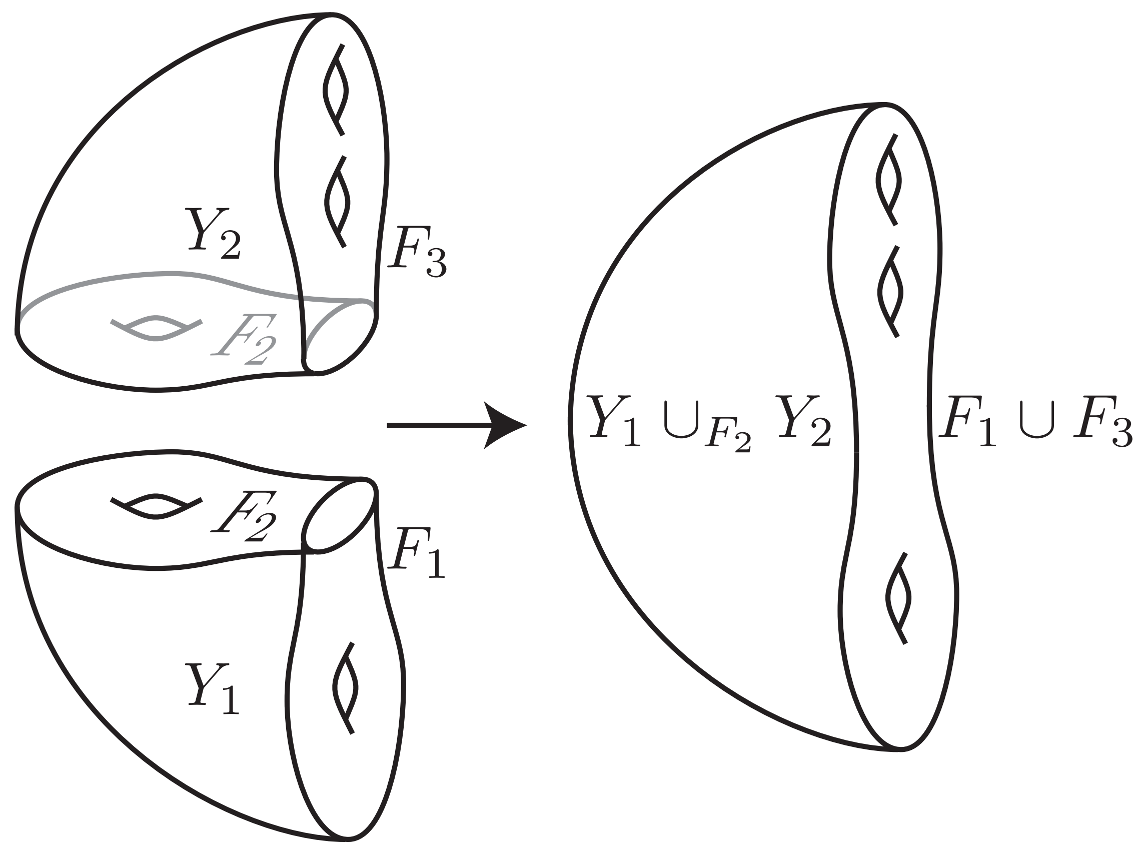
Cornered Floer Homology

Robert Lipshitz
May 20, 2013

Joint with Chris Douglas and Ciprian Manolescu

Gluing 3-manifolds with corners and Heegaard Floer homology



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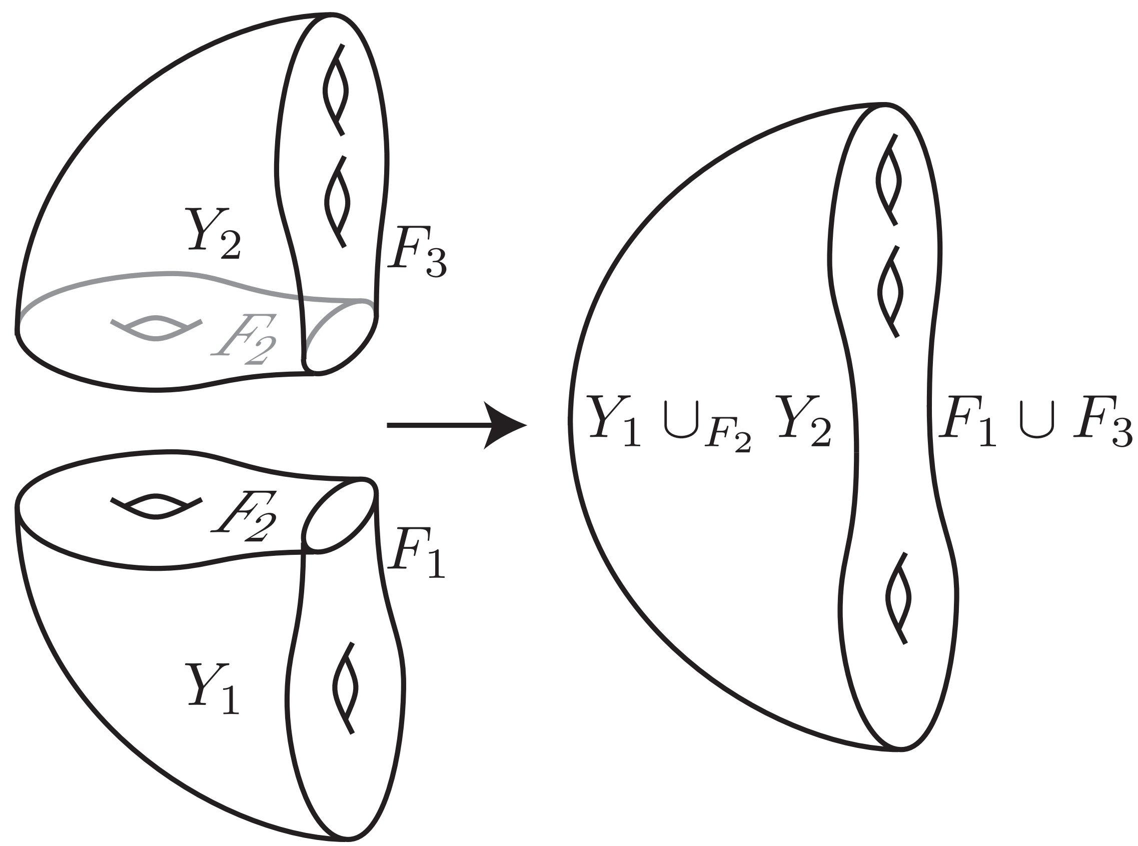
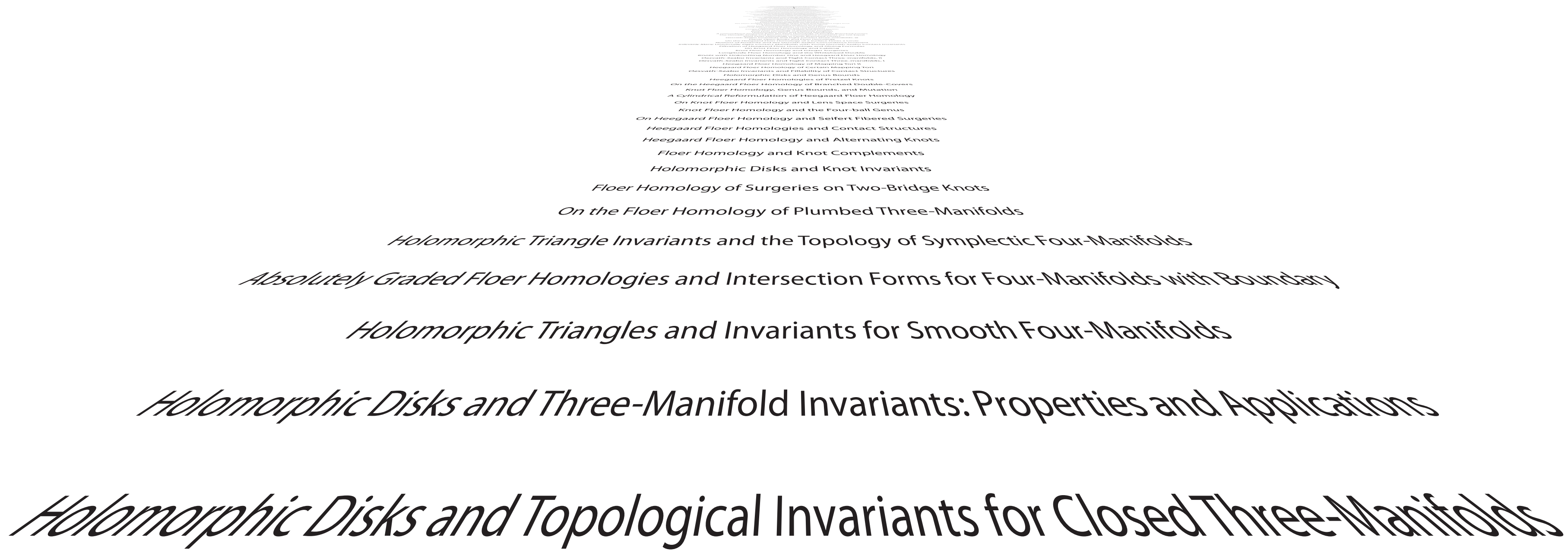


1. Background: bordered Floer and Heegaard Floer.

2. Cornered gluing from bordered Floer, via trimodules.

3. Cornered Floer homology.



Heegaard Floer and Bordered Floer

Heegaard Floer Homology

(Ozsváth-Szabó)

- Y^3 based 3-manifold
  $\longrightarrow \widehat{HF}(Y)$ abelian group.

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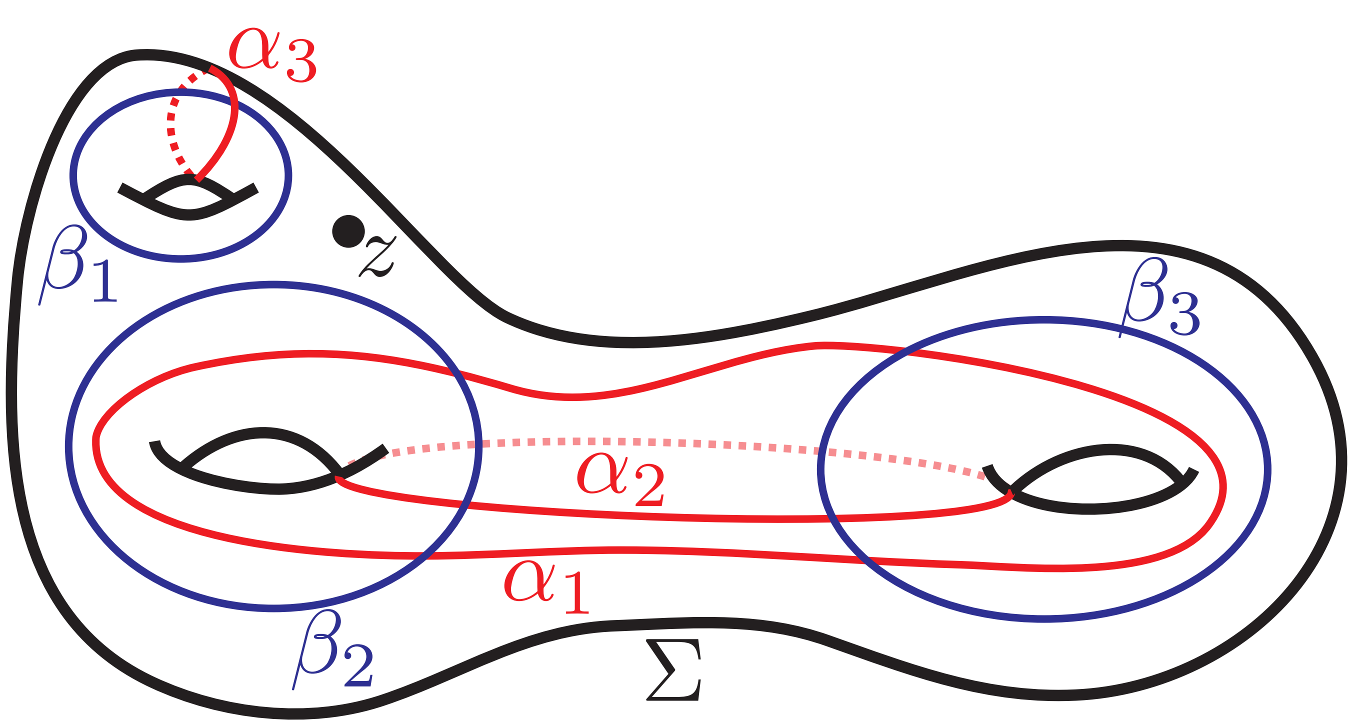

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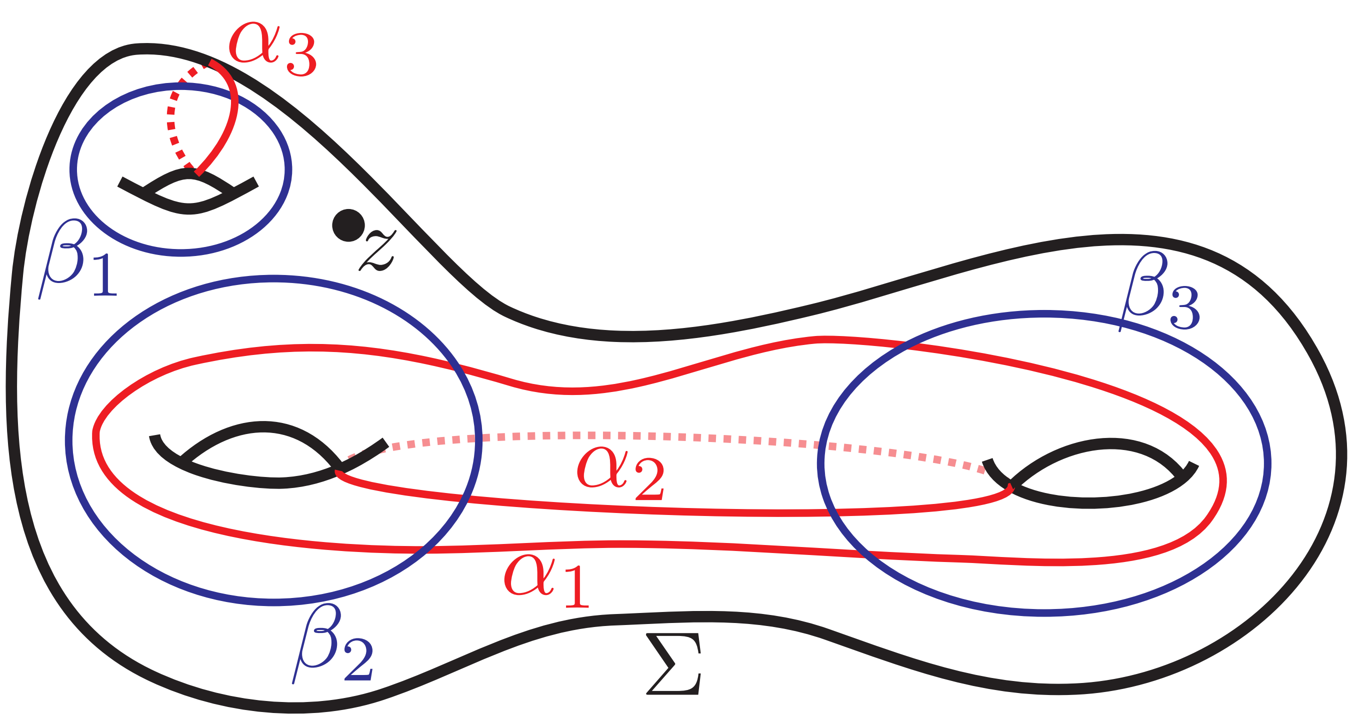
- Also comes in HF^-, HF^+ flavors.



Bordered Floer Homology

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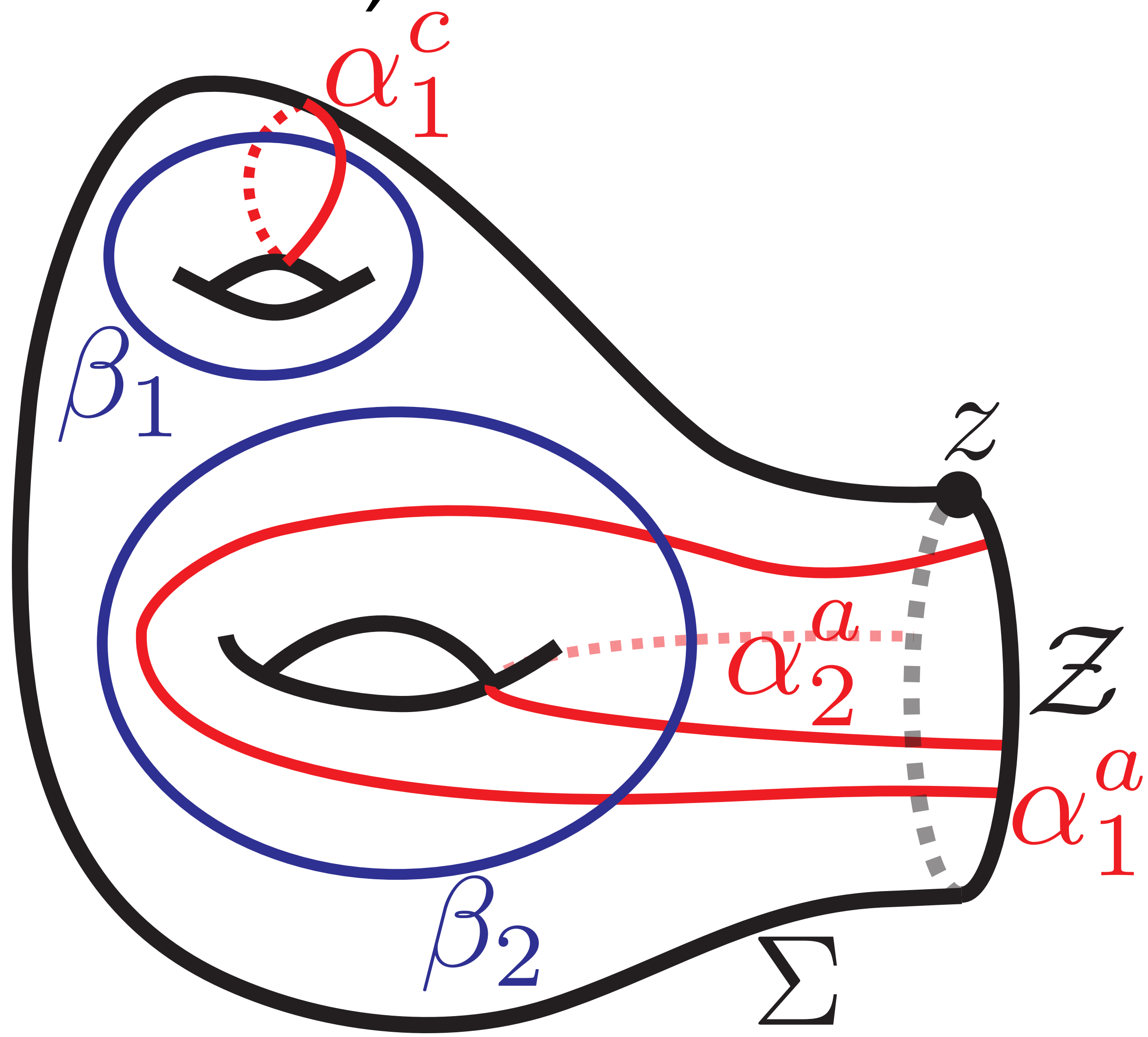

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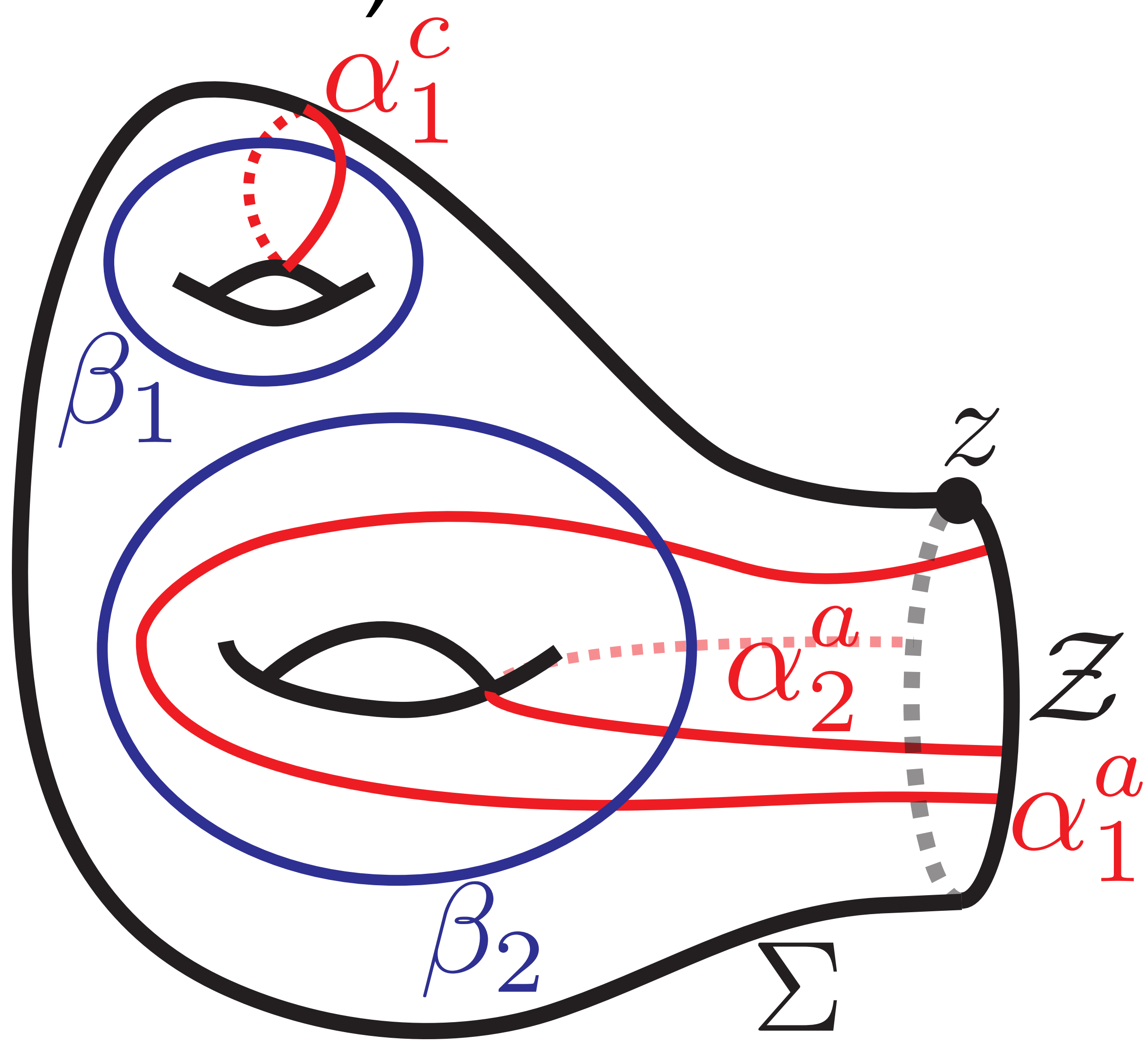
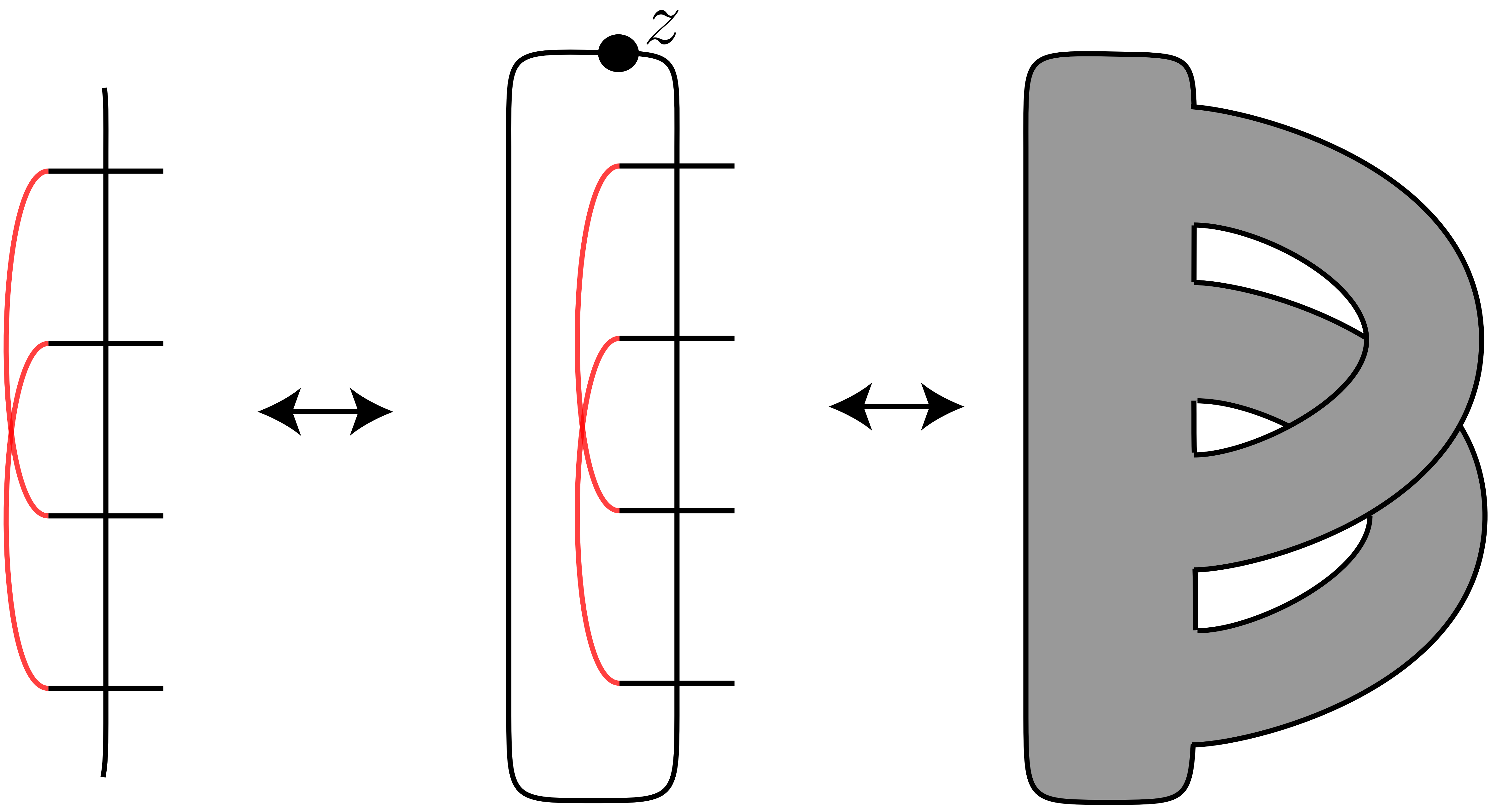
- Defined via bordered Heegaard diagrams
- Cobordisms $\longrightarrow$ Bimodules

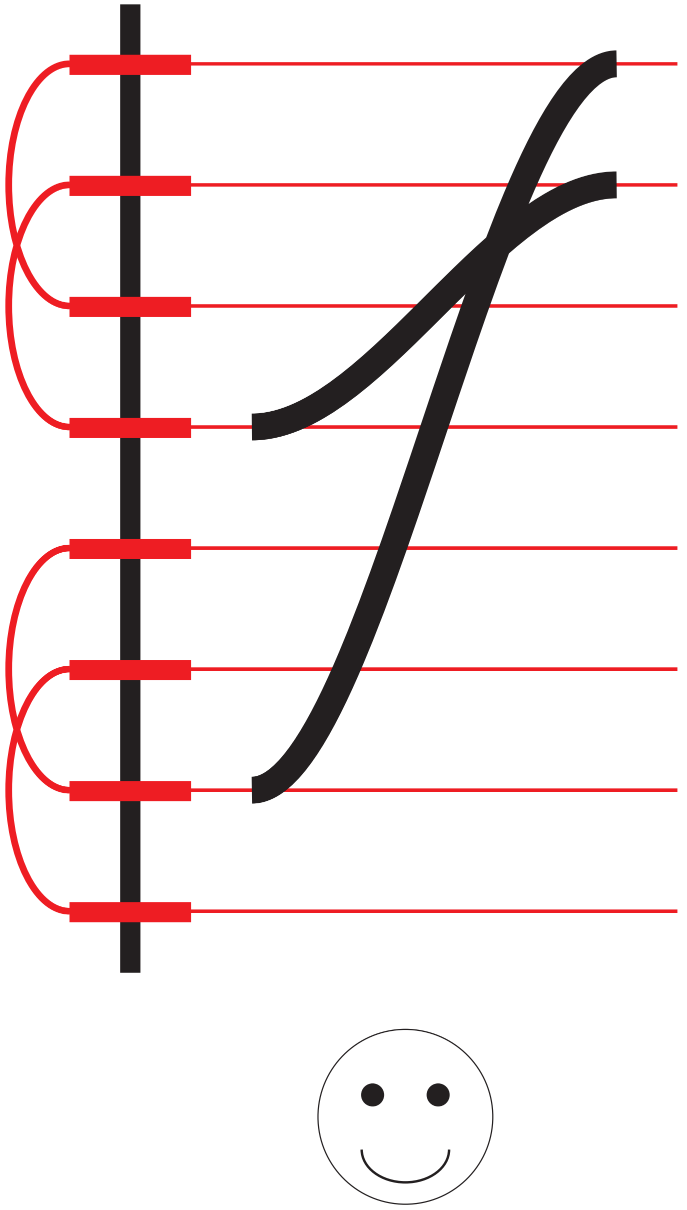


Pointed matched circles



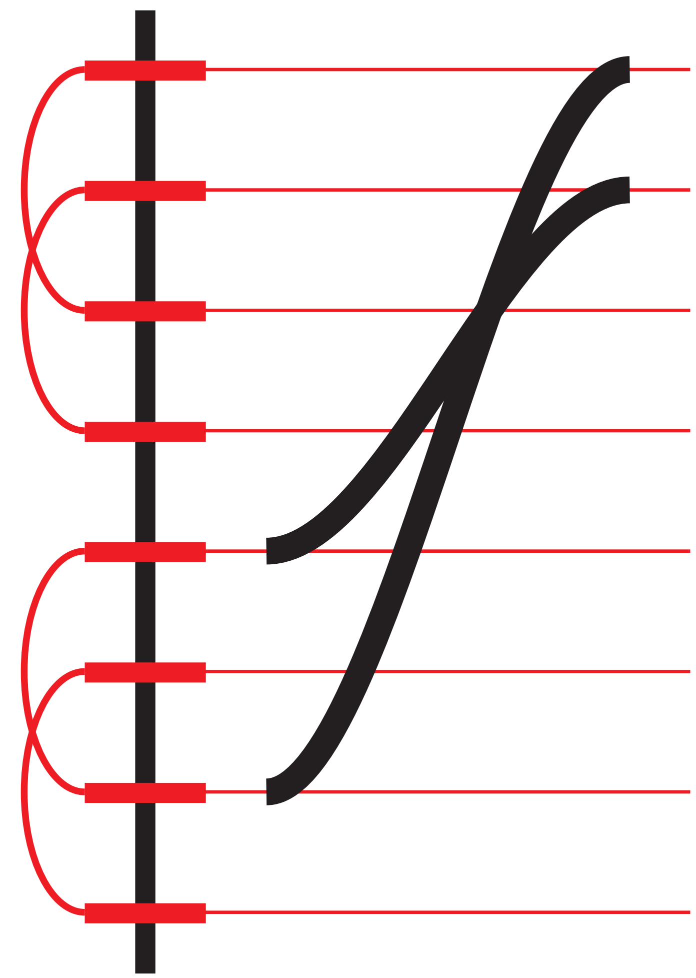
The Bordered Algebras

- Upward-veering strand diagrams.



The Bordered Algebras

- Upward-veering strand diagrams.
- No initial (resp. terminal) endpoints matched.



The Bordered Algebras

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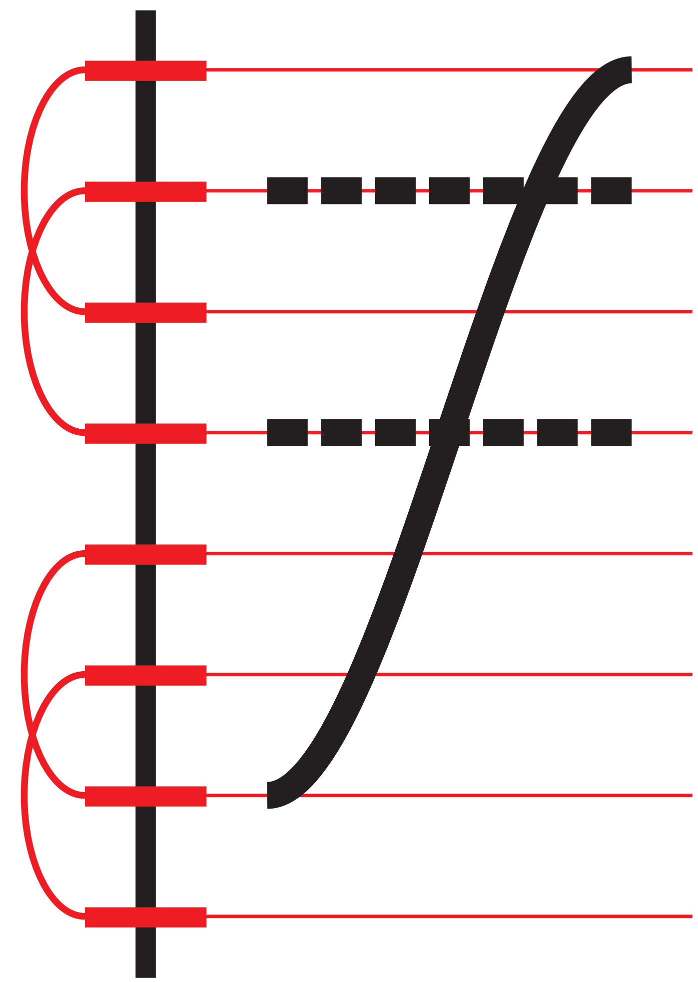
- Smeared horizontal lines



The Bordered Algebras

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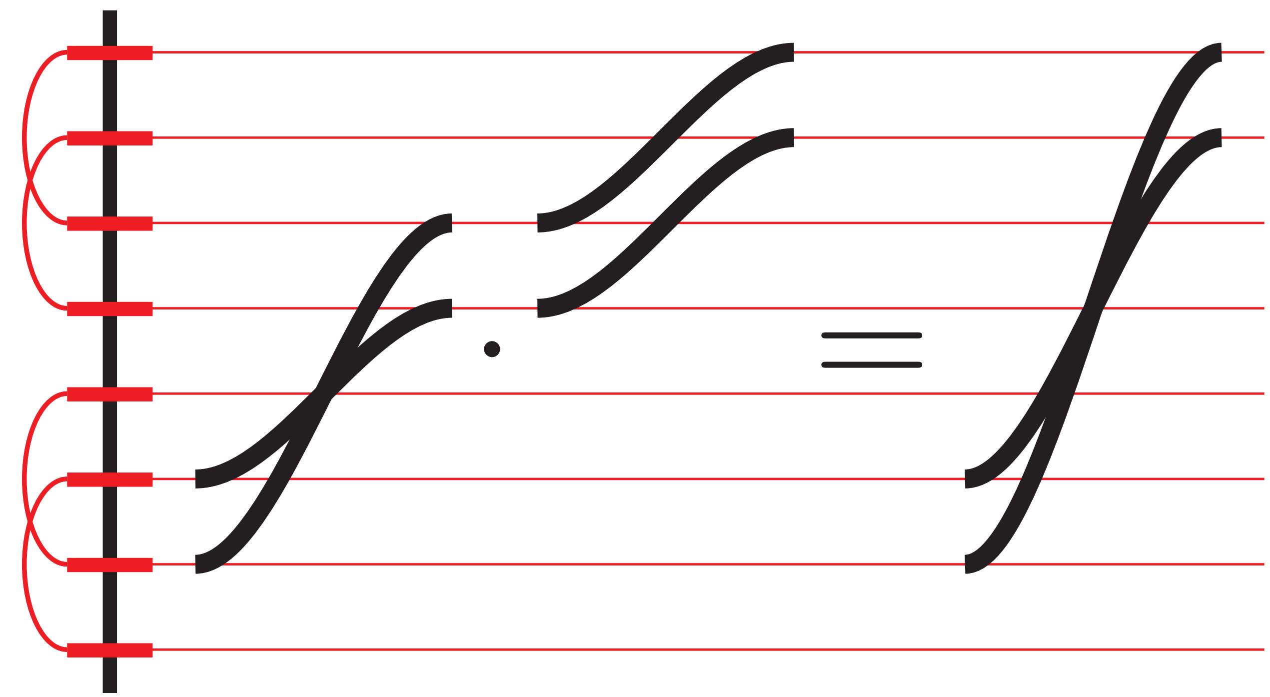
- Multiplication is concatenation.



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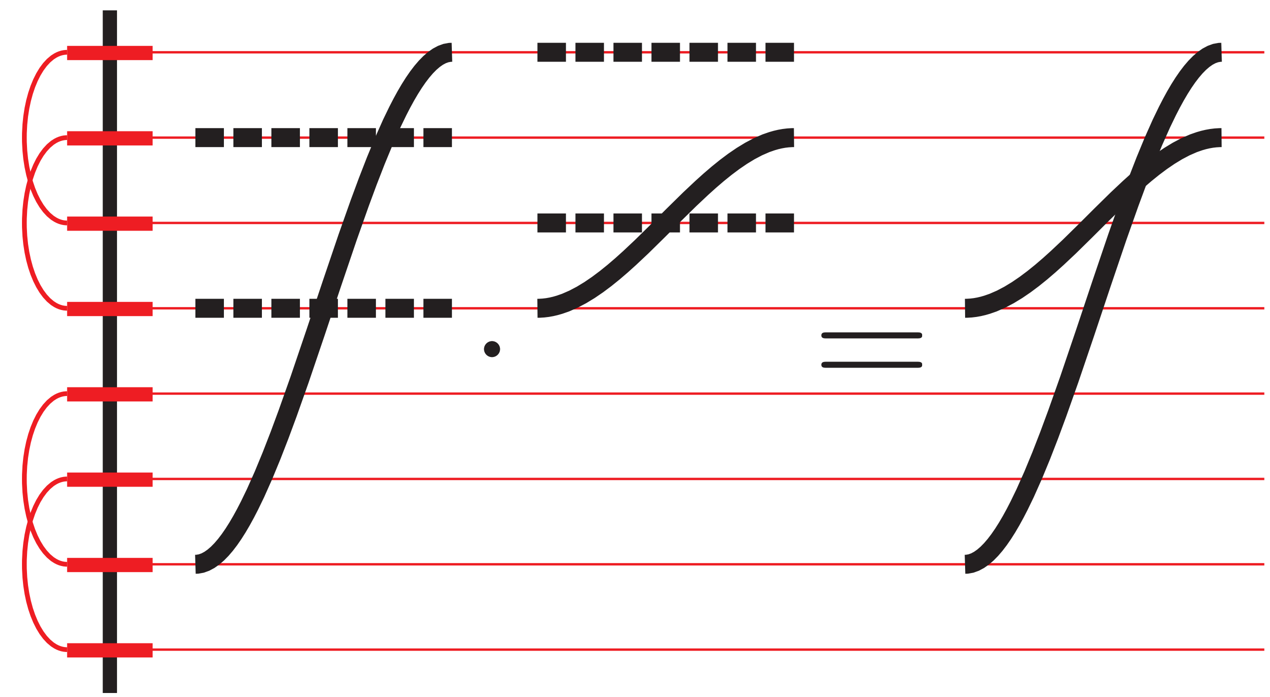

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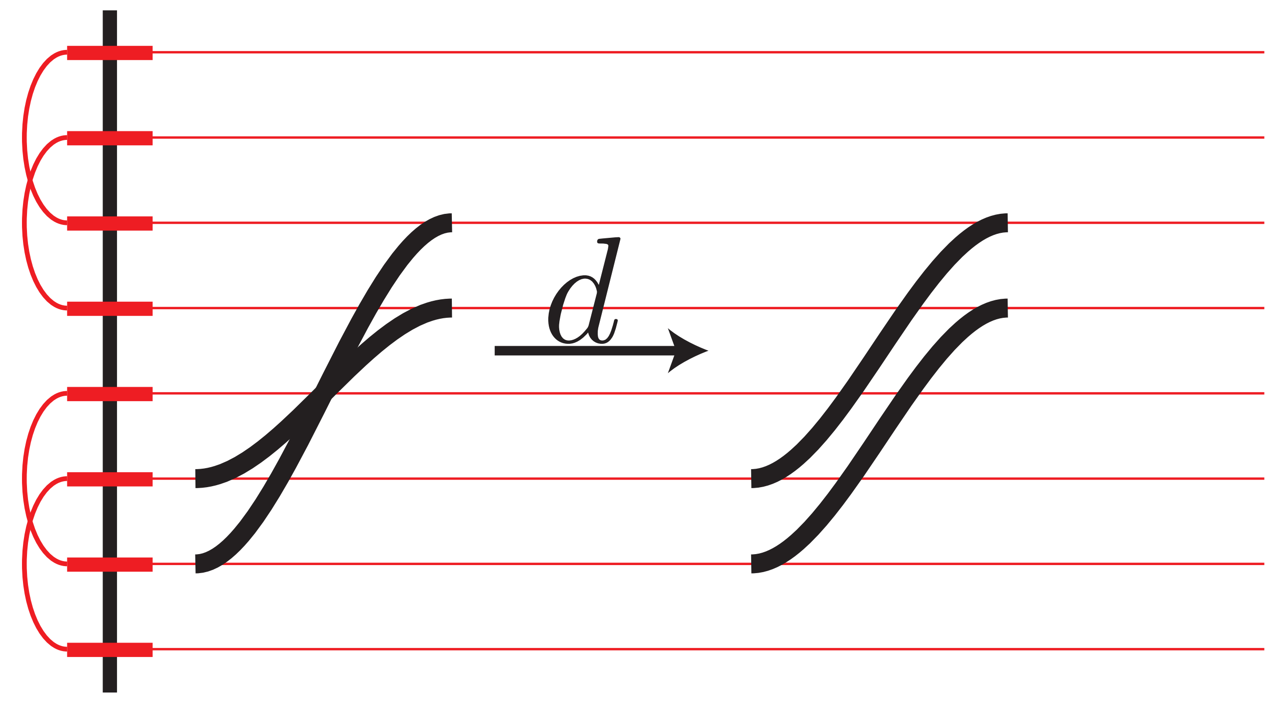

The Bordered Algebras

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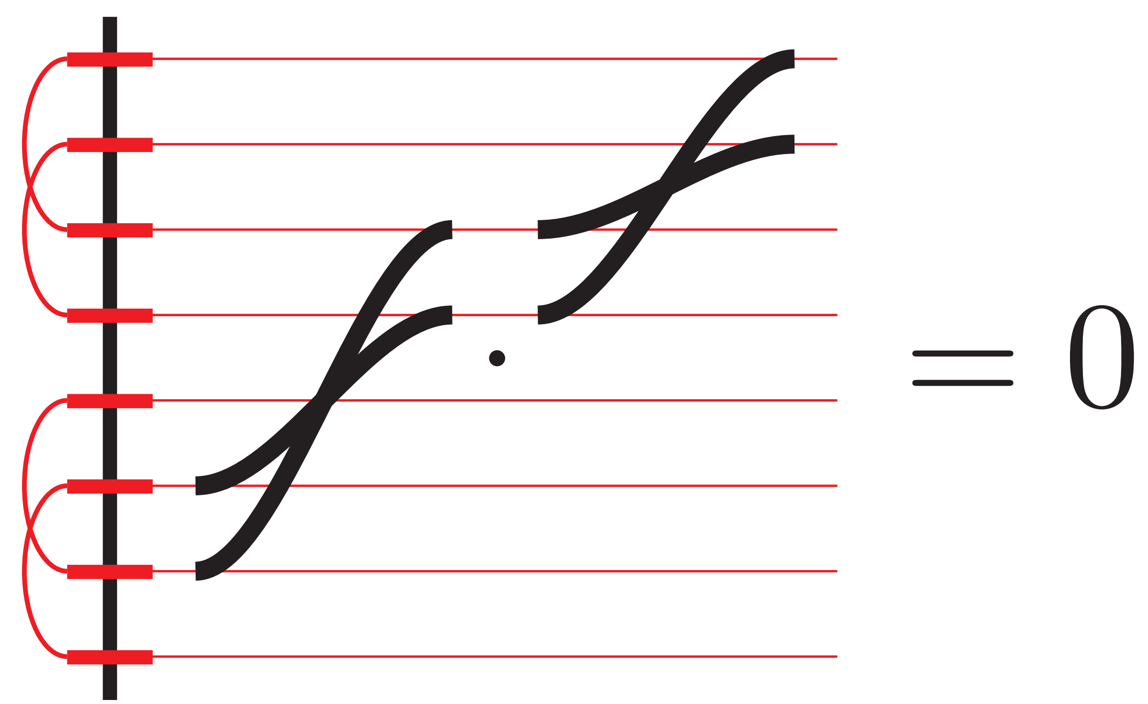
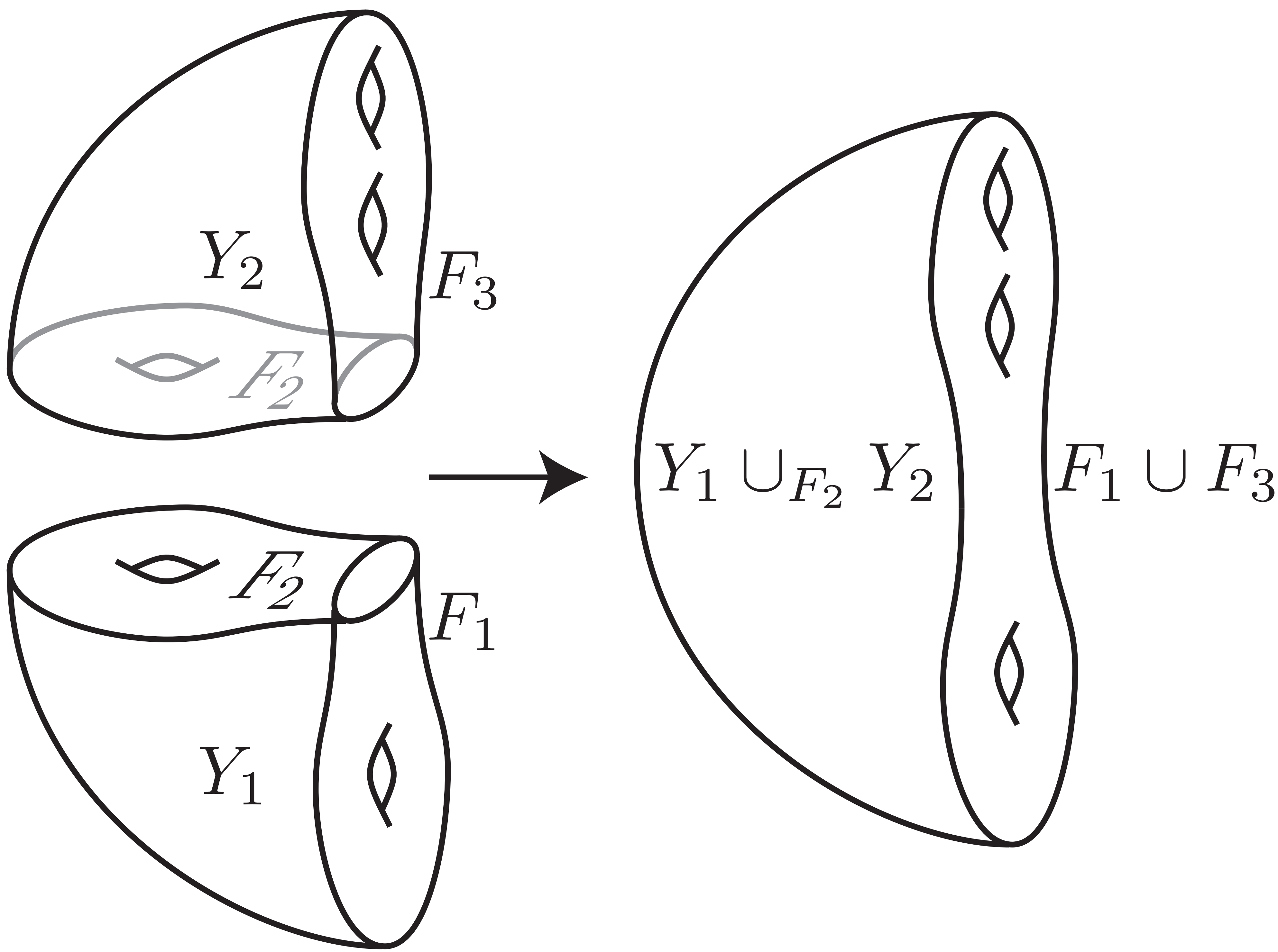
- Multiplication is concatenation.
- Differential smooths crossings.
- Double crossings = 0.

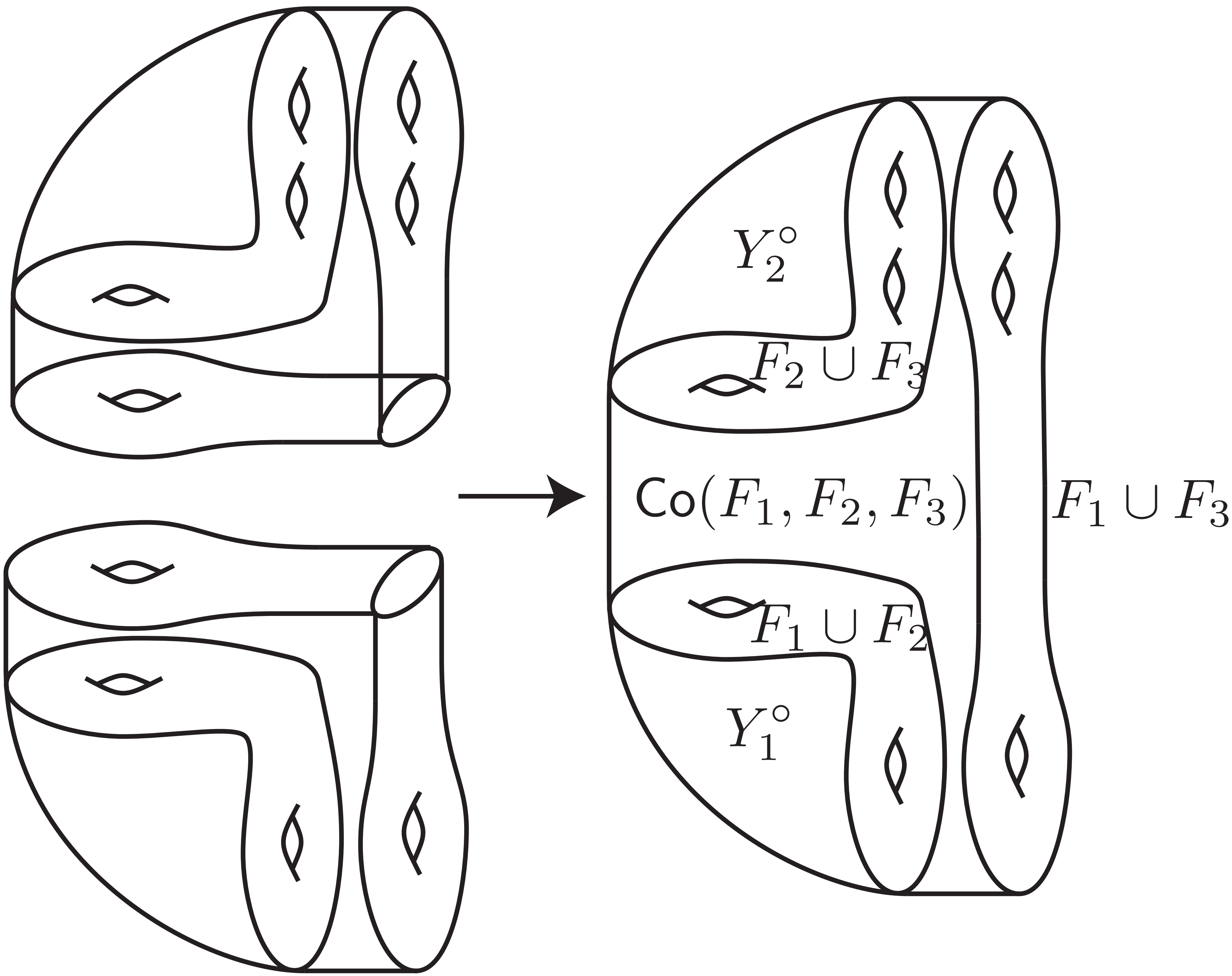


Cornered Gluing in Bordered Floer

Cornered Gluing...



...via Trimodules



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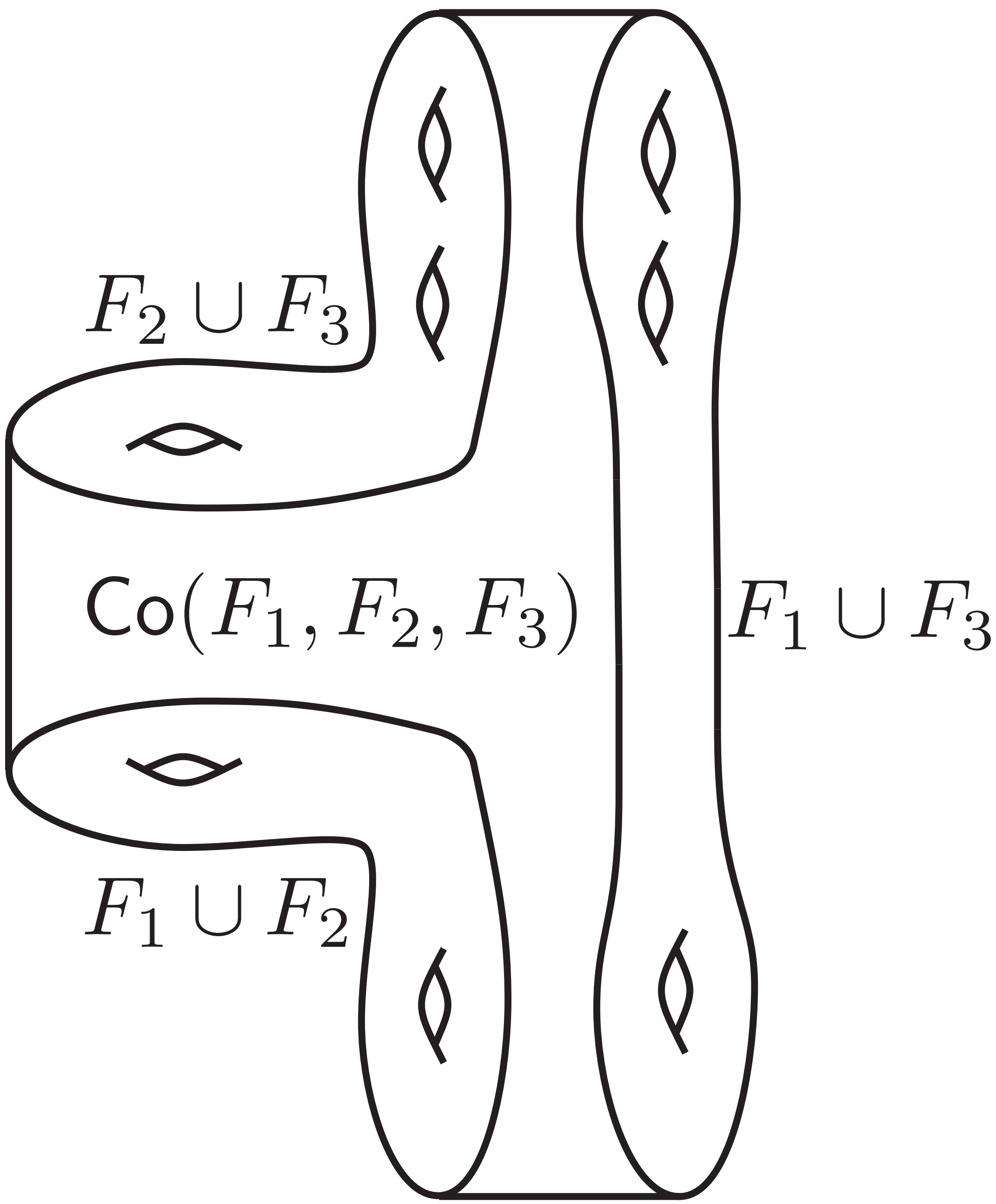

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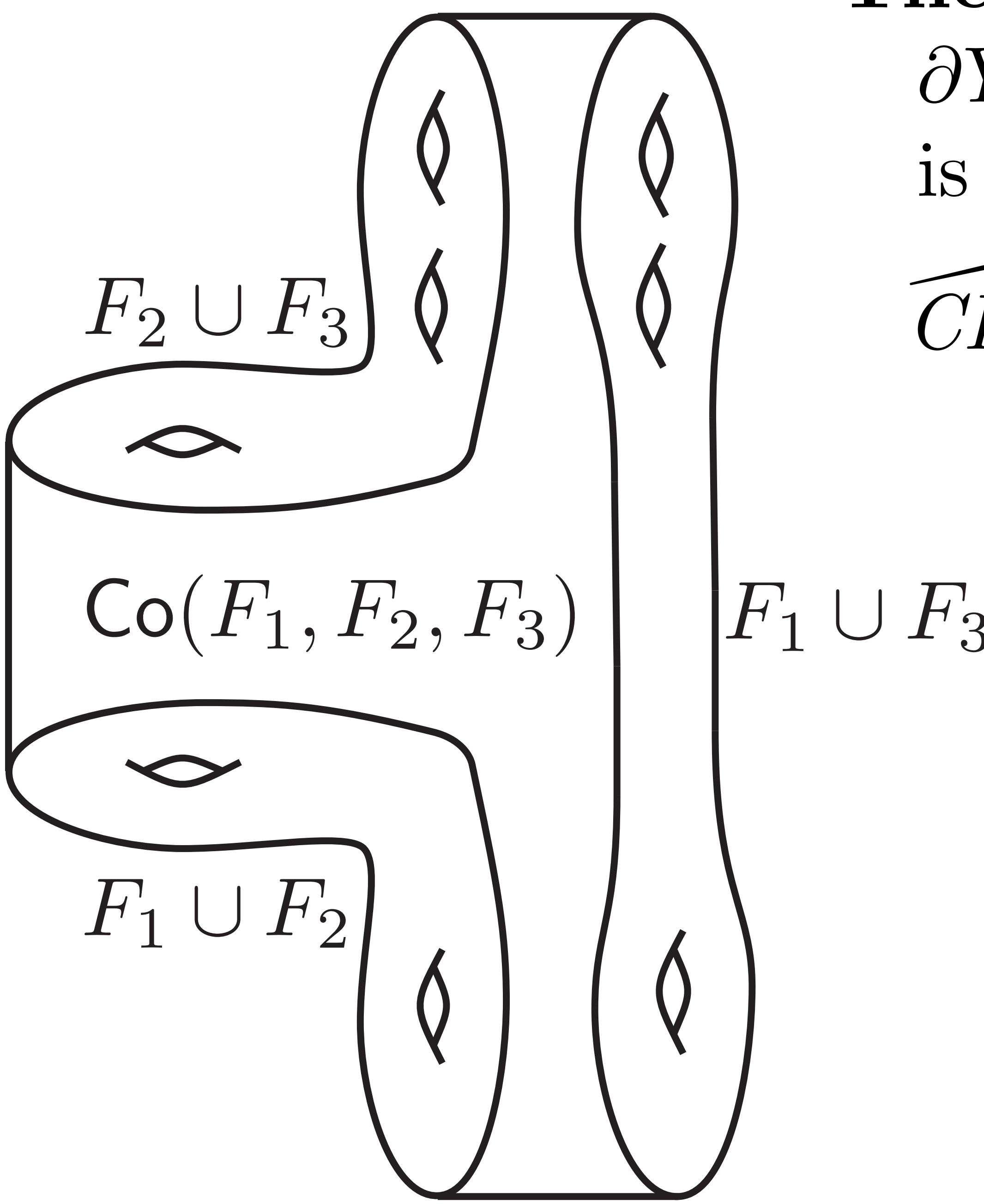

Theorem. If $\partial Y_1 = F_1 \cup F_2$ and $\partial Y_2 = F_2 \cup F_3$ then $\widehat{CFA}(Y_1 \cup_{F_2} Y_2)$ is quasi-isomorphic to

$$\widehat{CFA}(Y_1) \otimes_{\mathcal{A}(F_1 \cup F_2)} \widehat{CFA}(Y_2) \otimes_{\mathcal{A}(F_2 \cup F_3)} \widehat{CFDDA}(\mathsf{Co}(F_1, F_2, F_3))$$

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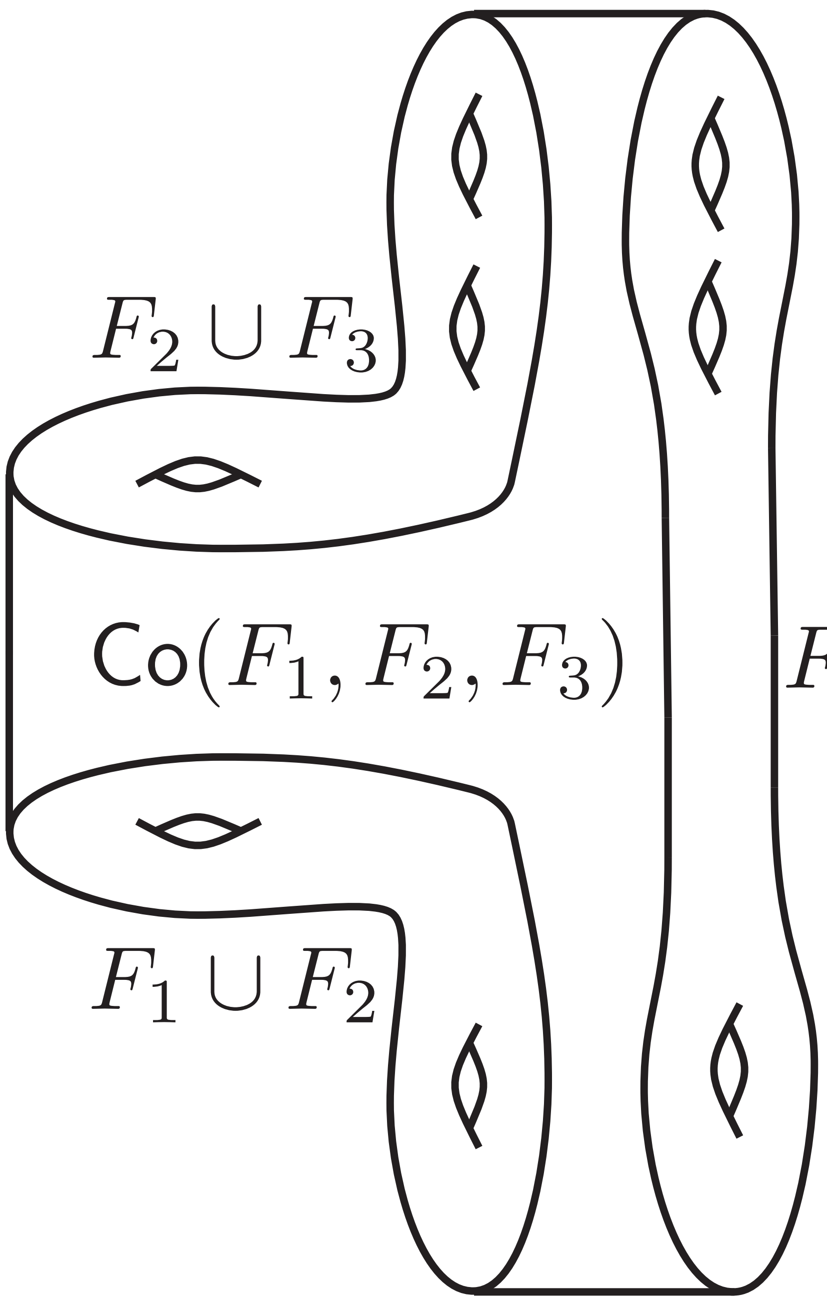

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$\widehat{CFD}(Y_1 \cup_{F_2} Y_2)$ is quasi-isomorphic to

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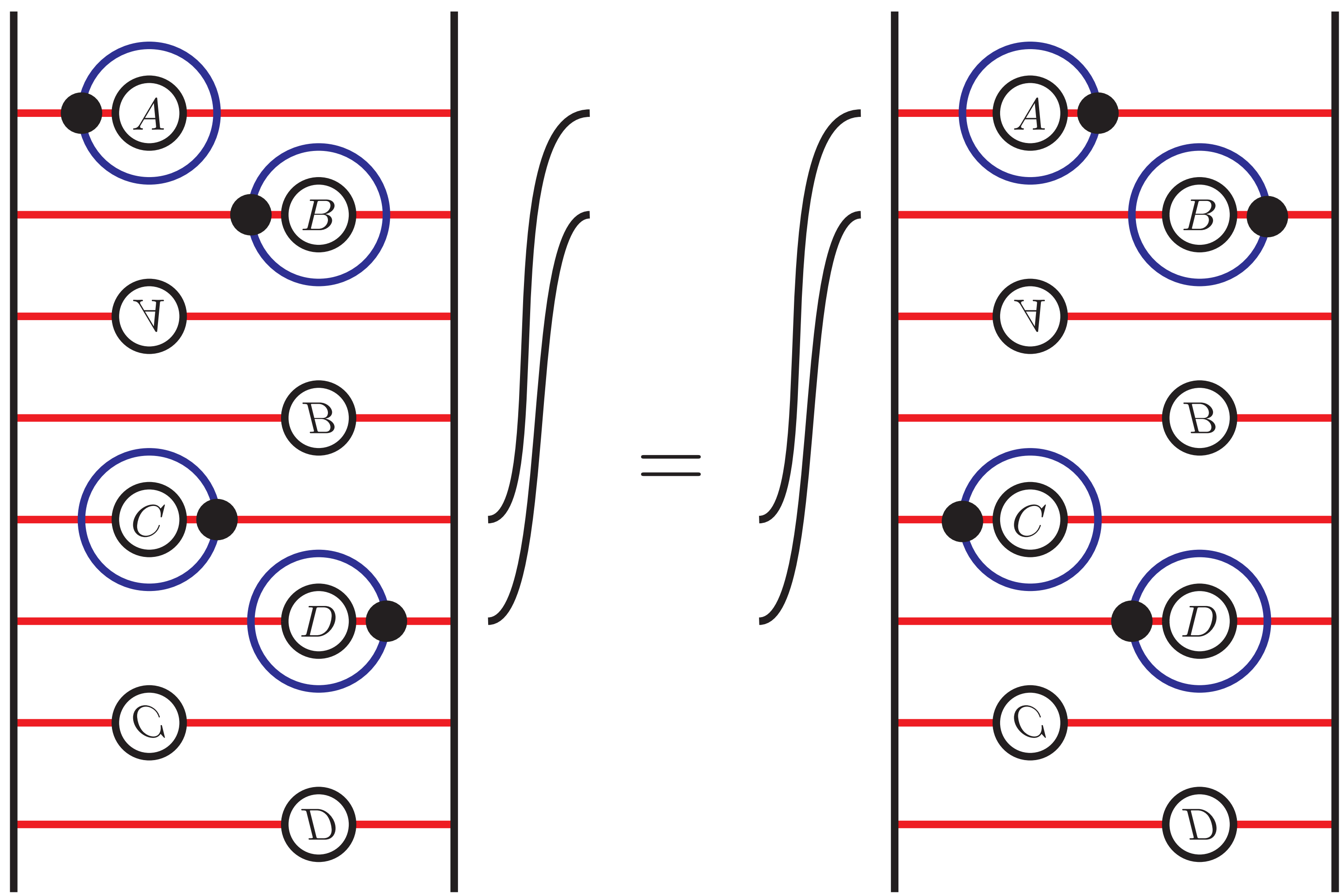
The *DA* Identity Bimodule

$\widehat{CFDA}([0,1] \times F)$

- $\mathcal{A}(F)$ as an $\mathcal{A}(F)$-bimodule

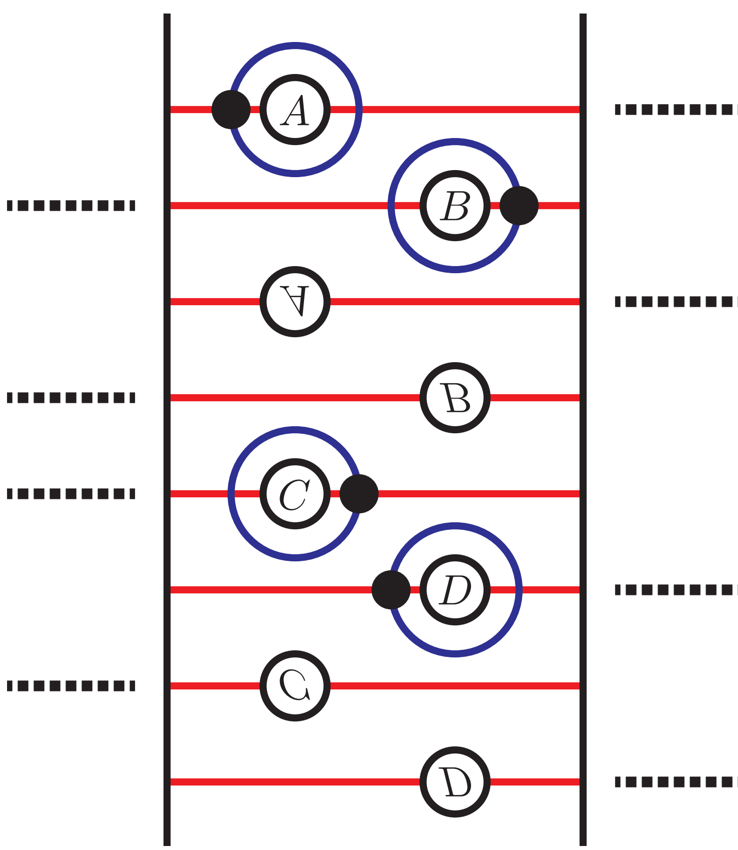


(LOT, "Bimodules in Bordered Floer Homology")

The *DD* Identity Bimodule

$$\widehat{CFDD}([0,1] \times F)$$

- Free (projective) module over $\mathcal{A}(F) \otimes \mathcal{A}(F)$ generated by *complementary idempotents* $I \otimes J$.



(LOT, "Computing HF-hat by factoring mapping classes")

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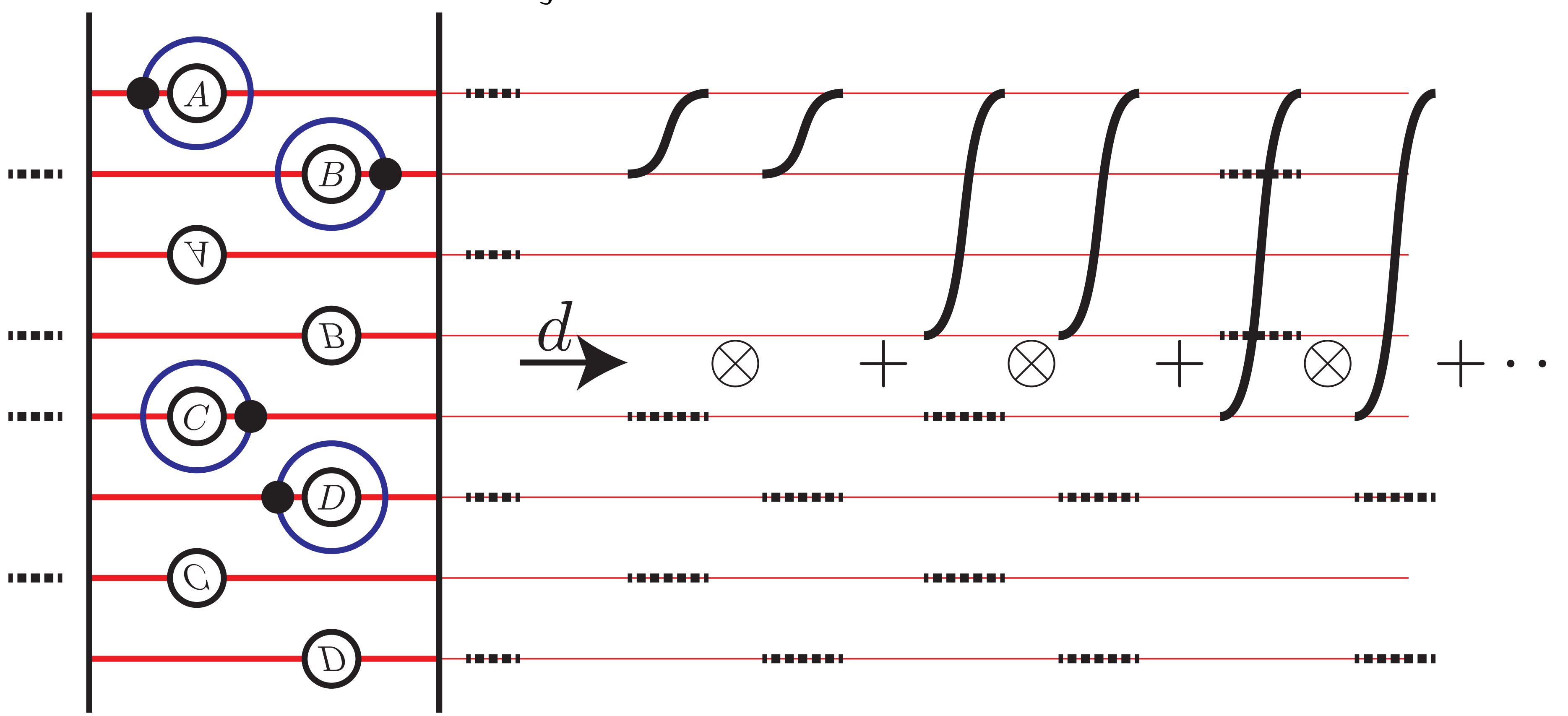
- $d(I \otimes J) = \sum_{\text{chords } \xi} I \cdot a(\xi) \otimes a(\xi) \cdot J$



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The *DDD* Cornering Trimodule

- Like the DD identity module.
- Generated by complementary idempotent triples.
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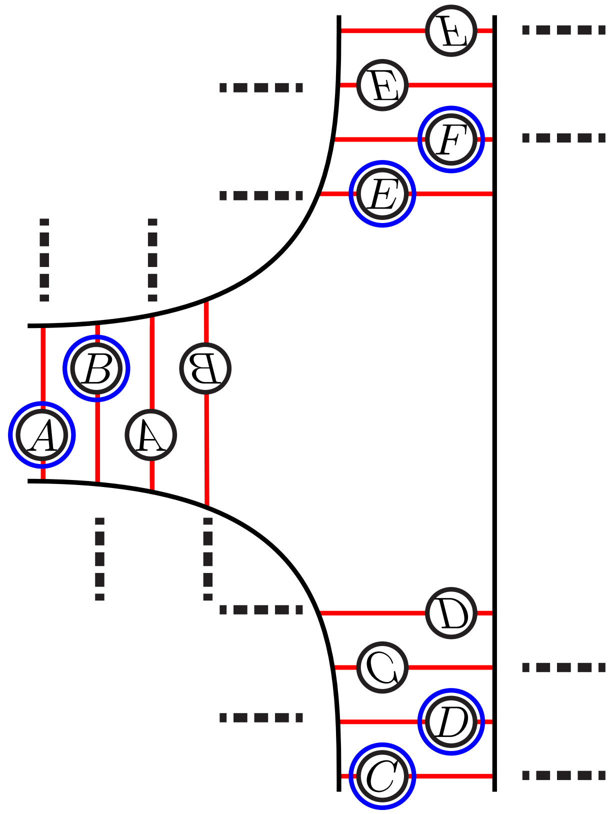

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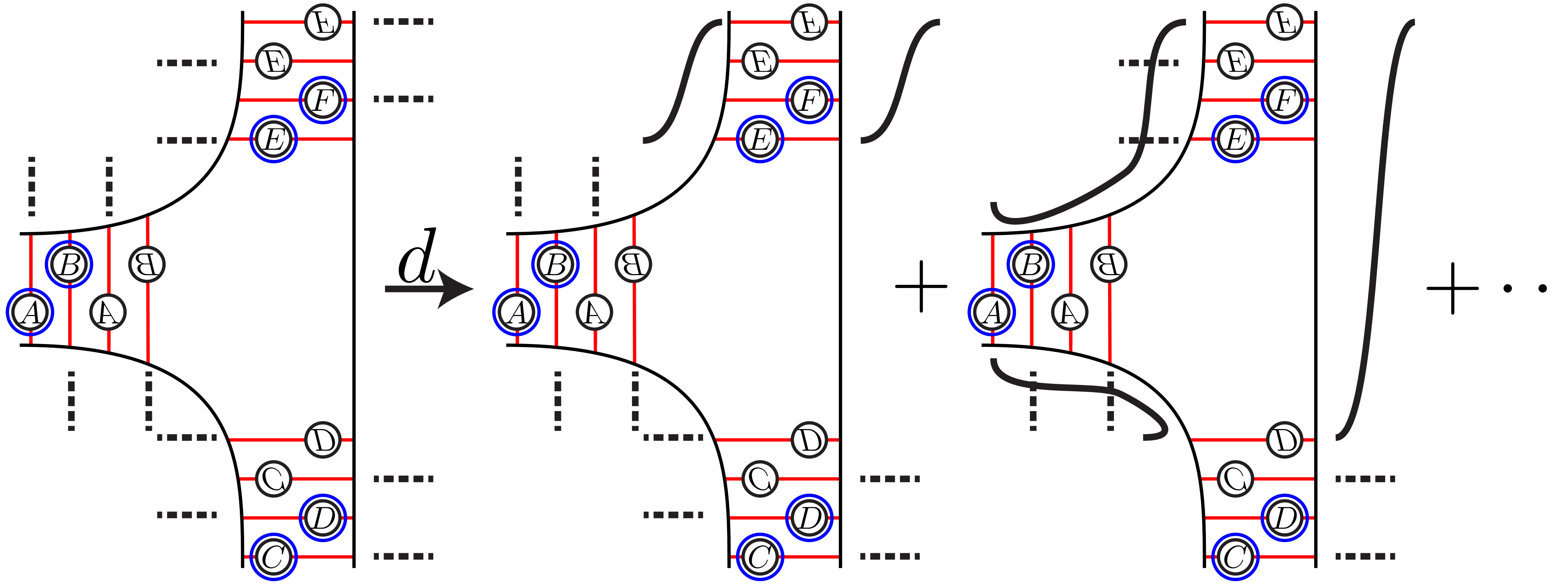
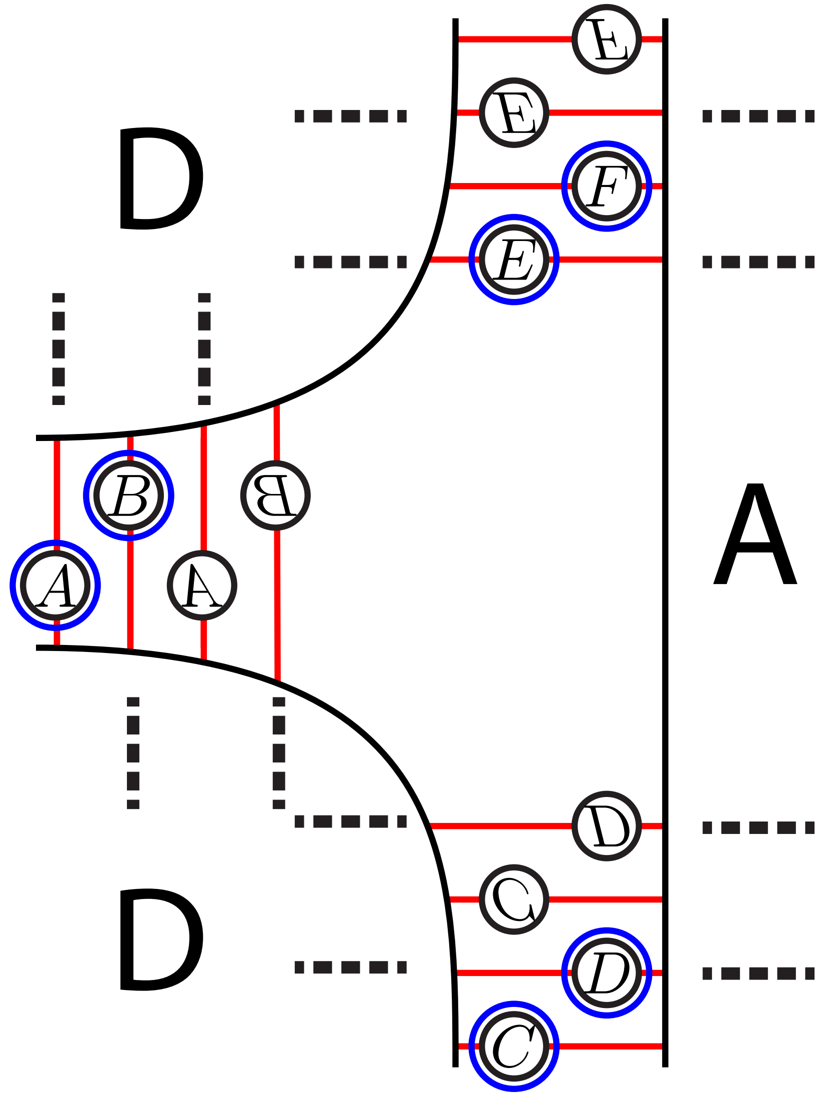


The *DDA* Cornering Trimodule

- A hybrid between the *DD* and *DA* identity modules.



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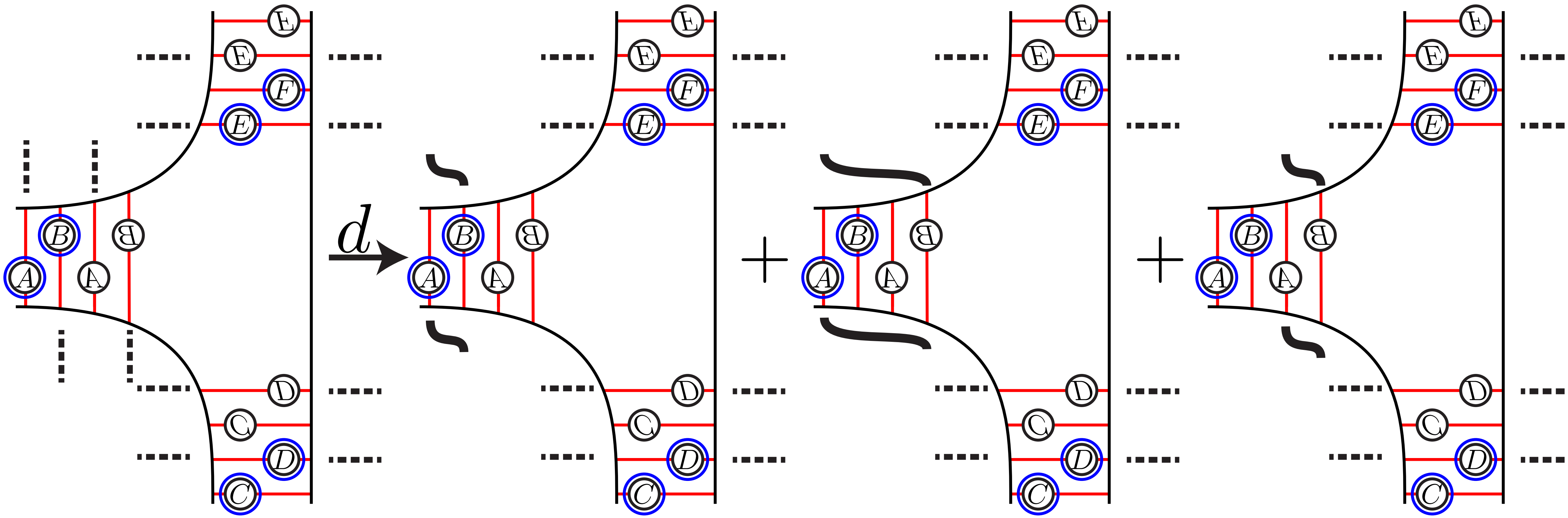

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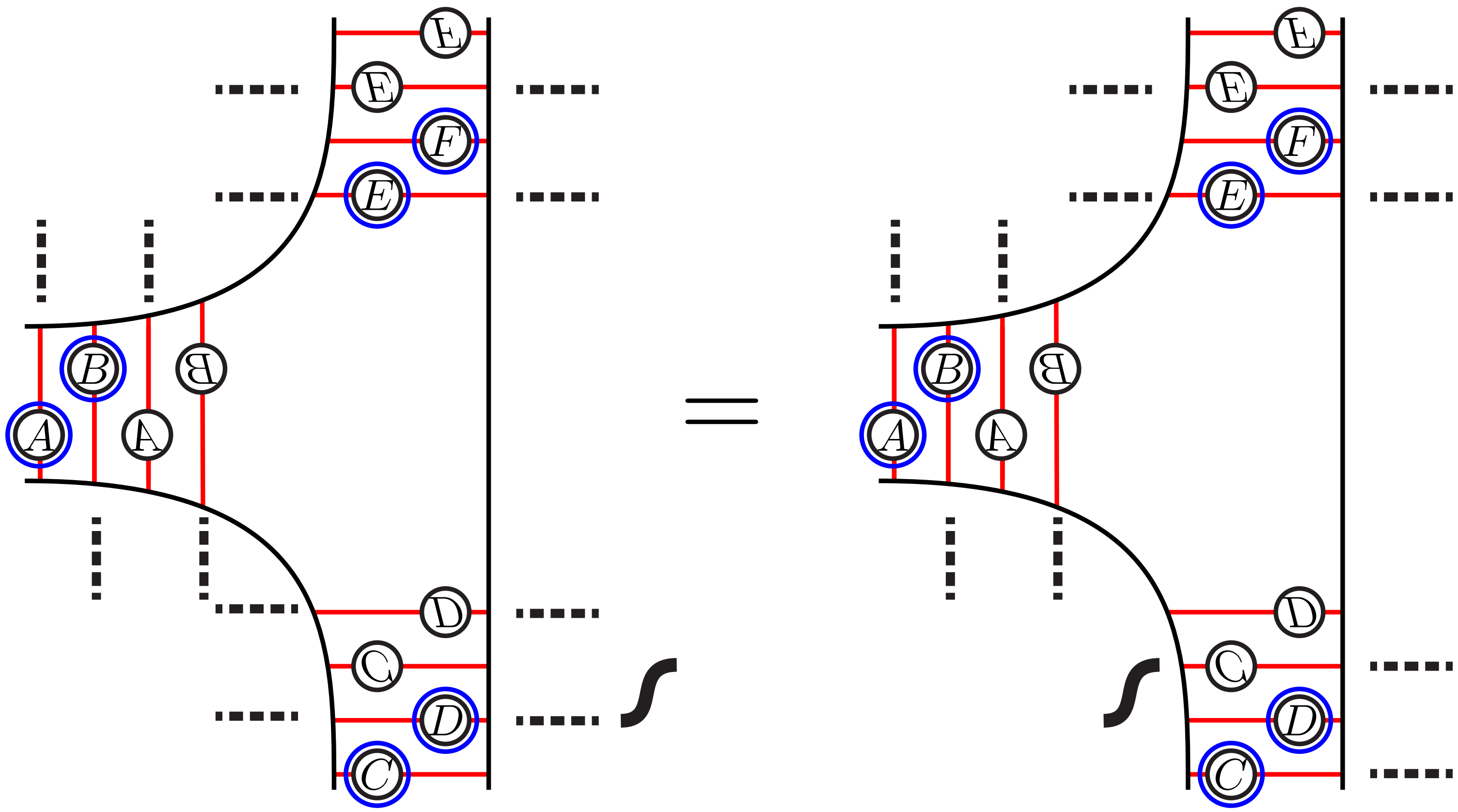

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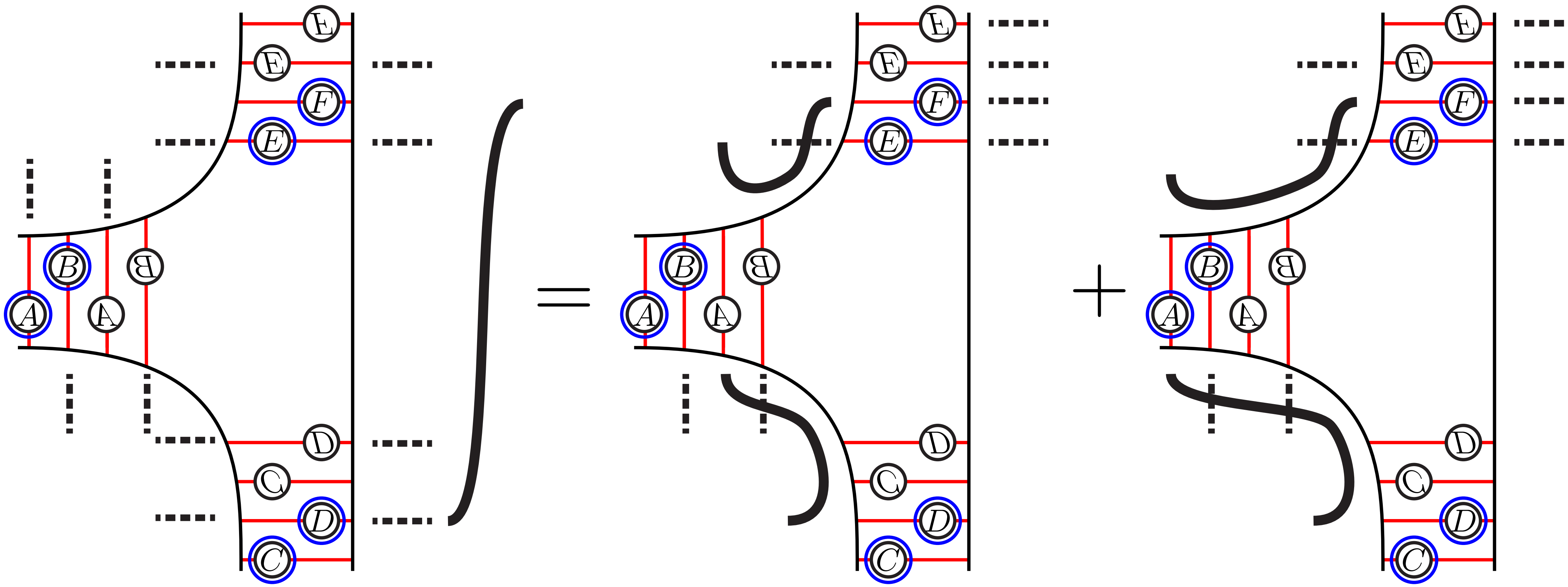

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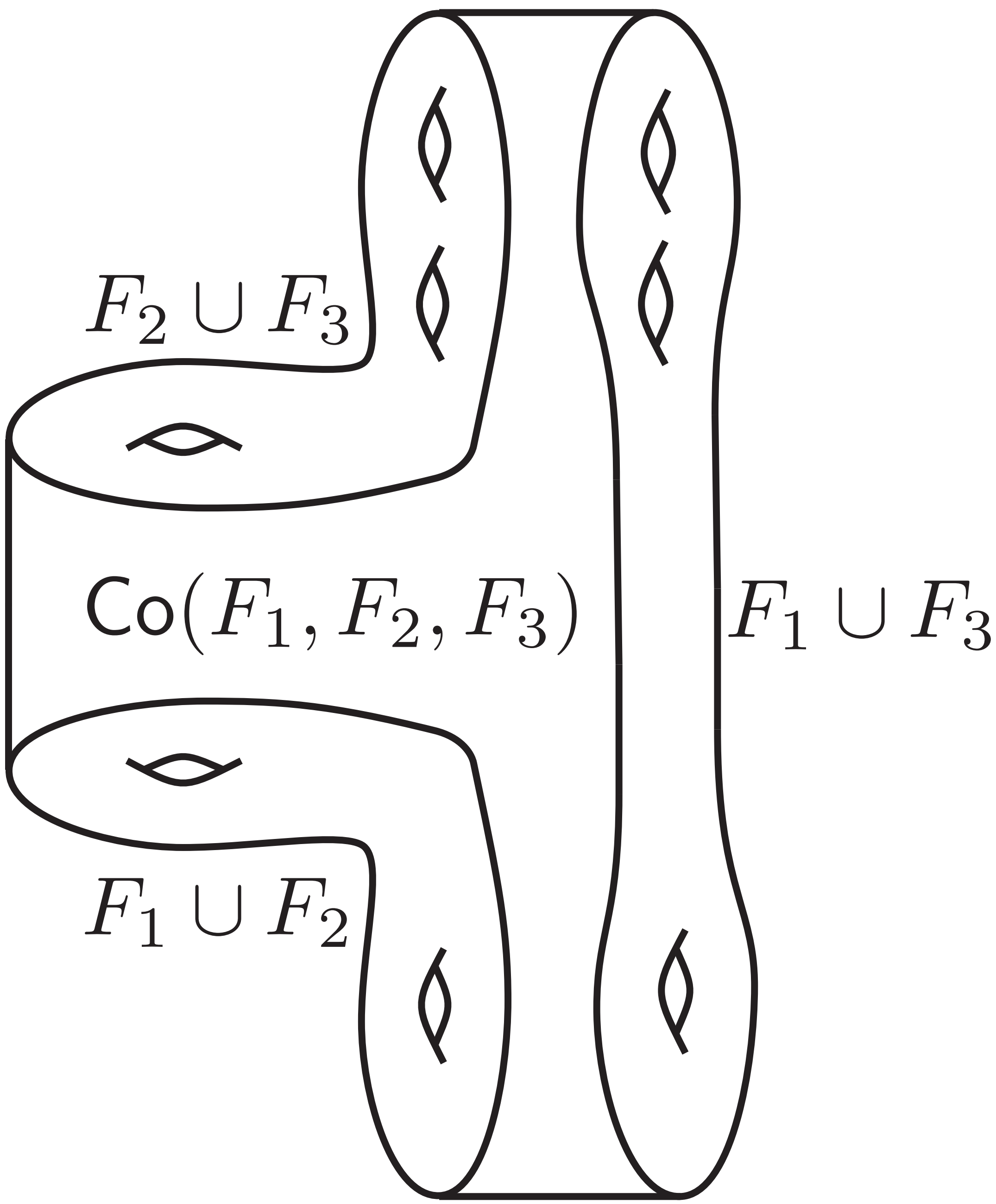

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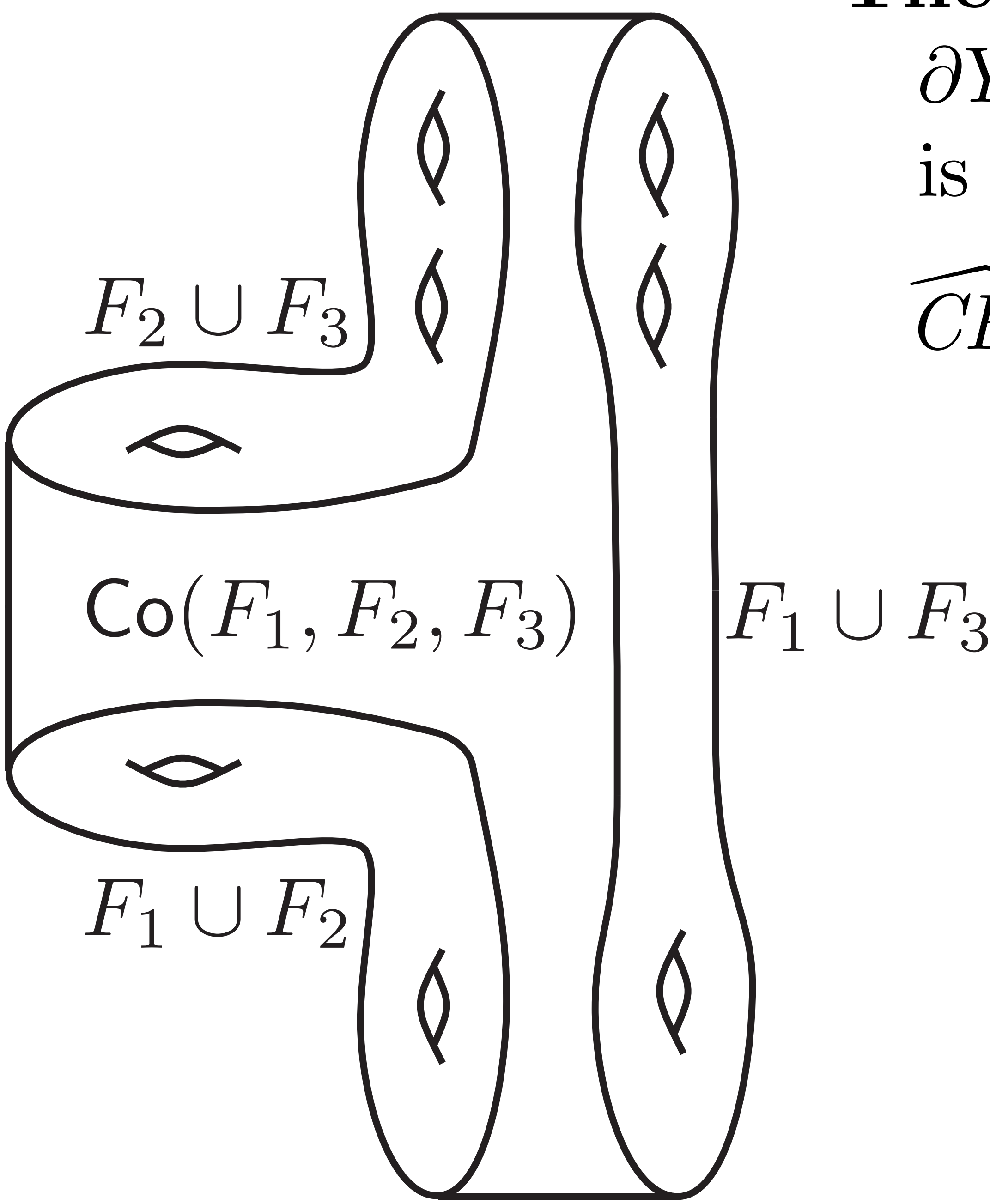

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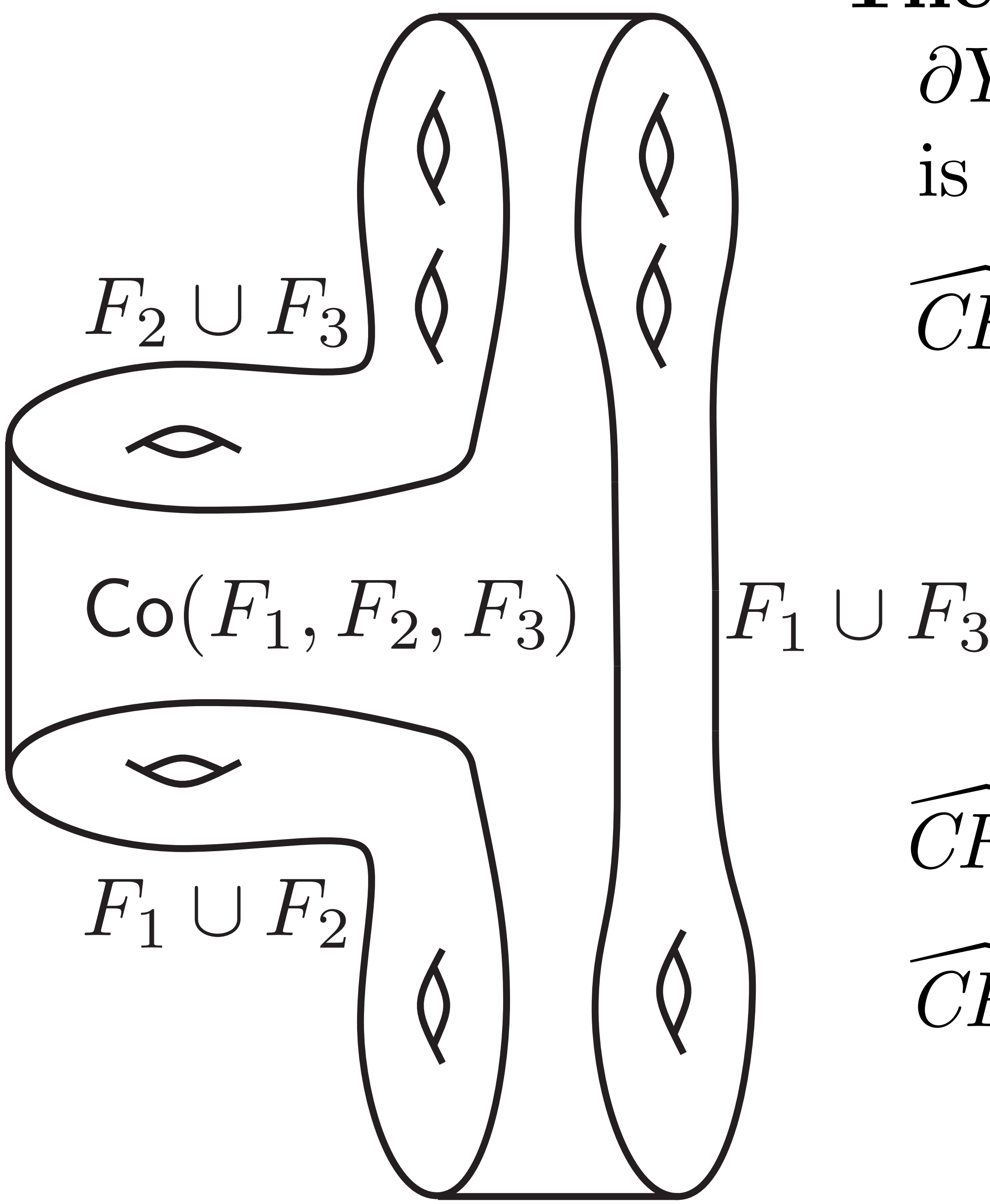

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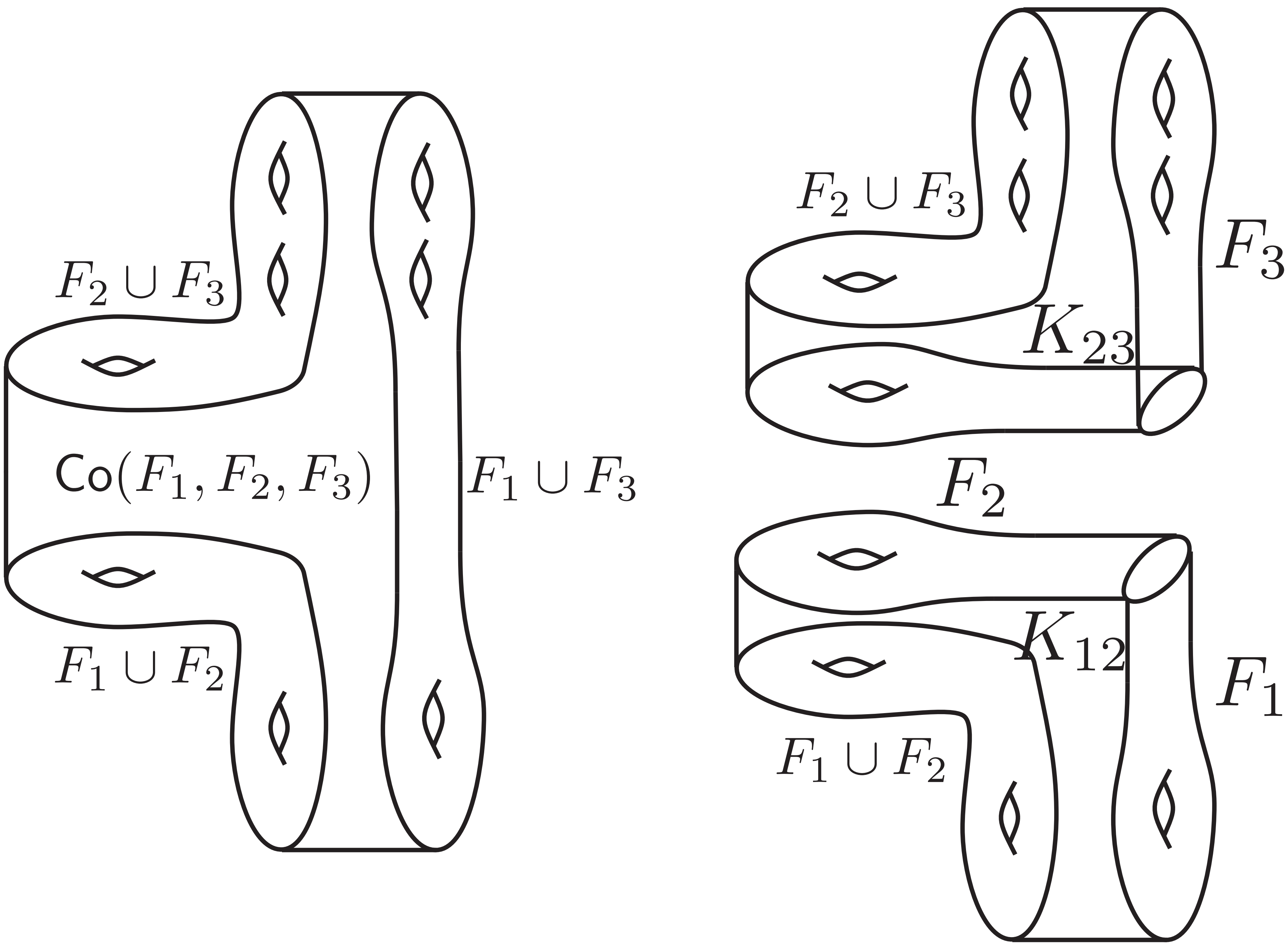
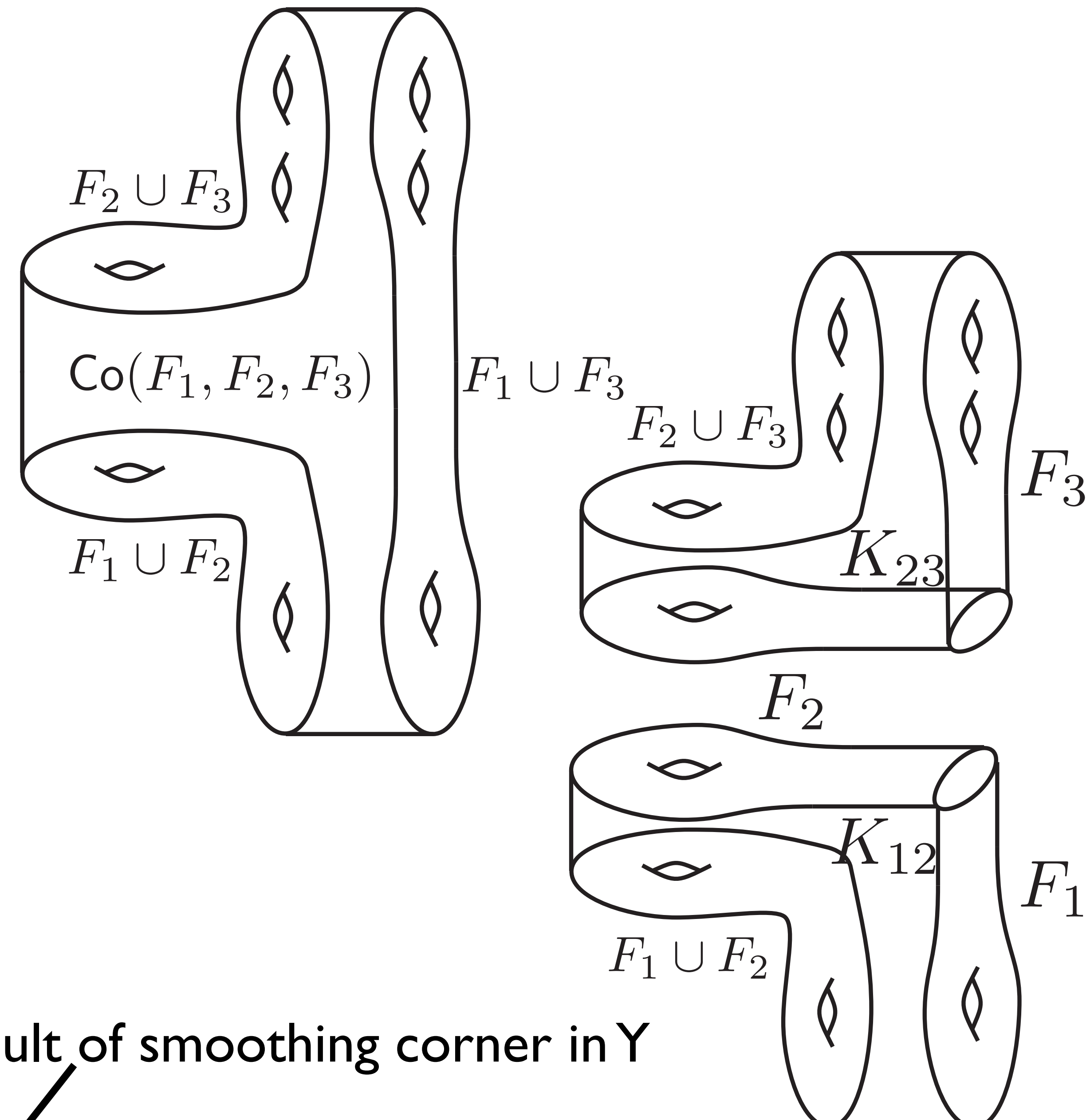
Towards a cornered invariant



Towards a cornered invariant

Need:



Result of smoothing corner in Y

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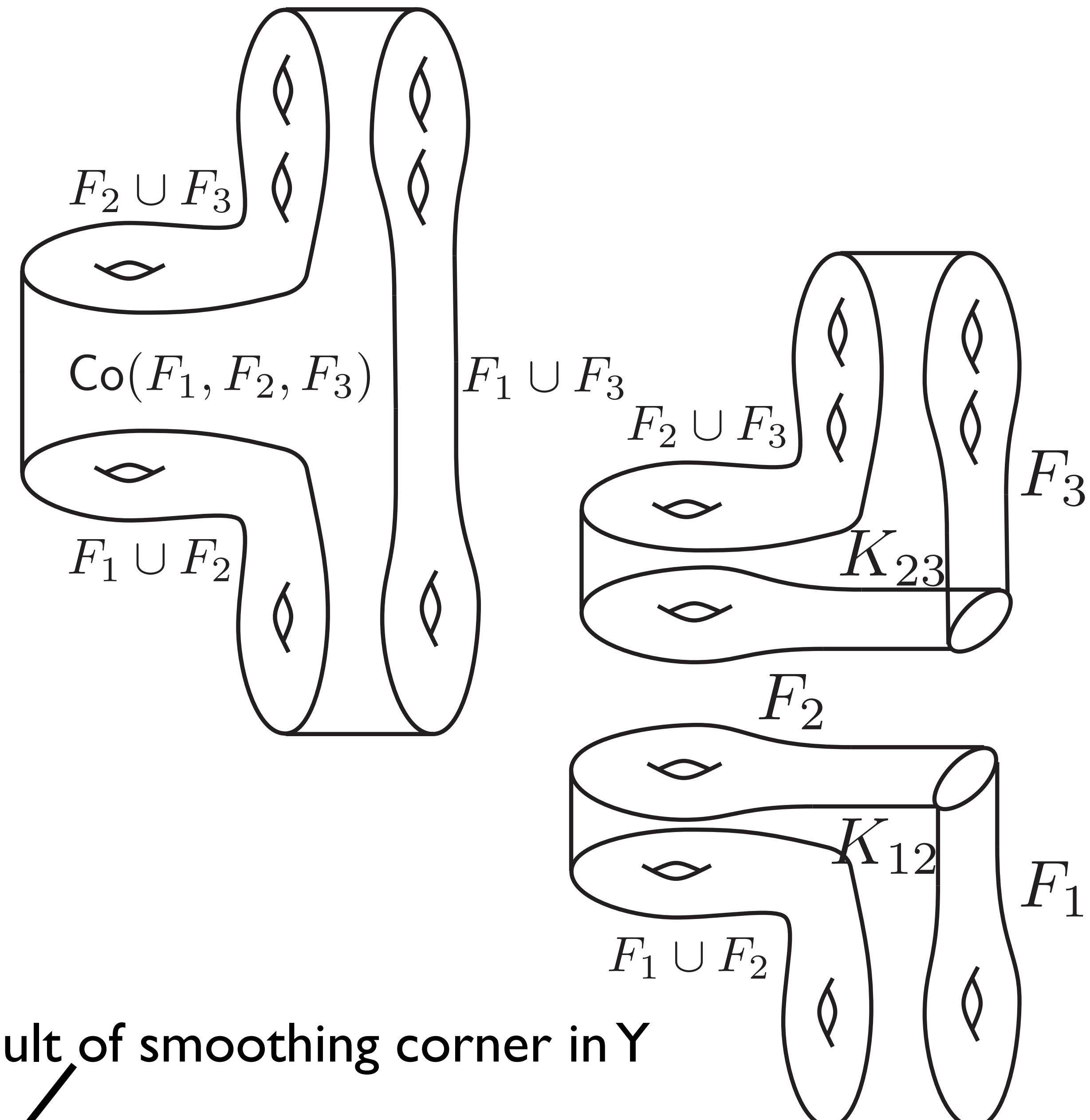

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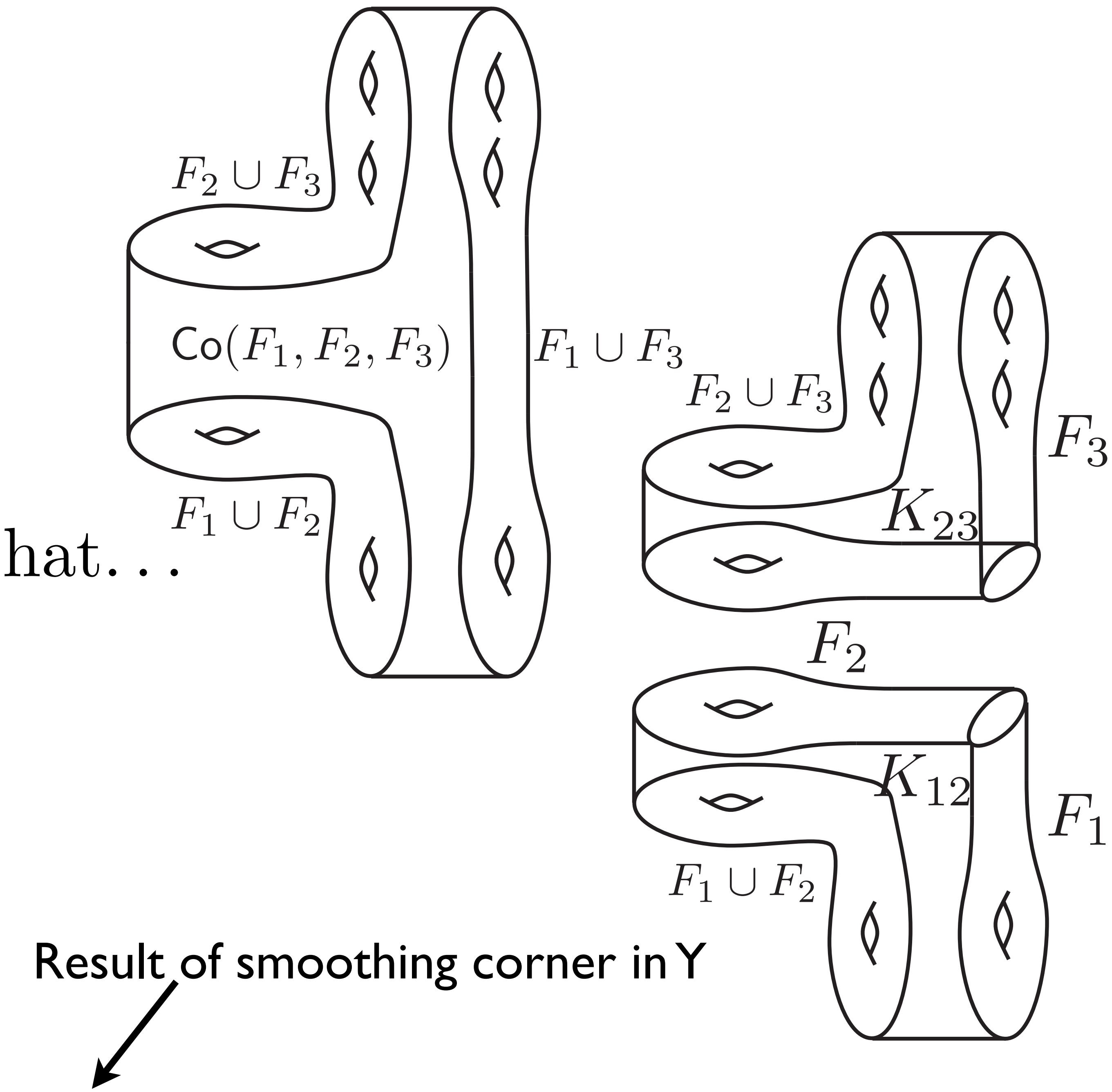

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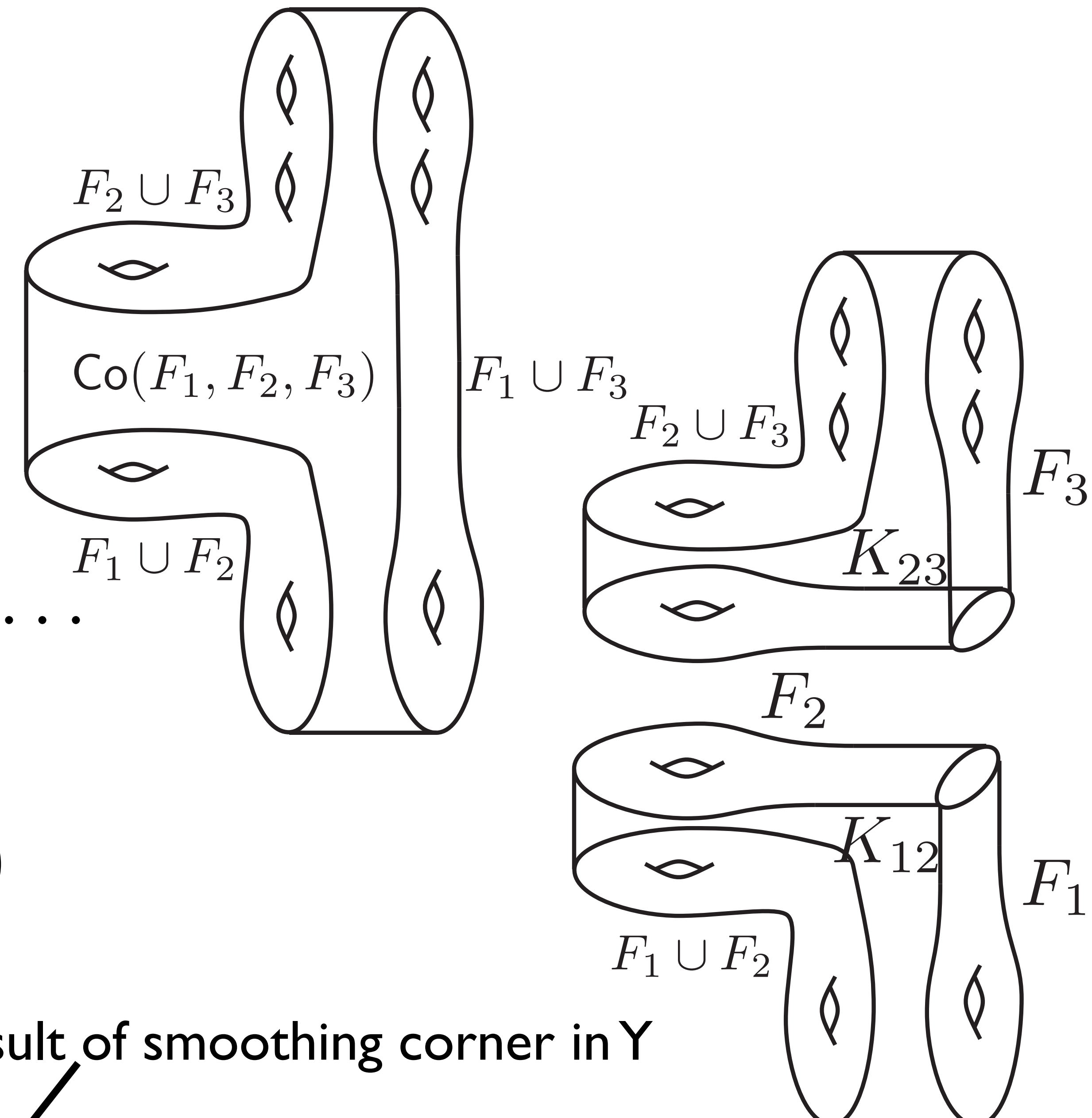

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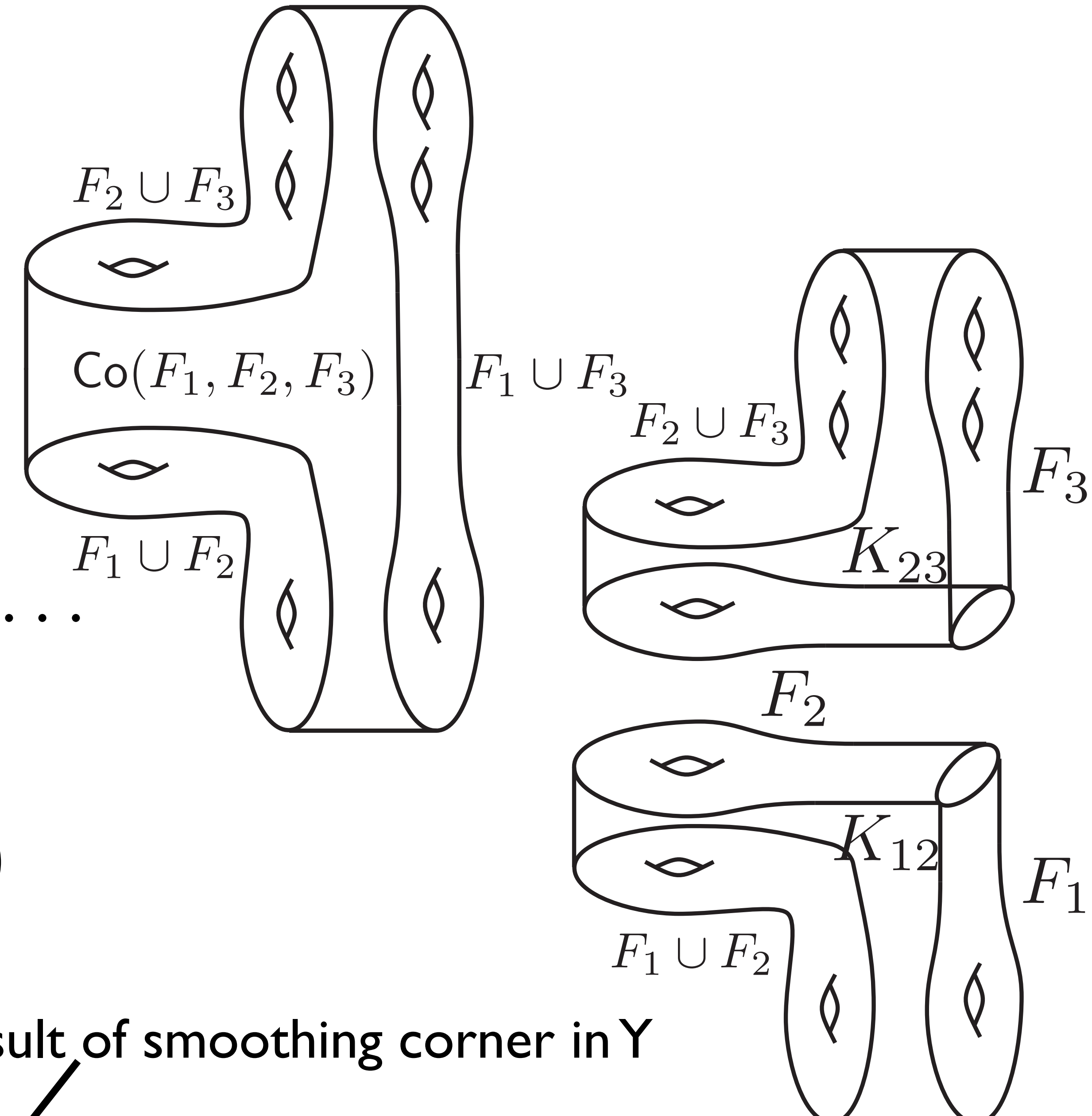
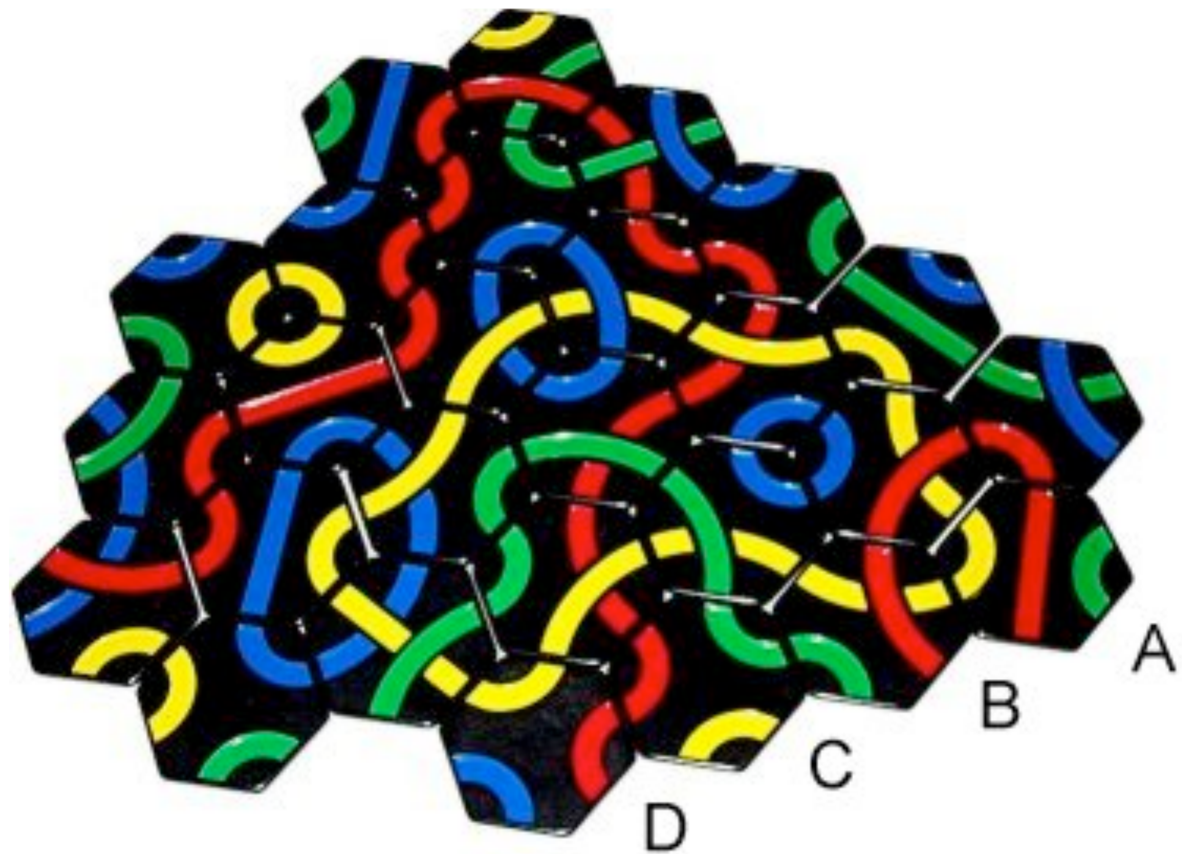


Then define:

Result of smoothing corner in Y

$$CF\{AA\}(Y) = \widehat{CFA}(Y^\circ) \otimes_{\mathcal{A}(F_1 \cup F_2)} C_{D\{AA\}}(K_{12})$$

$$CF\{DA\}(Y) = \widehat{CFA}(Y^\circ) \otimes_{\mathcal{A}(F_2 \cup F_3)} C_{D\{DA\}}(K_{23})$$



(Tantrix. Picture from thegamesjournal.com)

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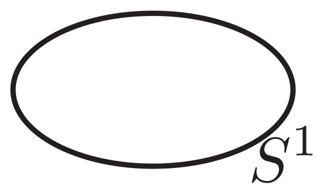
(Douglas-Manolescu, L-Douglas-Manolescu)



$\longrightarrow$ nilCoxeter 2-algebra $\mathcal{N}$ or $\mathcal{D}$.

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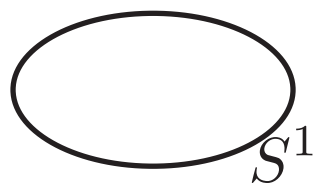

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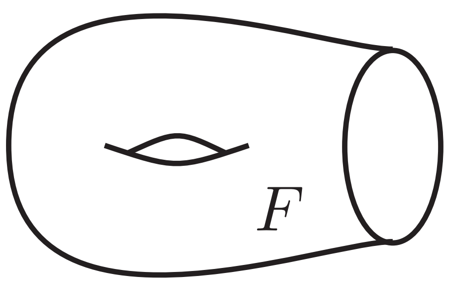

$\longrightarrow$ algebra-module $\mathcal{R}(F)$, $\mathcal{T}(F)$, $\mathcal{L}(F)$ or $\mathcal{B}(F)$.

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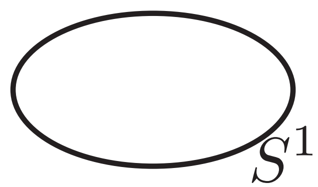

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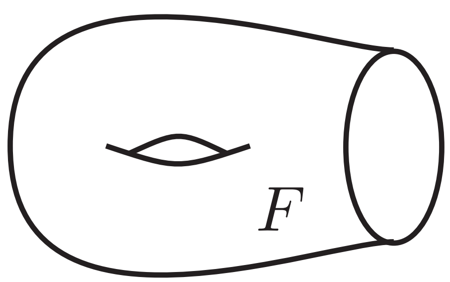
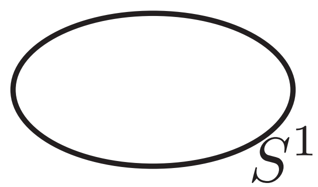


$\longrightarrow$ algebra-module $\mathcal{R}(F)$, $\mathcal{T}(F)$, $\mathcal{L}(F)$ or $\mathcal{B}(F)$.

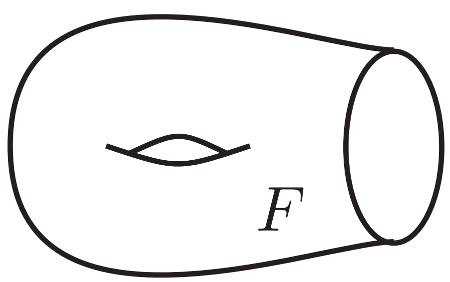
Theorem. $\mathcal{A}(F_1 \cup_{S^1} F_2) \cong \mathcal{T}(F_1) \otimes_{\mathcal{D}} \mathcal{B}(F_2) \cong \mathcal{R}(F_1) \otimes_{\mathcal{D}} \mathcal{L}(F_2)$.

Cornered Floer Homology

(Douglas-Manolescu, L-Douglas-Manolescu)



→ nilCoxeter 2-algebra $\mathcal{N}$ or $\mathcal{D}$.



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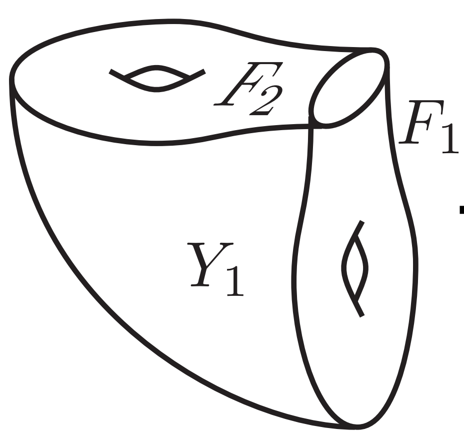

→ 2-module $\widehat{CF\{AA\}}(Y)$, $\widehat{CF\{DA\}}(Y)$, $\widehat{CF\{AD\}}(Y)$ or $\widehat{CF\{DD\}}(Y)$.

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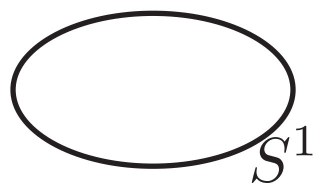
(Douglas-Manolescu, L-Douglas-Manolescu)



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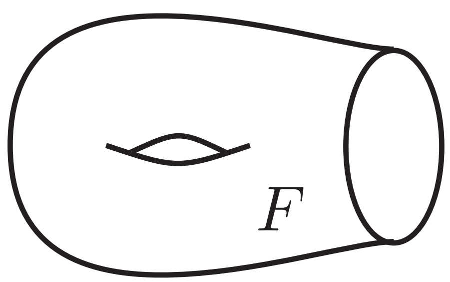

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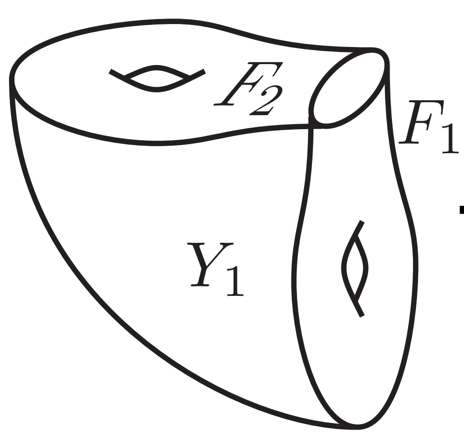

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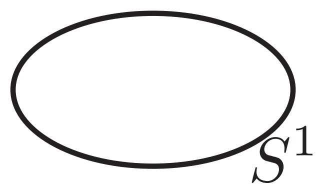

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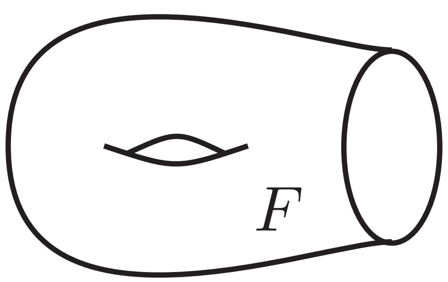

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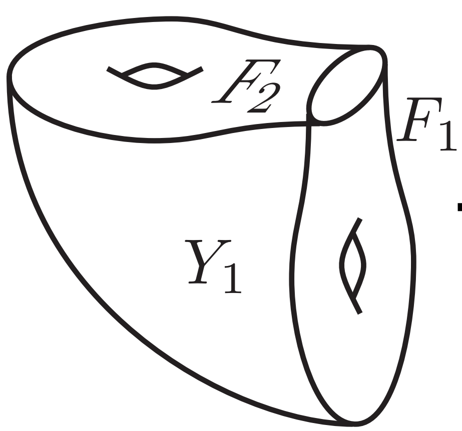

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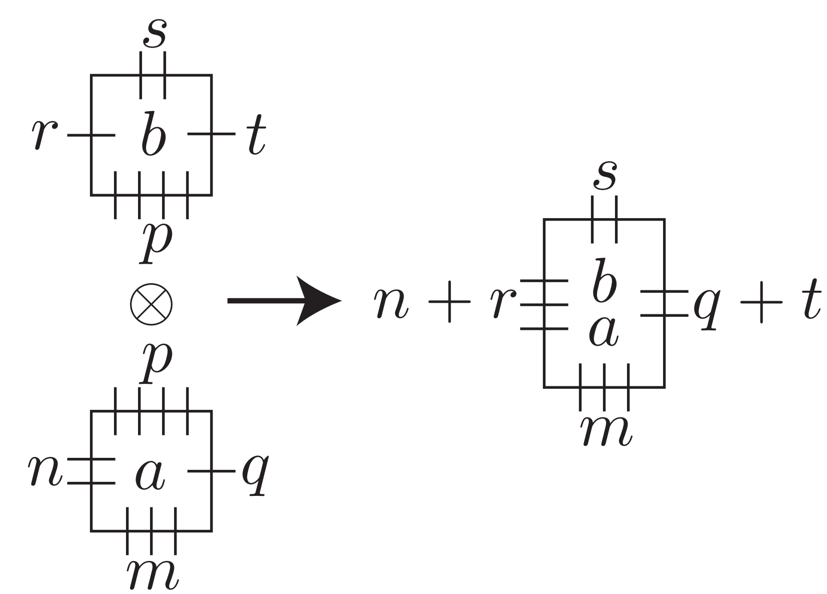
Two versions: sequential and planar.

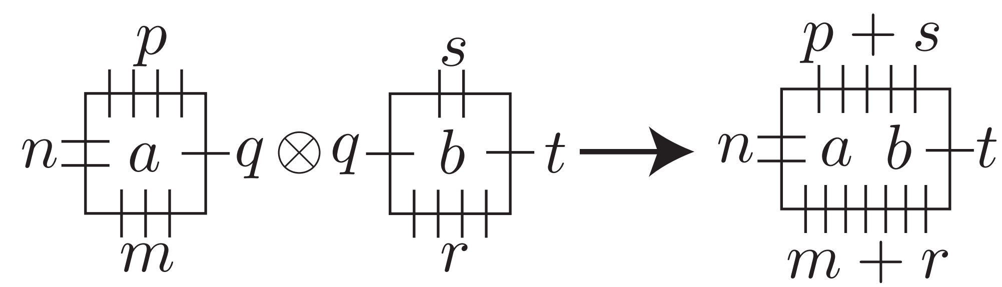
Abstract 2-algebra

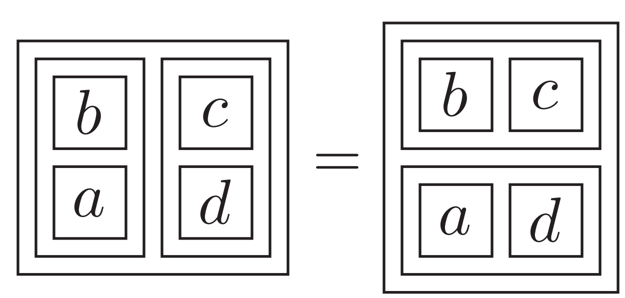
2-Algebra: Chain complexes $\underset{m}{\overset{p}{{}_n\mathcal{D}_q}}$ and maps

$$\begin{matrix}\underset{p}{\overset{s}{{}_r\mathcal{D}_t}} \\ \otimes \\ \underset{m}{\overset{p}{{}_n\mathcal{D}_q}}\end{matrix} \longrightarrow \underset{m}{\overset{s}{{}_{n+r}\mathcal{D}_{q+t}}}$$



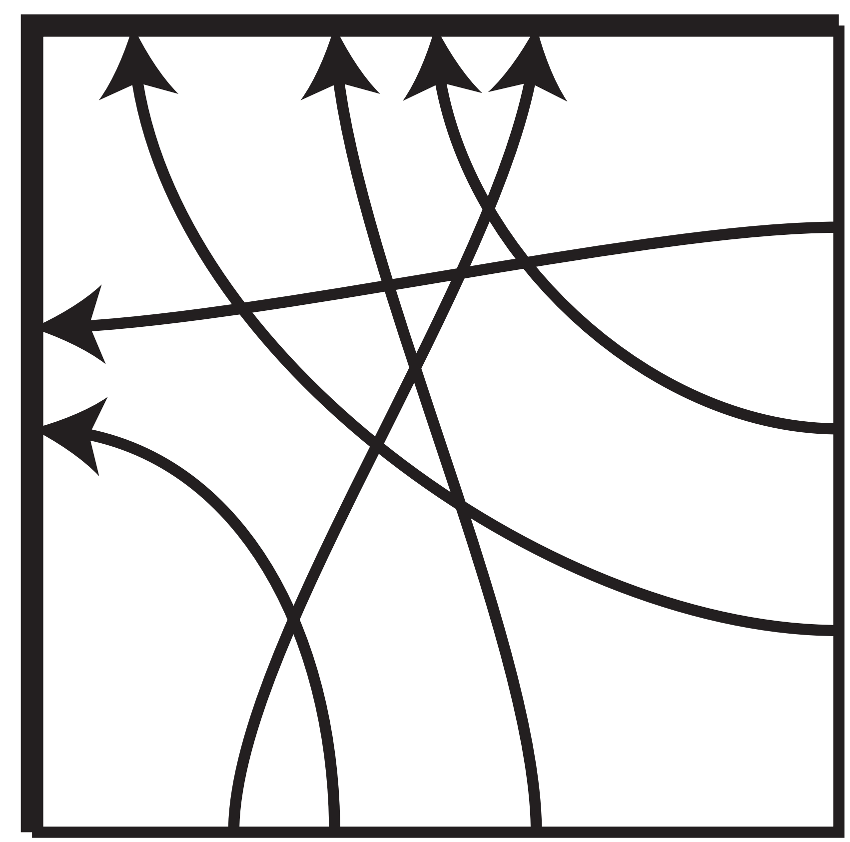
$$\underset{m}{\overset{p}{{}_n\mathcal{D}_q}} \otimes \underset{r}{\overset{s}{{}_q\mathcal{D}_t}} \longrightarrow \underset{m+r}{\overset{p+s}{{}_n\mathcal{D}_t}}$$





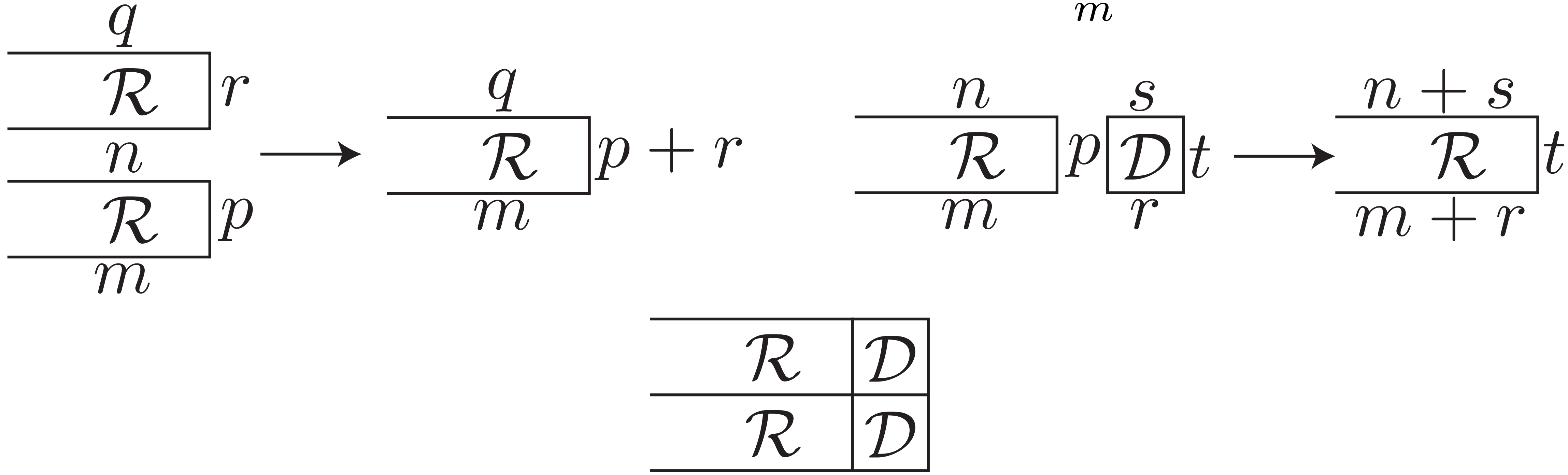
The nilCoxeter 2-algebra

- Multiplication is concatenation.
- Differential smooths crossings.
- Double crossings = 0.
- No closed components.



More abstract 2-algebra

Right algebra-module: Chain complexes ${\overset{p}{\underset{m}{\mathcal{R}}}}_q$ and maps



More abstract 2-algebra

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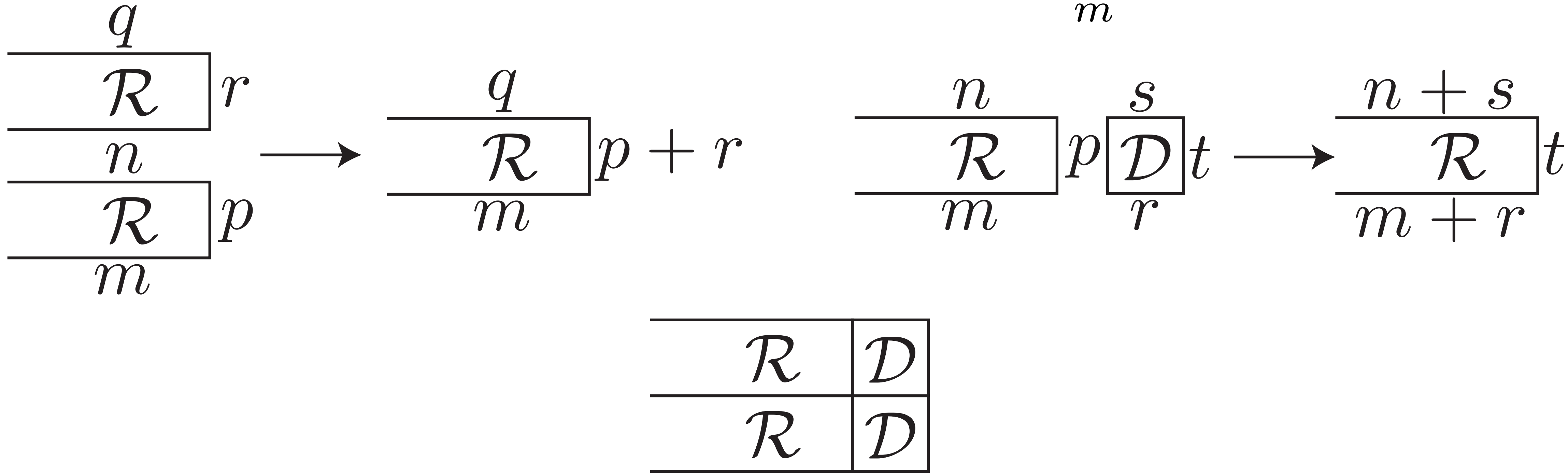
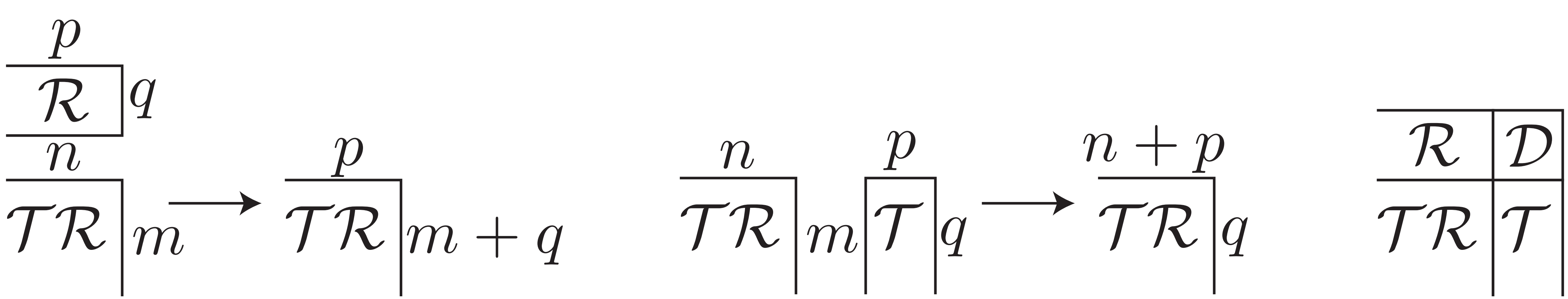
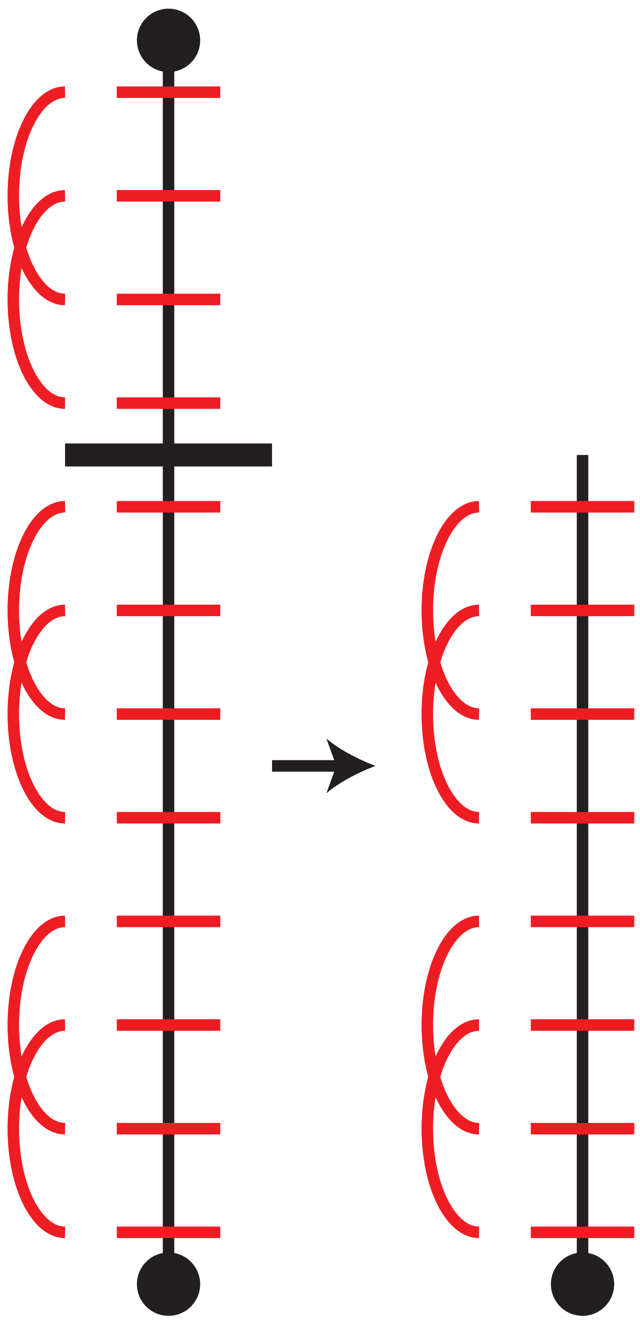


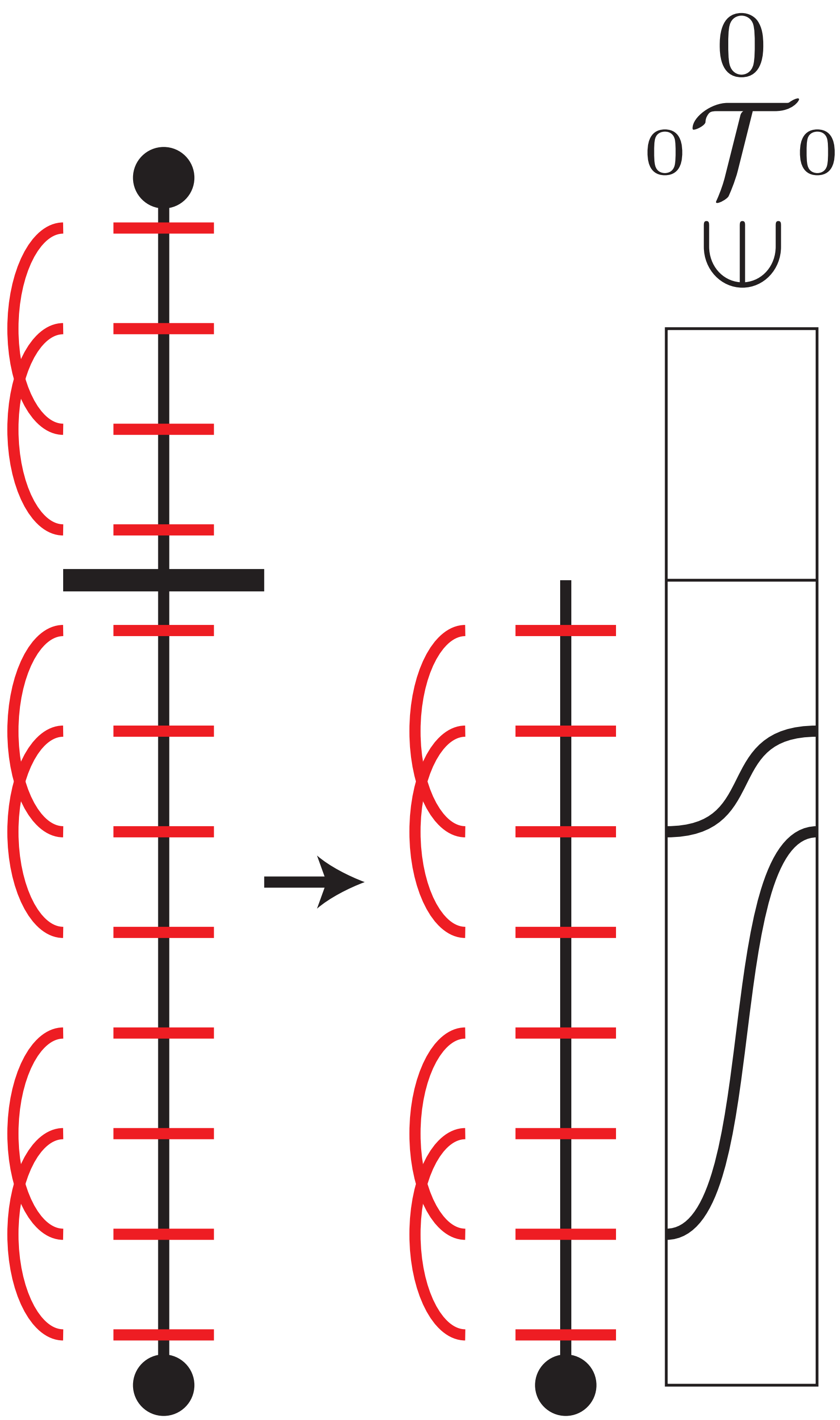
Top-right 2-module: Chain complexes $\overset{m}{\mathcal{T}\mathcal{R}}{}_n$ and maps



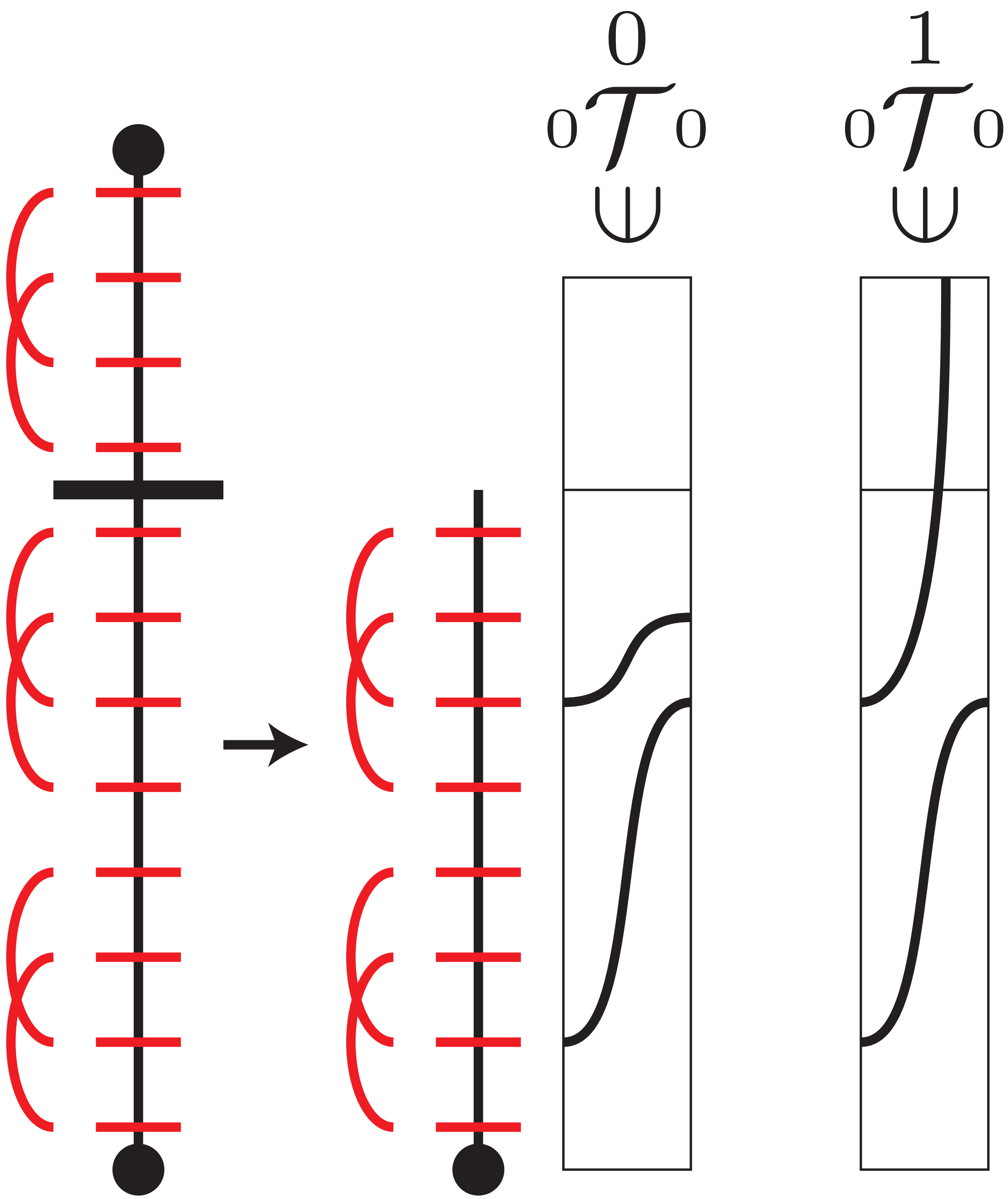
The algebra-modules



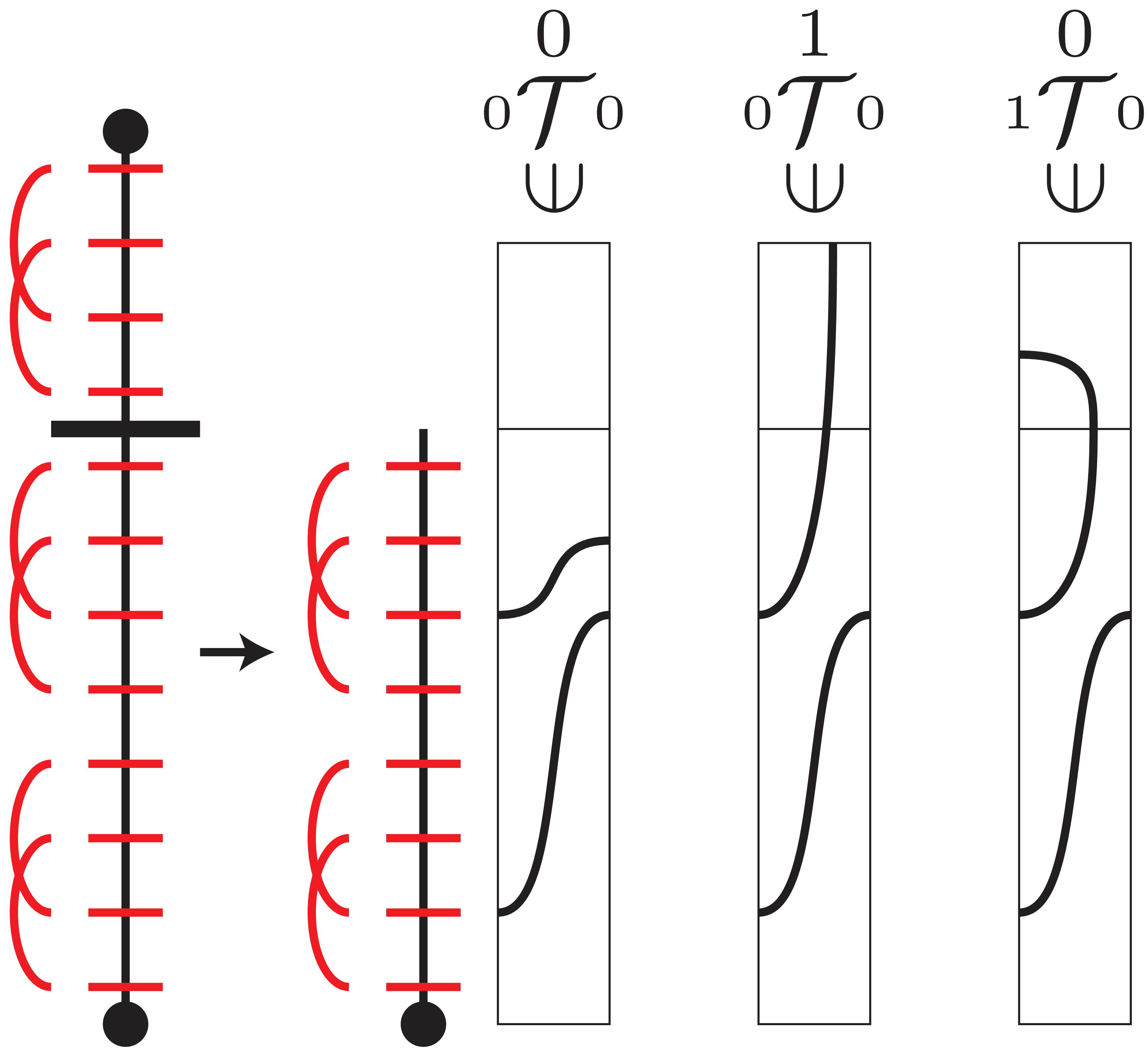
The algebra-modules



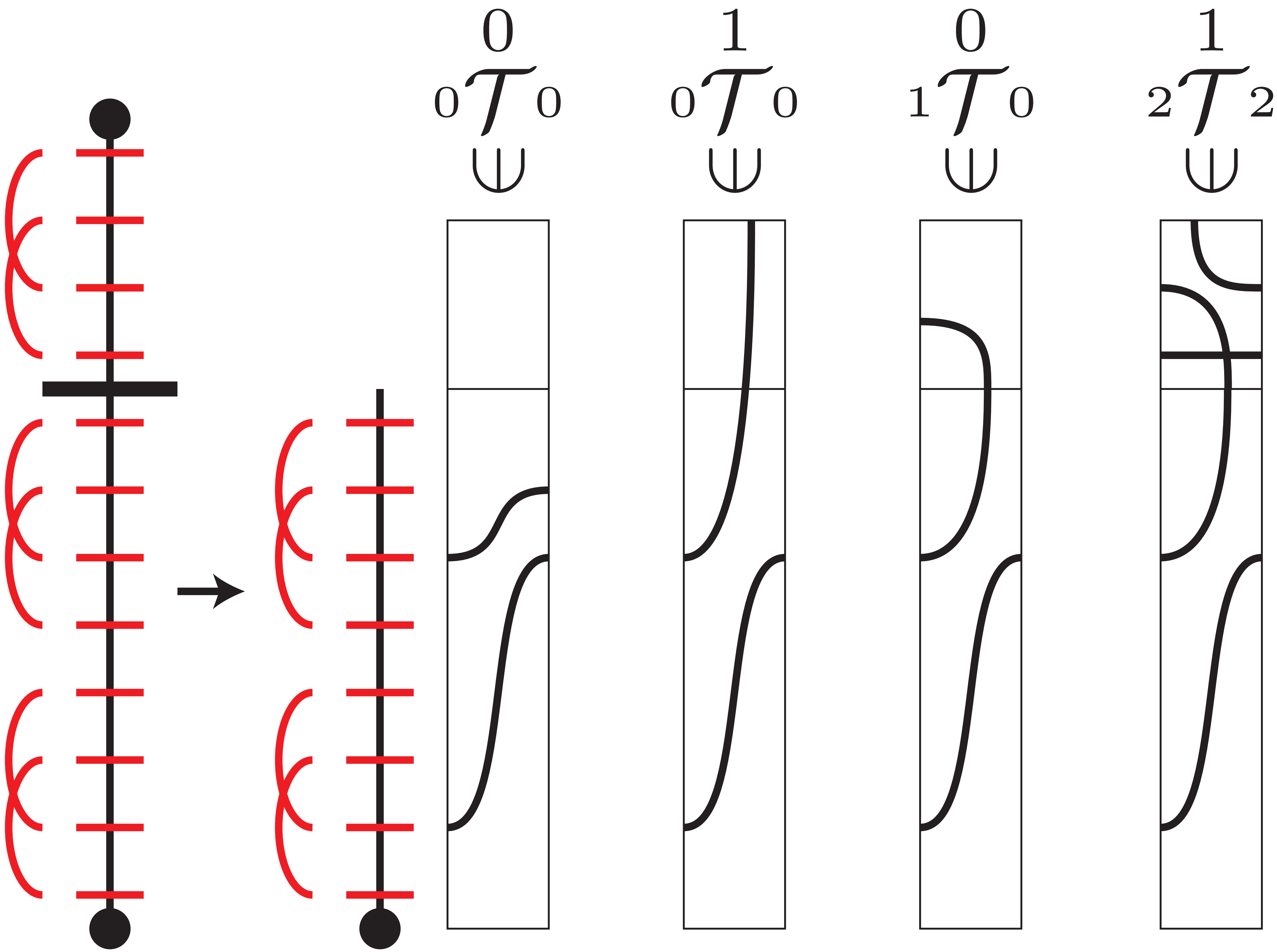
The algebra-modules



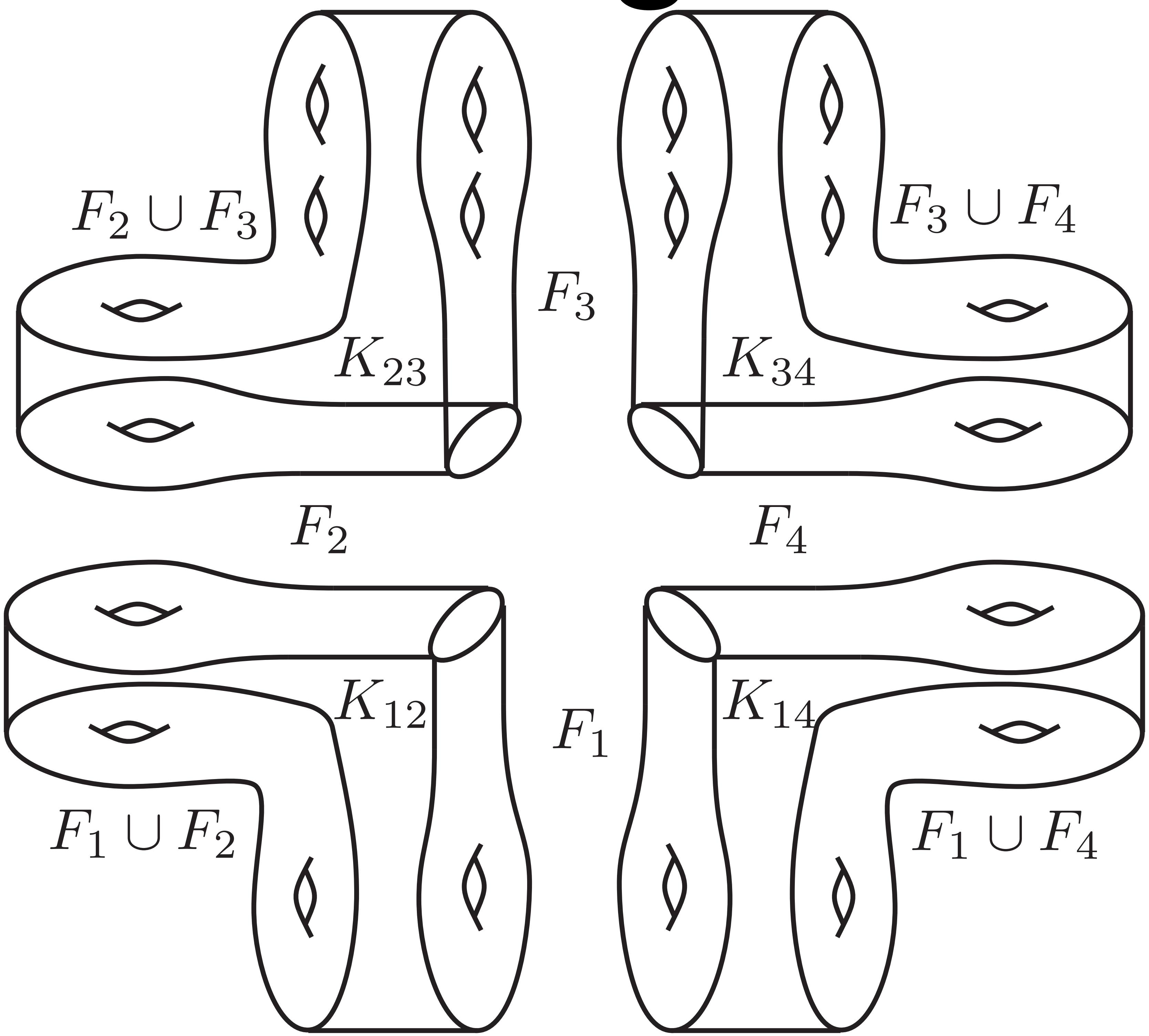
The algebra-modules

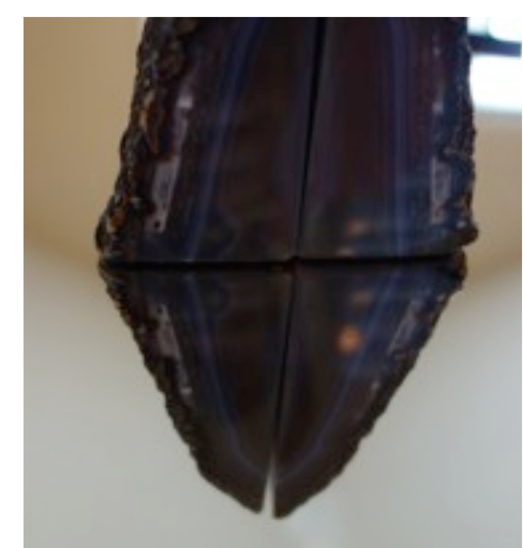


The algebra-modules

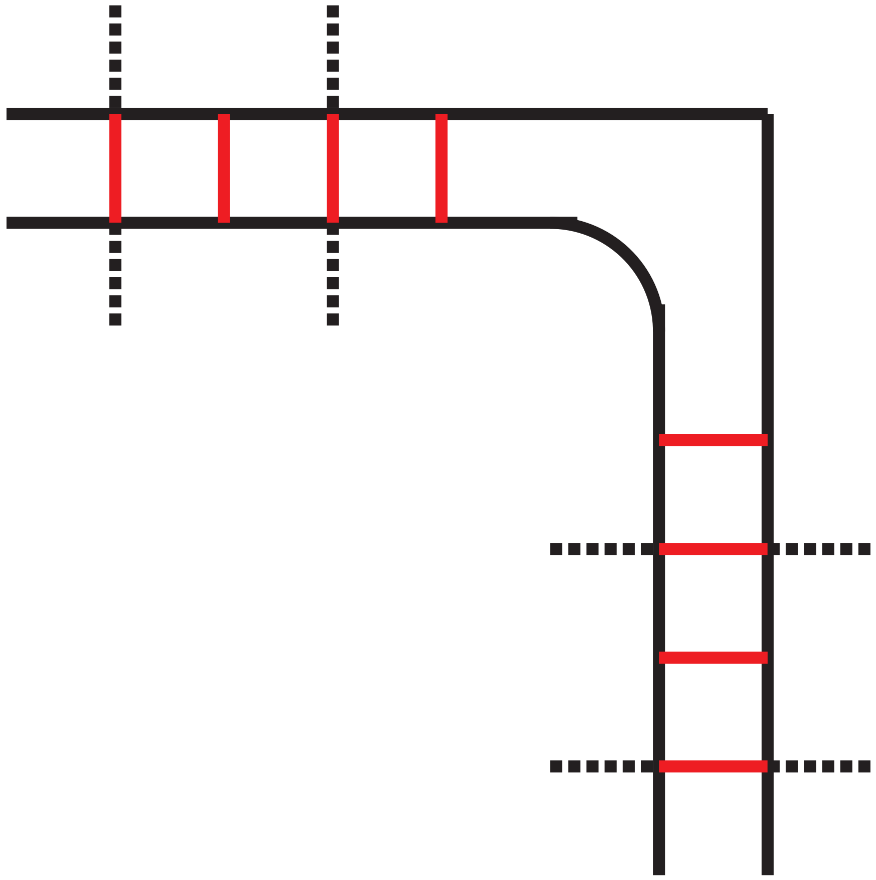


Cornering Pieces



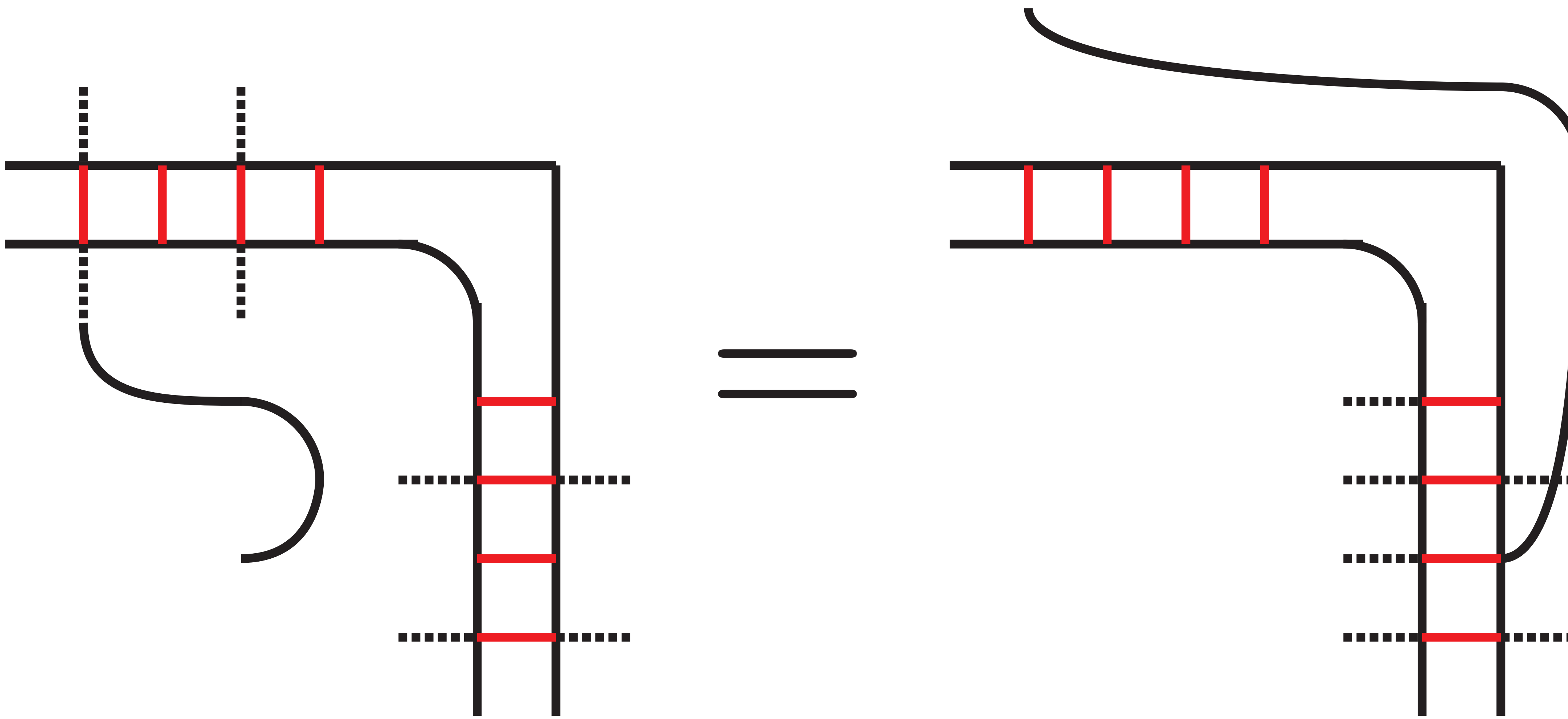


The *DAA* cornering 2-module



$C_{D\{AA\}}(K_{12})$

The *DAA* cornering 2-module



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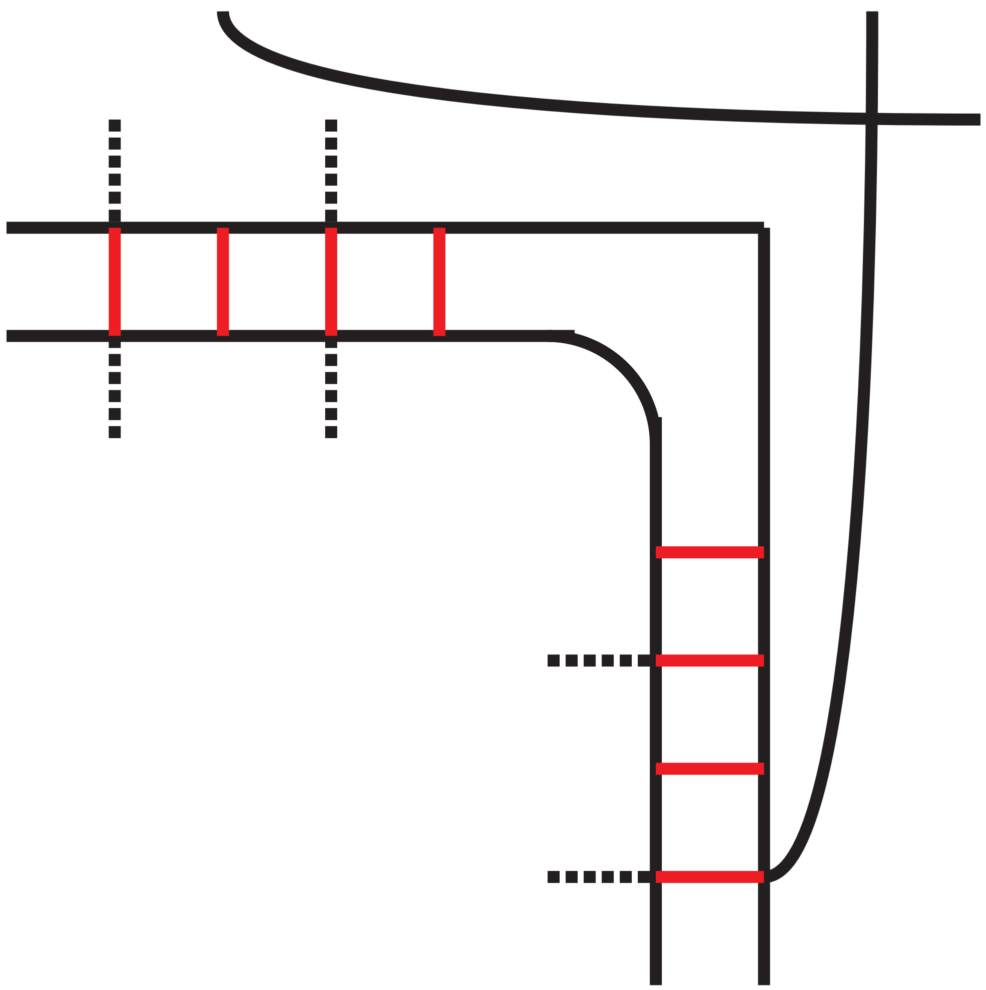
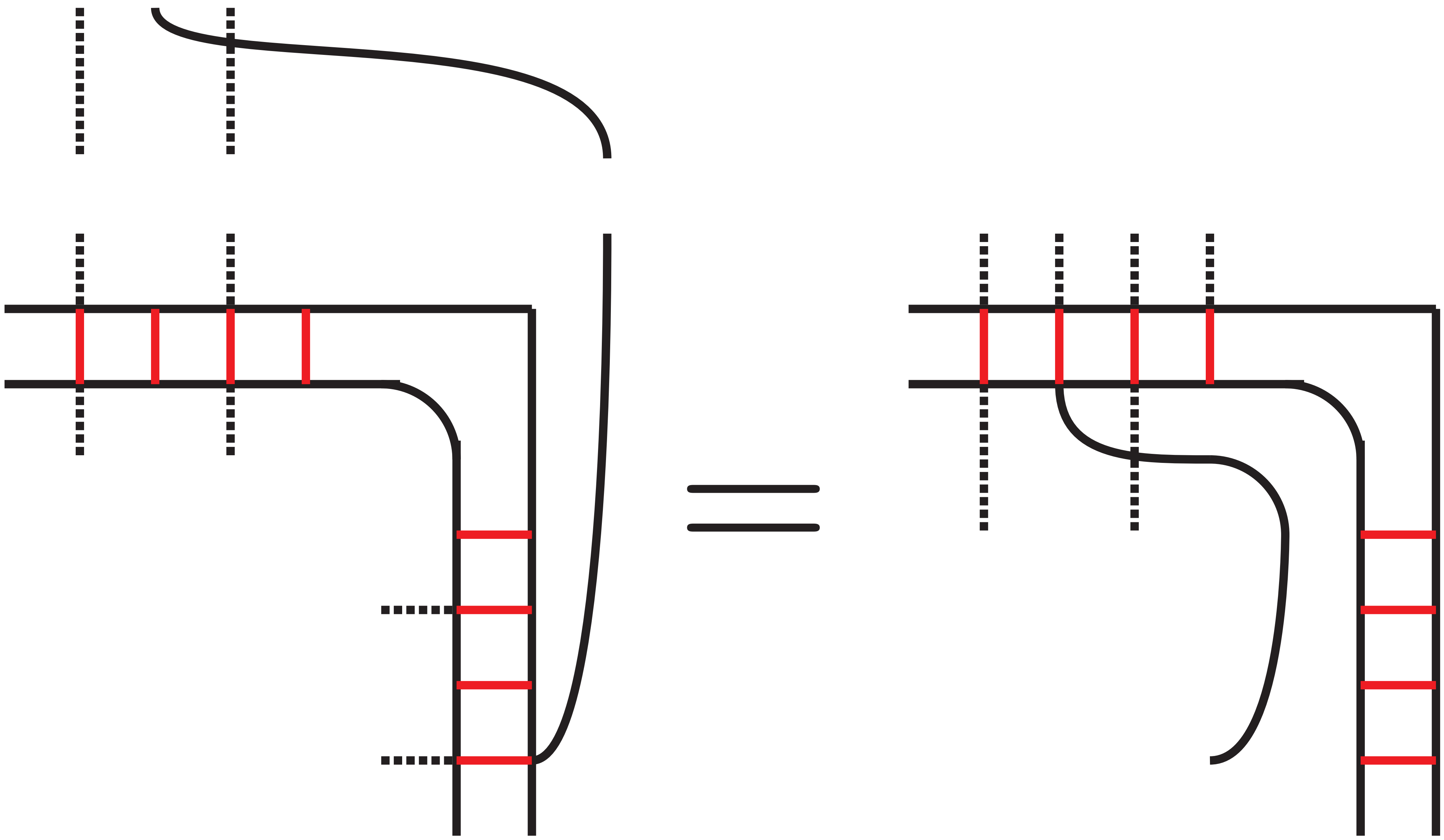


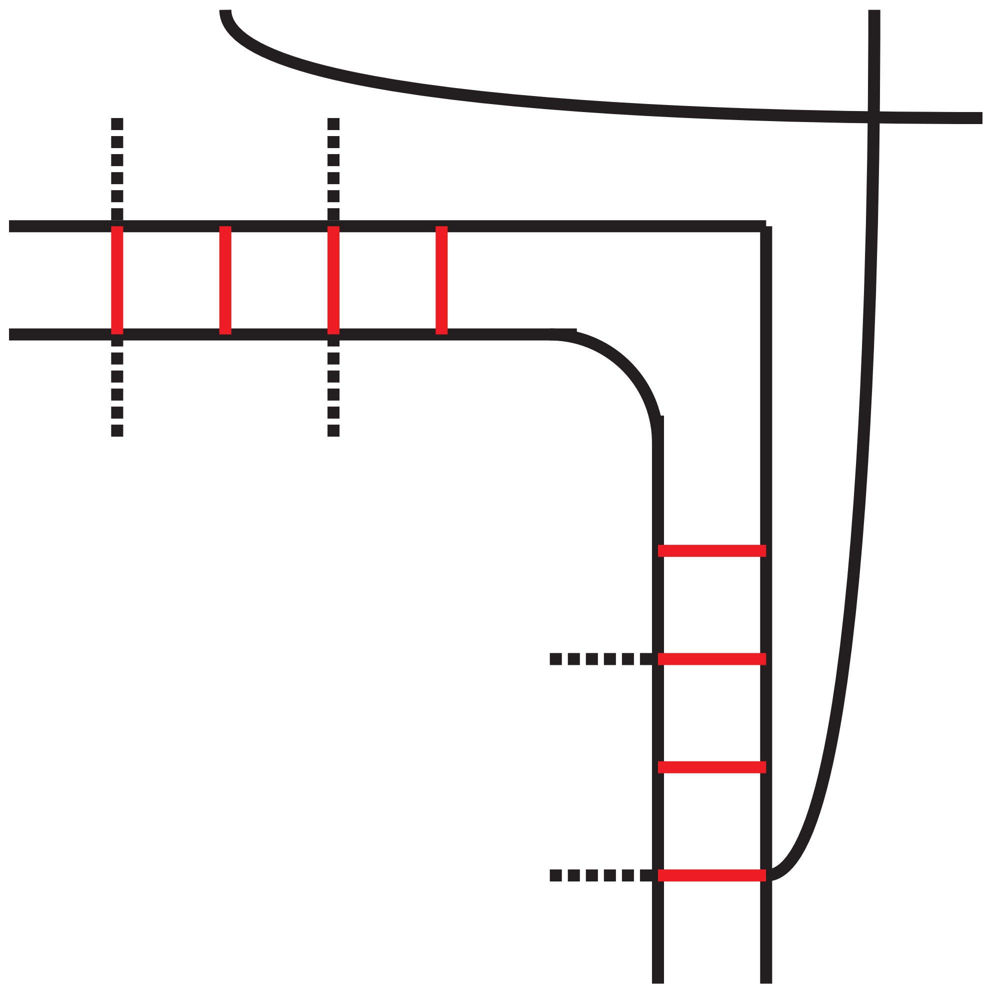
$C_{D\{AA\}}(K_{12})$

The *DAA* cornering 2-module



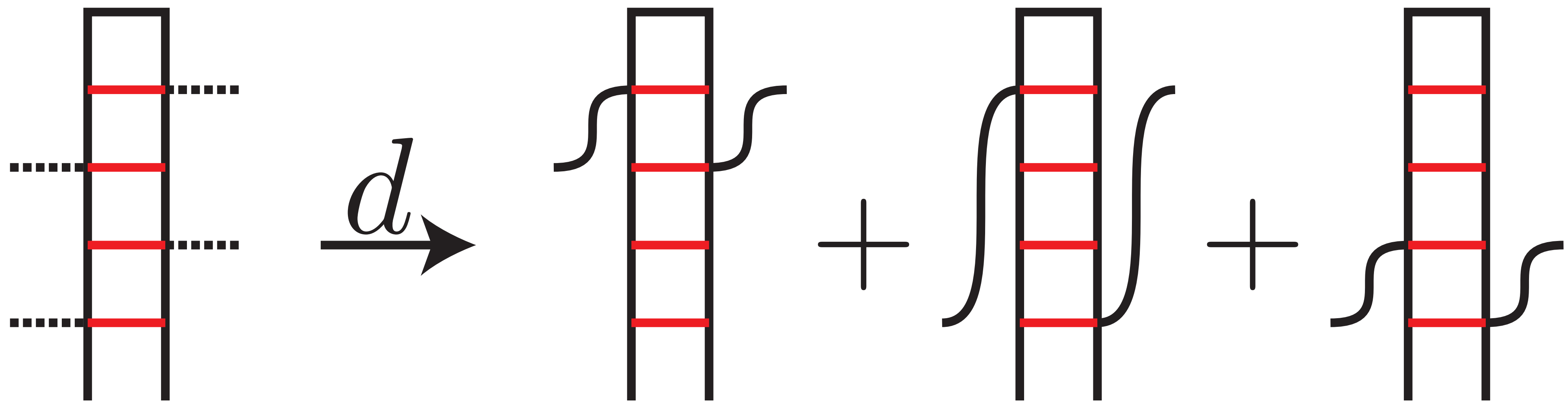
$C_{D\{AA\}}(K_{12})$

The *DAA* cornering 2-module



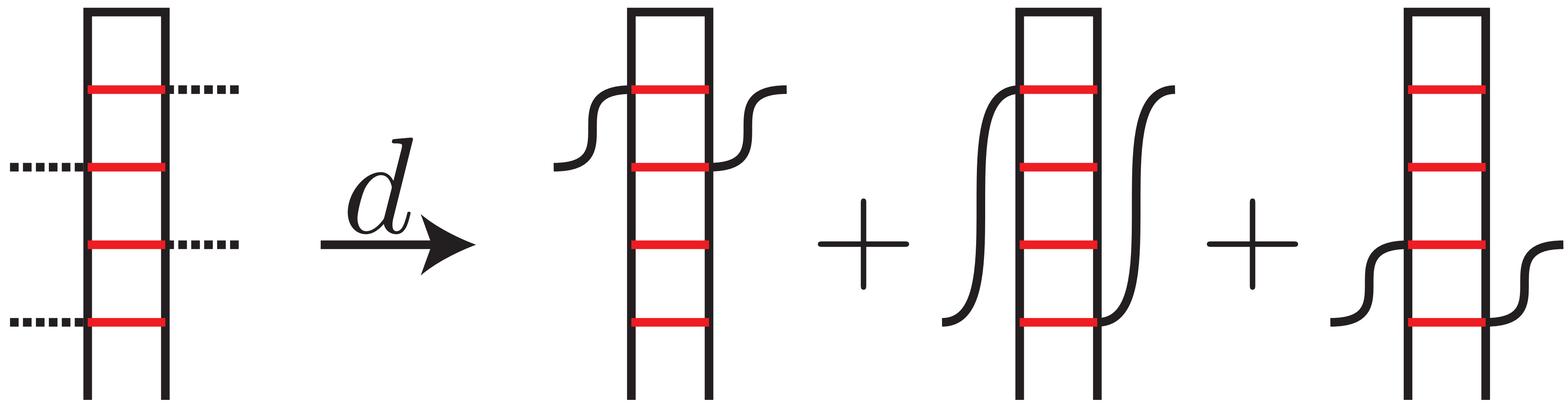
$$C_{D\{AA\}}(K_{12}) = \begin{array}{ccc} \overset{\bullet}{\underset{0}{\mathcal{R}}}\bullet & \otimes\ {}^{\bullet}\overset{0}{\underset{0}{\mathcal{D}}}{}^{\bullet} & \bullet\overset{0}{\underset{\bullet}{\mathcal{D}}}0 \\ & & \otimes\ {}_0\overset{\bullet}{\underset{\bullet}{\mathcal{D}}}{}_0 \\ & & {}_0\overset{\bullet}{\mathcal{T}}\bullet \end{array}$$

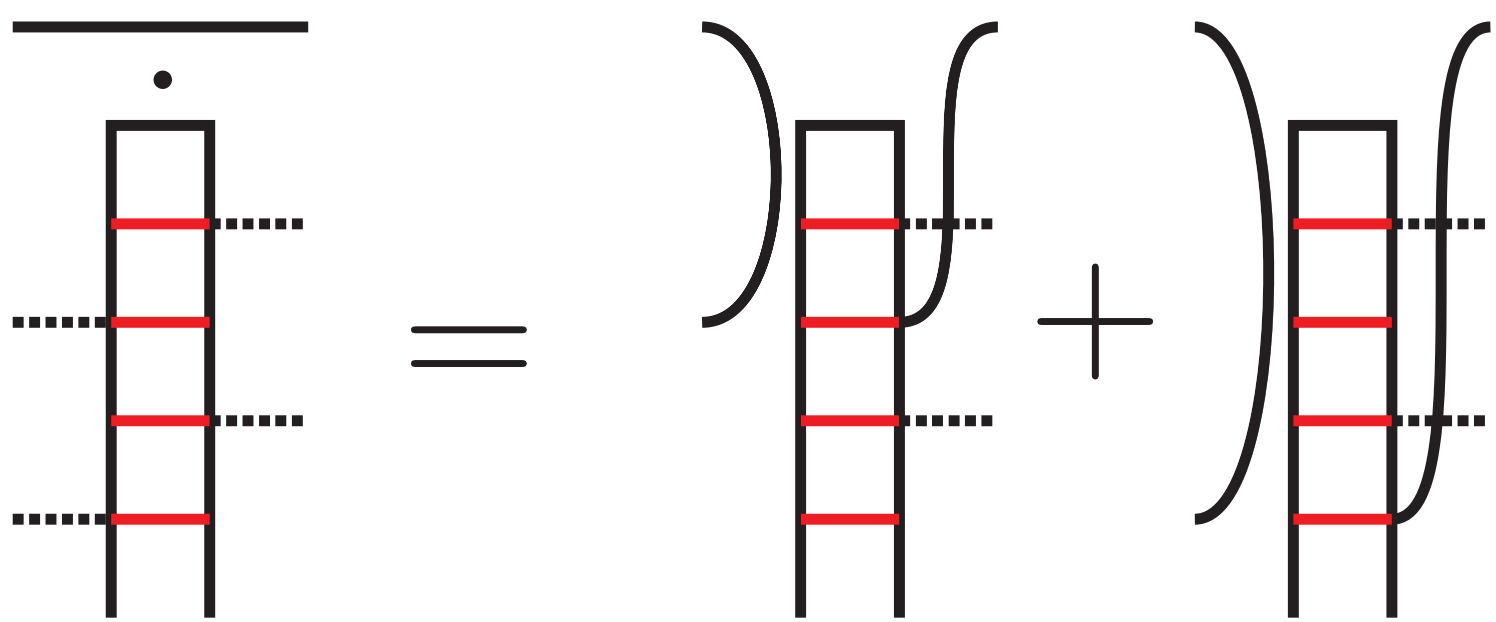
The *DD* Identity Module



$DD(\mathbb{I}_F)$

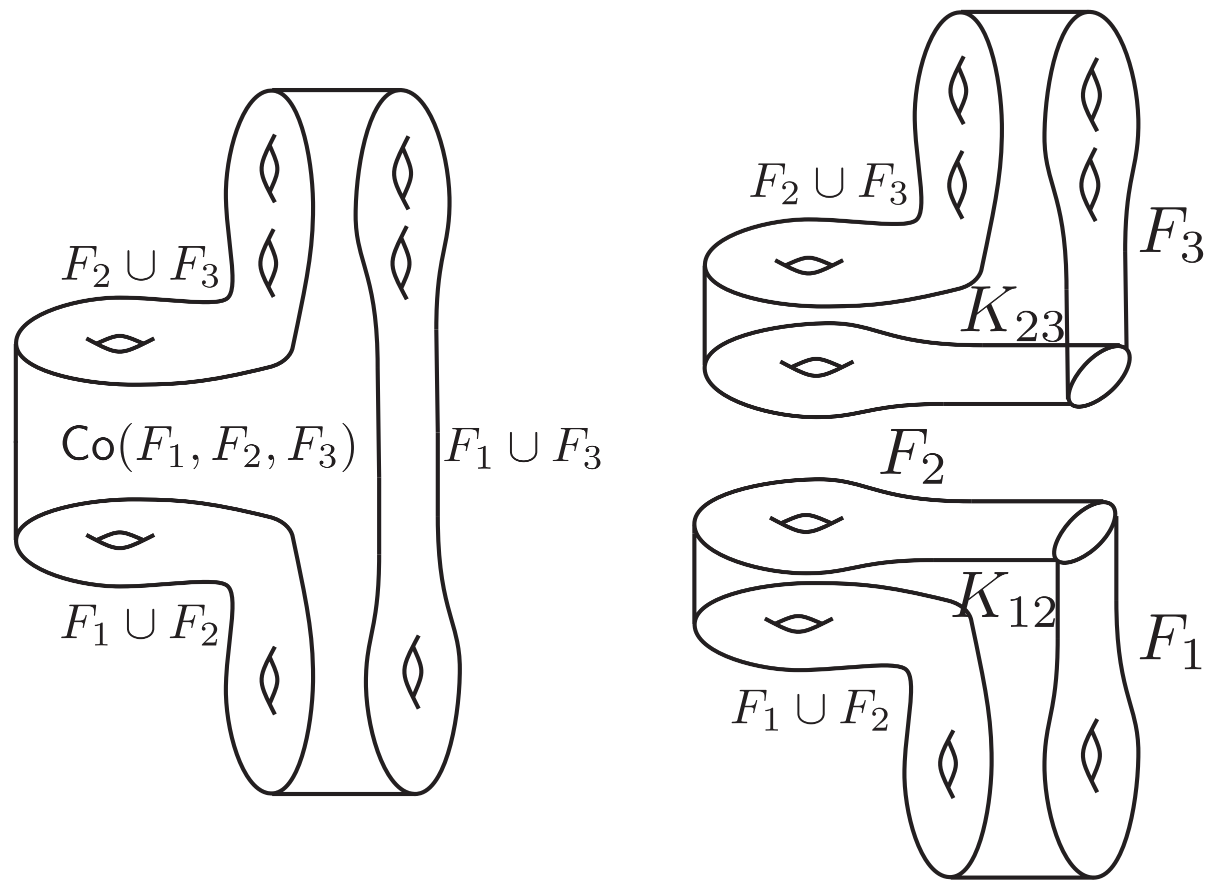
The *DD* Identity Module





$DD(\mathbb{I}_F)$

The other cornering pieces

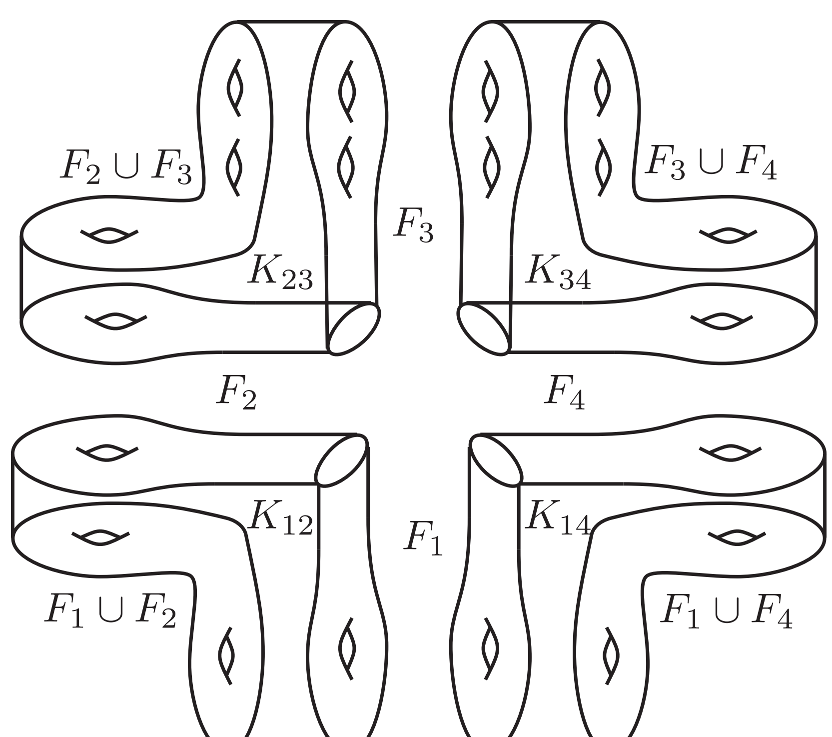


$$C_{D\{DA\}}(K_{23}) = C_{D\{AA\}}(K_{23}) \otimes_{\mathcal{R}(F_2)} DD(\mathbb{I}_{F_2})$$
$$C_{D\{AD\}}(K_{14}) = C_{D\{AA\}}(K_{14}) \otimes_{\mathcal{T}(F_1)} DD(\mathbb{I}_{F_1})$$
$$C_{D\{DD\}}(K_{34}) = C_{D\{AA\}}(K_{34}) \otimes_{\mathcal{T}(F_3)} DD(\mathbb{I}_{F_3}) \otimes_{\mathcal{R}(F_4)} DD(\mathbb{I}_{F_4})$$

The 2-modules

Define:

$$CF\{AA\}(Y) = \widehat{CFA}(Y^\circ) \otimes_{\mathcal{A}(F_1 \cup F_2)} C_{D\{AA\}}(K_{12})$$
$$CF\{DA\}(Y) = \widehat{CFA}(Y^\circ) \otimes_{\mathcal{A}(F_2 \cup F_3)} C_{D\{DA\}}(K_{23})$$
$$CF\{AD\}(Y) = \widehat{CFA}(Y^\circ) \otimes_{\mathcal{A}(F_1 \cup F_4)} C_{D\{AD\}}(K_{14})$$
$$CF\{DD\}(Y) = \widehat{CFA}(Y^\circ) \otimes_{\mathcal{A}(F_3 \cup F_4)} C_{D\{DD\}}(K_{34})$$



Invariance and pairing

Theorem. Up to quasi-isomorphism, $\widehat{CF}\{AA\}(Y)$, $\widehat{CF}\{DA\}(Y)$, $\widehat{CF}\{AD\}(Y)$ and $\widehat{CF}\{DD\}(Y)$ are invariants of Y.

Invariance and pairing

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Theorem.

$$C_{D\{AA\}}(K_{23}) \otimes_{\mathcal{R}(F_2)} C_{D\{DA\}}(K_{12}) \simeq \widehat{CFDDA}(\mathsf{Co}(F_1, F_2, F_3))$$

$$C_{D\{AD\}}(K_{23}) \otimes_{\mathcal{R}(F_2)} C_{D\{DD\}}(K_{12}) \simeq \widehat{CFDDD}(\mathsf{Co}(F_1, F_2, F_3))$$

Invariance and pairing

Theorem. Up to quasi-isomorphism, $\widehat{CF}\{AA\}(Y)$, $\widehat{CF}\{DA\}(Y)$, $\widehat{CF}\{AD\}(Y)$ and $\widehat{CF}\{DD\}(Y)$ are invariants of Y.

Theorem.

$$C_{D\{AA\}}(K_{23}) \otimes_{\mathcal{R}(F_2)} C_{D\{DA\}}(K_{12}) \simeq \widehat{CFDDA}(\mathsf{Co}(F_1, F_2, F_3))$$

$$C_{D\{AD\}}(K_{23}) \otimes_{\mathcal{R}(F_2)} C_{D\{DD\}}(K_{12}) \simeq \widehat{CFDDD}(\mathsf{Co}(F_1, F_2, F_3))$$

Corollary.

$$\widehat{CFA}(Y_1 \cup_F Y_2) \simeq \widehat{CF\{AA\}}(Y_1) \otimes_{\mathcal{R}(F)} \widehat{CF\{DA\}}(Y_2)$$

$$\widehat{CFD}(Y_1 \cup_F Y_2) \simeq \widehat{CF\{AD\}}(Y_1) \otimes_{\mathcal{R}(F)} \widehat{CF\{DD\}}(Y_2)$$