CALCULUS 1, NOTE
INTEGRATION

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5. **Integration**

The final topic we will talk about is integration. At the end, we will see that it corresponds to an inverse operation to differentiation. We start by directly doing this and introducing what an antiderivative is. As usual, when we mention a function $f\colon I \to \mathbb{R}$, we assume I is an open interval unless otherwise is specified.

5.1. **Antiderivative.** The definition is simple. Given a function, we just want to find another function whose derivative is the given one.

Definition 5.1. Let $f\colon I \to \mathbb{R}$ be a function. We say a function $F\colon I \to \mathbb{R}$ is an **antiderivative** of f if F is differentiable and $F' = f$ on I.

Example 5.2. We can get a few examples by our knowledge on derivatives.

(1) Let $f(x) = x$. Then an antiderivative of f is $x^2/2$.
(2) In general, if $f(x) = x^n$ for some $n \in \mathbb{N}$, then an antiderivative of f is $x^{n+1}/(n+1)$.
(3) Let $f(x) = \sin x$. We know $(\cos x)' = -\sin x$, so an antiderivative of f is $-\cos x$.
(4) Let $f(x) = 1/x$ for $x \neq 0$. Then an antiderivative of f is $\log |x|$.
(5) Let $f(x) = 2e^x$. Then f itself is an antiderivative of f.
(6) In all the examples above, we can freely add an arbitrary constant to those antiderivatives.

From above, we observe that when finding an antiderivative, we can add any constant at the end. In fact, this includes all the possibilities.

Lemma 5.3. Let $f\colon I \to \mathbb{R}$ be a function on an open interval. If F is an antiderivative of f, then any antiderivative of f is of the form $F + c$ for some constant $c \in \mathbb{R}$.

Proof. The proof is simple. If G is another antiderivative of f, then we get $G' = f = F'$. Thus, F and G are different by a constant. (e.g., see Corollary 4.33 in Note 4.) $\square$

Thus, if we can find one antiderivative, we can basically find all of them. Note that in Lemma 5.3, the function is defined on "one" interval. When the domain of a function is the union of more than one interval, then one may add different constants on different intervals.

Example 5.4. We look at the example given by $\log |x|$. Depending on the domains we specify, one may get different antiderivatives in general.

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(1) Suppose $f\colon (0,\infty) \to \mathbb{R}$ is defined by $f(x) = 1/x$. This is defined on one open interval, so any antiderivative of $f(x)$ is of the form $\log|x| + C = \log x + C$ for a constant C.

(2) Suppose $g\colon \mathbb{R}\setminus\{0\} \to \mathbb{R}$ is defined by $g(x) = 1/x$. Note that $\mathbb{R}\setminus\{0\} = (-\infty, 0) \cup (0,\infty)$ consists of two open interval. Thus, any antiderivative of g can be of the form

$$G(x) = \begin{cases} \log|x| + C_1 & \text{if } x > 0 \\ \log|x| + C_2 & \text{if } x < 0 \end{cases}$$

for some constants C_1 and C_2.

Although the defining expressions of f and g look the same, since their domains are different, they are different functions. In general, one has to be careful about this issue when a function is essentially defined in a piecewise manner (like g above).

When a function $f\colon I \to \mathbb{R}$ is defined on an open interval (so that the subtle situation in Example 5.4(2) does not happen), if an antiderivative F of f exists, then Lemma 5.3 tells us that a general form of antiderivatives of f is $F + C$ for some constant C. In this case, we write

$$\int f\,dx = F + C \text{ or}$$

$$\int f(x)dx = F(x) + C.$$

This is the **indefinite integral** of f on I.

Example 5.5. We now write down some indefinite integrals that we can derive directly.

(1) For $n \neq 1$, $\displaystyle\int x^n dx = \frac{1}{n+1}x^{n+1} + C$.

(2) On $(0,\infty)$ or $(-\infty,0)$, $\displaystyle\int x^{-1}dx = \log|x| + C$.

(3) For $a > 0$ with $a \neq 1$, $\displaystyle\int a^x dx = \frac{1}{\log a}a^x + C$.

(4) On $\mathbb{R}$, $\displaystyle\int \sin x\,dx = -\cos x + C$ and $\displaystyle\int \cos x\,dx = \sin x + C$.

(5) From Assignment 6, we know $(\log|\tan x + \sec x|)' = \sec x$, so if we consider $f(x) = \sec x$ for $x \in (-\pi/2, \pi/2)$, we have

$$\int f(x)dx = \log|\tan x + \sec x| + C.$$

Later we will see significant geometric meaning of indefinite integral. Before that, we can first talk about a direct application of it. In fact, the process of finding antiderivatives is solving a special form of **differential equations**, that is, an equation involving functions and their derivatives. For example, an equation

$$f'(x) = f(x)$$

asking for a function f on $\mathbb{R}$ as a solution is a differential equation. When the left hand side is f' and the right hand side does not involve, it is essentially the problem about finding antiderivatives. Practically, such a question comes with an **initial condition** that allows us to determine the constant in an indefinite integral.

Example 5.6. We will see some theoretical examples.

(1) We want to find $f\colon [0,\infty) \to \mathbb{R}$ such that
$$\begin{cases} f'(x) = 3x^2 + 4x & \text{for } x > 0 \\ f(0) = 4 \end{cases}.$$
The first condition tells us that
$$f(x) = x^3 + 2x^2 + C$$
for some $C \in \mathbb{R}$. Then we use the second condition, which implies
$$4 = f(0) = C.$$
Hence, we get
$$f(x) = x^3 + 2x^2 + 4.$$

(2) We want to find $f\colon [0,\infty) \to \mathbb{R}$ such that
$$\begin{cases} f'(x) = e^x + \sin x & \text{for } x > 0 \\ f(0) = 4 \end{cases}.$$
The first condition implies that
$$f(x) = e^x - \cos x + C$$
for some $C \in \mathbb{R}$. Then the second condition tells us
$$4 = f(0) = 1 - 1 + C.$$
Hence, we get
$$f(x) = e^x - \cos x + 4.$$

Example 5.7. We mention one example that is similar to many problems in physics. Suppose a particle is moving on the real line and suppose at time t, its velocity is $\cos t$. If the particle is at 1 when $t = 0$, we want to find the location of the particle at time t.

If we let $f(t)$ be the location of the particle at time t, then the problem can be translated to the equation
$$\begin{cases} f'(t) = \cos t \\ f(0) = 1 \end{cases}.$$
Then from the differential equation, we can solve
$$f(t) = \sin t + C$$
for some $C \in \mathbb{R}$. Then the initial condition implies
$$1 = f(0) = 0 + C,$$
so we finally get
$$f(t) = \sin t + 1,$$
which is the location of the particle at a general time t.

In general, the condition attached to a differential equation may not necessarily be about the initial time or x-value. Usually, any information at any time can determine the constant given by an indefinite integral.

5.2. **Area under a curve.** We are now going to study how to calculate the area between the graph of a given function and the x-axis. At the end, we will see that this process is related to finding antiderivatives.

There are two steps that generally work. We look at the region between the graph of a function and the x-axis.

(1) We divide the region into n rectangles and calculate their combined area.

(2) We let n tends to infinity.

We start with an example.

Example 5.8. We consider $f(x) = 2x$ for $x \in [0,1]$. The region between the graph of f and the x-axis is a triangle, and we know its area is $1 \cdot 2/2 = 1$. We are going to calculate this in a different way following the outline above.

We divide $[0,1]$ into n pieces

$$[x_0, x_1], [x_1, x_2], \cdots, [x_{n-1}, x_n],$$

with $x_i - x_{i-1} = 1/n$. For such an interval $[x_{i-1}, x_i]$, we look at the rectangle of height $f(x_i)$. Thus, an approximated area of this region is

$$\begin{aligned}
&(x_1 - x_0) \cdot f(x_1) + (x_2 - x_1) \cdot f(x_2) + \cdots + (x_n - x_{n-1}) \cdot f(x_n) \\
=&\frac{1}{n} \cdot f(x_1) + \frac{1}{n} \cdot f(x_2) + \cdots + \frac{1}{n} \cdot f(x_n) \\
=&\frac{1}{n} \cdot \frac{2}{n} + \frac{1}{n} \cdot \frac{4}{n} + \cdots + \frac{1}{n} \cdot \frac{2n}{n} \\
=&\frac{2}{n^2} (1 + 2 + \cdots + n) \\
=&\frac{2}{n^2} \cdot \frac{n(n+1)}{2}.
\end{aligned}$$

When $n \to \infty$, i.e., when n is very large, we could expect

$$\frac{2}{n^2} \cdot \frac{n(n+1)}{2} = \frac{n^2 + n}{n^2} = 1 + \frac{1}{n} \to 1 + 0 = 1.$$

Thus, the combined area should be one, the same as our expection.

The example above motivates a few things. First, since the process involves many summations, we introduce a notation to denote this. If we have a sequence of numbers $P_a, P_{a+1}, P_{a+2}, \cdots, P_{b-1}, P_b$ where $a < b$ are integers, then we will sometimes write

$$\sum_{i=a}^{b} P_i = P_a + P_{a+1} + \cdots + P_b.$$

For example, the sum we deal with in the example above can be written as

$$\sum_{i=1}^{n} (x_i - x_{i-1}) \cdot f(x_i).$$

When we want to take n to infinity as above, we will then write

$$\sum_{i=1}^{\infty} (x_i - x_{i-1}) \cdot f(x_i).$$

This is another point that needs clarification. When we are given a sequence of numbers

$$a_1, a_2, a_3, \cdots, a_n, \cdots,$$

we can similarly talk about their limit as $n \to \infty$, and we say

$$\lim_{n\to\infty} a_n = L$$

for some $L \in \mathbb{R}$ if given any restriction $R > 0$, there exists $N \in \mathbb{N}$ such that

$$|a_n - L| < R \text{ for } n \geq N.$$

This is similar to the limit of a function $f(x)$ when $x \to \infty$. Some obvious limits include

$$\lim_{n\to\infty} \frac{1}{n} = \lim_{n\to\infty} \frac{1}{n^2} = \lim_{n\to\infty} \frac{1}{n^3} = \lim_{n\to\infty} 2^{-n} = \lim_{n\to\infty} \frac{1}{\log n} = 0.$$

We remark that a basic property of the summation notation and the limits is the linearity. That is, if $P_a, P_{a+1}, \cdots, P_{b-1}, P_b$ and $Q_a, Q_{a+1}, \cdots, Q_{b-1}, Q_b$ are two sequences and $c \in \mathbb{R}$, then

$$\sum_{i=a}^{b} (P_i + c\, Q_i) = \sum_{i=a}^{b} P_i + c \sum_{i=a}^{b} Q_i,$$

and if the limits of the two sequences a_n and b_n exist, then

$$\lim_{n\to\infty} (a_n + c\, b_n) = \lim_{n\to\infty} a_n + c \lim_{n\to\infty} b_n.$$

In Example 5.8, on the interval $[x_{i-1}, x_i]$, we use $f(x_i)$ to approximate the height of the region. In general, if f is continuous, one can take any $y_i \in [x_{i-1}, x_i]$ and use $f(y_i)$ to approximate the height. The sequence x_i's also can be chosen in a more flexible way, as long as the maximal length of them tend to 0. However, to make things simpler, we first use the same way as in Example 5.8 to define the area when the function is continuous

Definition 5.9. Let $f\colon [a, b] \to \mathbb{R}$ be a continuous function. The **area** of the region below the graph of f from a to b is defined to be

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{b-a}{n} \cdot f\left(a + i \cdot \frac{b-a}{n}\right).$$

For any n, the approximate area in the limit is called a **right Riemann sum**.

As explained, this gives us the "signed" area between $x = a, x = b$, the x-axis, and the graph of $f(x)$. The area will be negative if the graph is below the x-axis.

Example 5.10. We look at another example $f(x) = x^2$ for $x \in [0, 2]$. This is not as obvious as Example 5.8, where we could just look at one single triangle. For any $n \in \mathbb{N}$, a right Riemann sum is

$$\sum_{i=1}^{n} \frac{2}{n} f\left(\frac{2i}{n}\right) = \sum_{i=1}^{n} \frac{2}{n} \cdot \frac{4i^2}{n^2} = \frac{8}{n^3} \sum_{i=1}^{n} i^2.$$

Here, we recall the formula

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}.$$

This then implies

$$\begin{aligned}\lim_{n\to\infty} \frac{8}{n^3}\sum_{i=1}^{n} i^2 &= \lim_{n\to\infty} \frac{8}{n^3} \cdot \frac{n(n+1)(2n+1)}{6} \\ &= \frac{4}{3} \cdot \lim_{n\to\infty} \frac{2n^3+3n^2+n}{n^3} \\ &= \frac{4}{3} \cdot \lim_{n\to\infty} \left(2 + \frac{3}{n} + \frac{1}{n^2}\right) = \frac{4}{3} \cdot 2 = \frac{8}{3}.\end{aligned}$$

Thus, the limit exists, and we get that the area under the graph of f is $8/3$.

Example 5.11. We consider $f(x) = e^x$ for $x \in [0, 2]$. For any $n \in \mathbb{N}$, a right Riemann sum is

$$\sum_{i=1}^{n} \frac{2}{n} f\left(\frac{2i}{n}\right) = \frac{2}{n}\sum_{i=1}^{n} e^{2i/n}.$$

Note that this is a geometric series, and we recall that

$$a + ar + ar^2 + \cdots + ar^{n-1} = \sum_{i=0}^{n-1} ar^i = a \cdot \frac{r^n - 1}{r-1}.$$

This implies

$$\frac{2}{n}\sum_{i=1}^{n} e^{2i/n} = \frac{2}{n} \cdot e^{2/n} \cdot \frac{e^{2/n \cdot n} - 1}{e^{2/n} - 1} = \frac{2}{n} \cdot \frac{e^2 - 1}{1 - e^{-2/n}}$$

and hence

$$\lim_{n\to\infty} \frac{2}{n} \cdot \frac{e^2-1}{1-e^{-2/n}} = (e^2 - 1) \lim_{n\to\infty} \frac{2/n}{1 - e^{-2/n}}.$$

To evaluate the limit, we can look at the "stronger" limit

$$\lim_{x\to\infty} \frac{2/x}{1-e^{-2/x}},$$

which is the same as

$$\lim_{x\to 0^+} \frac{x}{1-e^{-x}} = \lim_{x\to 0^+} \frac{1}{e^{-x}} = 1$$

by L'Hôpital's rule. Thus, we finally get that the area is $e^2 - 1$.

We remark that the limit of a sequence and the limit of a function are essentially two different notions. However, when the sequence comes from a function evaluated at discrete points $1, 2, 3, \cdots, n, \cdots$, then the existence of this function limit will imply the existence of the sequential limit. That is, if there is a function $f\colon (0, \infty) \to \mathbb{R}$ such that

$$\lim_{x\to\infty} f(x) = L$$

exists, then in particular, the limit

$$\lim_{n\to\infty} f(n) = L$$

also exists. In fact, this implies that any limit of $f(x_n)$ exists as $n \to \infty$ if $\lim\limits_{n\to\infty} x_n = \infty$.

5.3. **Definite integral.** We define the notion of signed area in the previous section using a particular way of dividing the region under the graph of a continuous function. In general, if the function is not continuous, this may not give us a good way to define its area.

Example 5.12. We look at a pathological example. Let $f\colon [0,1] \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 0 & \text{if } x \in \mathbb{Q} \\ 1 & \text{if } x \notin \mathbb{Q} \end{cases},$$

where we recall that $\mathbb{Q}$ is the set of rational numbers (i.e., fractions). If we calculate a right Riemann sum using Definition 5.9, we only get 0. However, the set of irrational numbers is in fact much larger than the set of rational numbers. Thus, this region is not expected to have zero area.

In general, as we briefly, we will freely choose the sample point in the subintervals we divide into. We write down a formal definition, and then we will indicate that in most situations, we can still work with the method given in Definition 5.9.

Definition 5.13. Let $f\colon [a,b] \to \mathbb{R}$ be a function. For any $n \in \mathbb{N}$, we choose a sample point

$$x_i^n \in \left[a + (i-1) \cdot \frac{b-a}{n}, a + i \cdot \frac{b-a}{n}\right]$$

for each $i = 1, \cdots, n$. We then define the **definite integral** of f from a to b to be

$$\int_a^b f(x)dx := \lim_{n\to\infty} \sum_{i=1}^n \frac{b-a}{n} \cdot f(x_i^n)$$

if the limit exists. That is, given any restriction $R > 0$, there exists $N \in \mathbb{N}$ such that

$$\left| \int_a^b f(x)dx - \sum_{i=1}^n \frac{b-a}{n} \cdot f(x_i^n) \right| < R$$

for $n \geq N$ and any sample points x_i^n's we choose. When this limit exists, we say f is **integrable** on $[a,b]$.

If we know a function is integrable, then we can choose any sample points to calculate the definite integral. In particular, we can go back to the right Riemann sum in Definition 5.9. We write down a useful fact that basically covers most of the situations we will encounter.

Theorem 5.14. Let f be a continuous function on a closed bounded interval $[a,b]$. Then f is integrable on $[a,b]$.

This is a result that cannot be proven with our knowledge in Calculus I (as in the case of the extreme value theorem), so we will just assume it. It then tells us that when we are working with a continuous function on a closed and bounded interval, it is integrable and we can use right Riemann sum to calculate the definite integral. This justifies the algorithm given in Definition 5.9.

From Theorem 5.14, we know that integrability is a weaker property than continuity. It is possible that an integrable function is not continuous.

We summarize some general properties of definite integrals for integrable functions.

Proposition 5.15. Let f and g be integrable functions on $[a,b]$.

(1) $\displaystyle\int_a^a f(x)dx = 0.$

(2) $\displaystyle\int_b^a f(x)dx = -\int_a^b f(x)dx.$

(3) $\displaystyle\int_a^b (f + c\,g)\,dx = \int_a^b f\,dx + c\int_a^b g\,dx$ for any $c \in \mathbb{R}$.

(4) If $f \geq g$ on $[a, b]$, then $\displaystyle\int_a^b f\,dx \geq \int_a^b g\,dx$. In particular,

- if $f \geq 0$ on $[a, b]$, then $\displaystyle\int_a^b f\,dx \geq 0$;
- if $m \leq f(x) \leq M$ for $x \in [a, b]$, then $\displaystyle m(b-a) \leq \int_a^b f\,dx \leq M(b-a)$.

Example 5.16. We see some concrete examples for continuous functions.

(1) We calculate $\int_0^2 (x^3 - x)dx$. From the linearity property, we can do the two integrals separately. First, by calculating the area of a triangle, we can directly get

$$\int_0^2 x\,dx = 2 \cdot 2 \cdot \frac{1}{2} = 2.$$

Next, we use right Riemann sums to calculate the integral of x^3. For $n \in \mathbb{N}$, we have

$$\sum_{i=1}^{n} \frac{2}{n} \cdot \left(\frac{2i}{n}\right)^3 = \frac{16}{n^4}\sum_{i=1}^{n} i^3.$$

We recall the formula

$$\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2,$$

so the Riemann sum is

$$\frac{16}{n^4} \cdot \left(\frac{n(n+1)}{2}\right)^2,$$

and hence

$$\int_0^2 x^3 dx = \lim_{n\to\infty} \frac{16}{n^4} \cdot \left(\frac{n(n+1)}{2}\right)^2 = 4 \cdot \lim_{n\to\infty} \frac{n^4 + 2n^3 + n^2}{n^4} = 4.$$

Therefore,

$$\int_0^2 \left(x^3 - x\right) dx = 4 - 2 = 2.$$

(2) We calculate $\int_0^2 \sqrt{16 - x^2}$. Note that from the form, one could expect that the Riemann sum may not be easy to calculate. However, we note that the graph of $\sqrt{16 - x^2}$ on $[-4, 4]$ is a semicircle. Therefore, we can calculate the area directly. We can first calculate

$$\int_2^4 \sqrt{16 - x^2}dx = \frac{1}{6} \cdot 4^2\pi - \frac{1}{2} \cdot 2 \cdot 2\sqrt{3} = \frac{8}{3}\pi - 2\sqrt{3}.$$

Thus,

$$\int_0^2 \sqrt{16-x^2}dx = \int_0^4 \sqrt{16-x^2}dx - \int_2^4 \sqrt{16-x^2}dx = \frac{1}{4}\cdot 4^2\pi - \left(\frac{8}{3}\pi - 2\sqrt{3}\right) = \frac{4}{3}\pi + 2\sqrt{3}.$$

Example 5.17. We end the section by mentioning two examples that are not integrable.

(1) We look at the function given in Example 5.12. That is, we consider $f\colon [0,1] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 0 & \text{if } x \in \mathbb{Q} \\ 1 & \text{if } x \notin \mathbb{Q} \end{cases}.$$

Note that no matter how large a given positive integer n is, we can find sample points x_i^n and y_i^n each subinterval, that is,

$$x_i^n, y_i^n \in \left[\frac{i-1}{n}, \frac{i}{n}\right]$$

such that x_i^n is a rational number while y_i^n is an irrational number. Therefore, we get

$$\sum_{i=1}^n \frac{1}{n} f(x_i^n) = 1$$

while

$$\sum_{i=1}^n \frac{1}{n} f(y_i^n) = 0.$$

As a result, the limit does not exist when $n \to \infty$, and hence the function is not integrable on $[0,1]$.

(2) We look at the function $g\colon [0,1] \to \mathbb{R}$ defined by

$$g(x) = \begin{cases} \frac{1}{x} & \text{for } x \in (0,1] \\ 0 & \text{for } x = 0 \end{cases}.$$

Now for each $n \in \mathbb{N}$, we can choose

$$x_1^n = \frac{1}{n^2}$$

as the first sample point, from which the first term in a Riemann sum will be

$$\frac{1}{n}\cdot g\left(x_1^n\right) = \frac{1}{n}\cdot\frac{1}{x_1^n} = \frac{1}{n}\cdot n^2 = n.$$

Note that the rest of the terms in such a Riemann sum are positive, so no matter what sample points x_i^n's we choose for the rest, we get

$$\sum_{i=1}^n \frac{1}{n} g(x_i^n) > \frac{1}{n}\cdot g\left(x_1^n\right) = n.$$

As a result, the limit does not exist because $\lim_{n\to\infty} n = \infty$ does not exist.

We know that to construct a non-integrable example, based on Theorem 5.14, one has to work on functions that are not continuous. For the first example, the function oscillates infinitely between 0 and 1 everywhere, and hence there's no way to "pin down" a single area using rectangles. For the second example, the function blows up too fast as $x \to 0^+$.

5.4. **Fundamental theorem of calculus.** We will talk about the most important theorem in calculus. It is the fundamental theorem of calculus. It has two versions, and it connects indefinite integrals and definite integrals. People call the two versions FTC 1 and FTC 2, and we will call them FTC in differential version and FTC in integral version.

Theorem 5.18 (FTC, differential version)**.** Let f be a continuous function on $[a, b]$. Consider $g\colon [a, b] \to \mathbb{R}$ defined by

$$g(x) := \int_a^x f(t)dt.$$

Then g is continuous on $[a, b]$, is differentiable on (a, b), and it satisfies $g' = f$.

First, the function g is well-defined by Theorem 5.14. We've also seen why this differential form of FTC is expected, since for small h,

$$\frac{g(x+h)-g(x)}{h} = \frac{1}{h}\int_x^{x+h} f(x)dx \approx \frac{1}{h}\cdot\frac{(f(x)+f(x+h))\,h}{2} \approx f(x).$$

We will make this rigorous in the proof.

Proof. About the differentiability, as mentioned above, we want to estimate

$$\frac{g(x+h)-g(x)}{h} = \frac{1}{h}\int_x^{x+h} f(t)dt.$$

To estimate this integral,for a fixed $x \in (a, b)$ we consider two functions

$$M(h) := \max_{[x-|h|,x+|h|]} f \text{ and } m(h) := \min_{[x-|h|,x+|h|]} f,$$

which exist (for $|h|$ small) by the extreme value theorem. The continuity of f implies that

$$\lim_{h\to 0} M(h) = \lim_{h\to 0} m(h) = f(x). \tag{5.19}$$

Now we use the two functions to estimate the quotient. By the properties in Proposition 5.15, we have

$$\frac{1}{h}\cdot m(h)\cdot h \le \frac{1}{h}\int_x^{x+h} f(t)dt \le \frac{1}{h}\cdot M(h)\cdot h,$$

which means

$$m(h) \le \frac{1}{h}\int_x^{x+h} f(t)dt \le M(h).$$

Note that these inequalities still hold when $h < 0$. Thus, by (5.19) and the squeeze theorem, we conclude

$$\lim_{h\to 0}\frac{g(x+h)-g(x)}{h} = \lim_{h\to 0}\frac{1}{h}\int_x^{x+h} f(t)dt = f(x),$$

and this proves the differentiability of g and $g' = f$ on (a, b).

About the continuity (at the endpoints), it is equivalent to show

$$\lim_{h\to 0^+}\int_a^{a+h} f(x)dx = \lim_{h\to 0^+}\int_{b-h}^{b} f(x)dx = 0.$$

This can be shown by looking at the same functions M and m as above but the extrema are taken over $[a, a+h]$ and $[b-h, b]$. $\square$

The differential version of FTC implies the following integral version. In practice, we may use the integral version more often.

Theorem 5.20 (FTC, integral version)**.** Let f be a continuous function on $[a, b]$. If F is an antiderivative of f, then

$$\int_a^b f(x)dx = F(b) - F(a).$$

Here, we say F is an antiderivative of f if F is a continuous function on $[a, b]$ such that$F' = f$ on (a, b).

Note that in this integral version, we can choose an arbitrary antiderivative F of f. In particular, it tells us that we can use any antiderivatives to evaluate a definite integral of f. Its proof, however, directly follows from our characterization of antiderivatives on a single interval, that is, Lemma 5.3. Note that the lemma also works for antiderivatives on a closed and bounded interval.

Proof. By the differential version of FTC (Theorem 5.18), we know that the function $g\colon [a, b] \to \mathbb{R}$ defined by

$$g(x) := \int_a^x f(t)dt$$

is an antiderivative of f. By lemma 5.3, we can find $C \in \mathbb{R}$ such that

$$g(x) = F(x) + C.$$

This implies

$$\begin{aligned}\int_a^b f(x)dx = g(b) &= g(b) - g(a)\\ &= (F(b) + C) - (F(a) + C)\\ &= F(b) - F(a).\end{aligned}$$

This proves what we want. □

As mentioned, since we could use any antiderivative to evaluate a definite integral, we usually just take the one with constant term 0. We may sometimes use the notation

$$F(x)|_a^b := F(b) - F(a).$$

Example 5.21. Based on the fundamental theorem of calculus, we can now evaluate most of the definite integrals we did before in a simpler way using their antiderivatives. We now redo all of them.

(1) We look at $\int_0^2 x^2 dx$. An antiderivative of x^2 is $\frac{1}{3}x^3$, so

$$\int_0^2 x^2 dx = \frac{1}{3}x^3|_0^2 = \frac{1}{3}(8 - 0) = \frac{8}{3}.$$

(2) We look at $\int_0^2 e^x dx$. An antiderivative of e^x is itself, so

$$\int_0^2 e^x dx = e^x|_0^2 = e^2 - 1.$$

(3) We look at $\int_0^2 \left(x^3 - x\right) dx$. An antiderivative of $x^3 - x$ is $\dfrac{1}{4}x^4 - \dfrac{1}{2}x^2$, so

$$\int_0^2 \left(x^3 - x\right) dx = \left(\frac{1}{4}x^4 - \frac{1}{2}x^2\right)|_0^2 = 2.$$

(4) For $\int_0^2 \sqrt{16 - x^2}dx$, an antiderivative of the integrand is not easy to find. We will do this later if there is time.

Besides the previous examples, which all rely on certain summation formulas, we can now evaluate all definite derivatives as long as we know antiderivatives (or indefinite integrals) of the integrands.

Example 5.22. We see some examples that are not easy to deal with using Riemann sums.

(1) We look at $\int_0^\pi \sin x \, dx$. An antiderivative of $\sin x$ is $-\cos x$, so

$$\int_0^\pi \sin x \, dx = (-\cos x)|_0^\pi = 1 - (-1) = 2.$$

One can see this by looking at a particle moving from $(1, 0)$ to $(-1, 0)$ on the semicircle with unit speed. The horizontally projected distance is exactly this integral, which is 2.

(2) We look at $\int_1^4 \frac{1}{x} dx$. An antiderivative of $1/x$ is $\log x$ when $x > 0$, so

$$\int_1^4 \frac{1}{x} dx = \log x|_1^4 = \log 4 - \log 1 = \log 4.$$

(3) We look at $\int_0^4 \sqrt{x}dx$. An antiderivative of $\sqrt{x}$ is $\frac{2}{3}x^{3/2}$, so

$$\int_0^4 \sqrt{x}dx = \frac{2}{3}x^{3/2}|_0^4 = \frac{2}{3} \cdot 8 = \frac{16}{3}.$$

The fundamental theorem of calculus also allows us to differentiate a function defined using definite integrals.

(4) Let $g(x) := \displaystyle\int_0^{x^2} \sin^4 t \, dt$. We want to differentiate $g(x)$. It is not easy to find an antiderivative of $\sin^4 x$, so using the integral version of FTC is not a good strategy. We can however use the differential version along with the chain rule. To be specific, we could consider $F(u) := \int_0^u \sin^4 t dt$ and $G(x) := x^2$, which implies $g(x) = F \circ G(x)$. Thus,

$$g'(x) = F'(G(x)) \cdot G'(x) = \sin^4 t|_{t=x^2} \cdot 2x = \sin^4 x^2 \cdot 2x.$$

When one wants to use the theorem, it is important to make sure that the integrand is continuous on the whole closed interval.

(5) We look at $\int_{-1}^1 \frac{1}{x} dx$. One is not allowed to use the FTC to conclude

$$\int_{-1}^1 \frac{1}{x} dx = \log|x| \,|_{-1}^1 = \log 1 - \log 1 = 0$$

since $1/x$ is not a continuous function on $[-1, 1]$. In fact, by Example 5.17(2), this integral does not exist.

5.5. **Substitution Rule.** From the fundamental theorem of calculus, we know that to evaluate a definite integral, it is sufficient to know how to find the indefinite integral, i.e., an antiderivative. There are a few useful techniques to find antiderivatives. Here, we talk about the one that is based on the chain rule.

Recall that the chain rule basically says

$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x).$$

If we take the indefinite integrals of both sides, we get

$$f \circ g(x) + C = \int f'(g(x)) \cdot g'(x) dx,$$

where we obtain the left hand side by the fundamental theorem of calculus. This tells us that if we could recognize a derivative $g'(x)$ in an integrand, then we could use it as a new variable to integrate $f'(g(x))$. In other words, if we consider a new variable $u = g(x)$, then we can view the process above as

$$\int f'(g(x)) g'(x) dx = \int f'(u) \cdot \frac{du}{dx} dx = \int f'(u) du = f(u) + C.$$

Thus, this method is also called the u-substitution method.

Example 5.23. In practice, one could choose to write down the new function u or not. We see some examples.

(1) We look at $\int \cos 2x \, dx$. We have seen that it is $\frac{1}{2} \sin 2x + C$. With the substitution method, we can consider $u = 2x$ since it appears in cos. Then we get $du = 2dx$, so we can do it by

$$\int \cos 2x \, dx = \int \cos u \cdot \frac{1}{2} du = \frac{1}{2} \sin u + C = \frac{1}{2} \sin 2x + C.$$

If one is familiar with this process, one can do it directly by

$$\int \cos 2x \, dx = \int \cos 2x \cdot \frac{1}{2} d(2x) = \frac{1}{2} \sin 2x + C.$$

(2) We look at $\int x^4 e^{x^5+3} dx$. Since $x^5 + 3$ appears in the exponent, we consider a new variable $u = x^5 + 3$ which gives $du = 5x^4 dx$. Thus,

$$\int x^4 e^{x^5+3} dx = \int e^u \cdot \frac{1}{5} du = \frac{1}{5} e^u + C = \frac{1}{5} e^{x^5+3} + C.$$

(3) We look at $\int \sqrt{1+x^2} x^3 dx$. Since $1 + x^2$ appears in the square root, we consider $u = 1 + x^2$, and hence $du = 2xdx$. Thus,

$$\begin{aligned}\int \sqrt{1+x^2} x^3 dx = \int \sqrt{u} \cdot \frac{1}{2}(u-1) du &= \frac{1}{2} \int \left(u^{3/2} - u^{1/2} \right) \\ &= \frac{1}{2} \left(\frac{2}{5} u^{5/2} - \frac{2}{3} u^{3/2} \right) + C \\ &= \frac{1}{5} \left(1+x^2\right)^{5/2} - \frac{1}{3} \left(1+x^2\right)^{3/2} + C.\end{aligned}$$

(4) We look at $\int \tan x \, dx$ when $\cos x > 0$. Note that $\tan x = \sin x / \cos x$ and we know $(\cos x)' = -\sin x$. Thus, we consider $u = \cos x$ and hence $du = -\sin x dx$. We then get

$$\int \tan x \, dx = \int \frac{-1}{u} du = -\log |u| + C = -\log \cos x + C.$$

Next, we talk about how to use the substitution technique to evaluate a definite integral. If one uses a new variable u, then the only additional thing to do is to get new upper and lower bounds of the integral. This still follows from the chain rule, which says $(f \circ g)'(x) = f'(g(x)) \cdot g'(x)$, and hence for any $a, b \in \mathbb{R}$,

$$f(g(b)) - f(g(a)) = \int_a^b f'(g(x)) \cdot g'(x) dx.$$

Thus, if one considers a new variable $u = g(x)$, then the new upper and lower bounds are $u_1 = g(a)$ and $u_2 = g(b)$, and we have

$$f(u_2) - f(u_1) = \int_a^b f'(g(x)) \cdot g'(x) dx = \int_{u_1}^{u_2} f'(u) du.$$

Example 5.24. We see some examples.

(1) We look at $\int_0^{\pi/2} \cos 2x \, dx$. As in its indefinite integral, we let $u = 2x$, which implies $du = 2dx$ and the new bounds are $2 \cdot 0 = 0$ and $2 \cdot (\pi/2) = \pi$. Thus, we get

$$\int_0^{\pi/2} \cos 2x \, dx = \int_0^{\pi} \cos u \cdot \frac{1}{2} u = \frac{1}{2} \sin u|_0^{\pi} = 0 - 0 = 0.$$

(2) We look at $\int_1^{e^2} \frac{\log x}{x} dx$. Since $(\log x)' = 1/x$, we consider $u = \log x$, which implies $du = dx/x$ and the new bounds are $\log 1 = 0$ and $\log e^2 = 2$. Thus,

$$\int_1^{e^2} \frac{\log x}{x} dx = \int_0^2 u \, du = \frac{1}{2} u^2|_0^2 = 2.$$

(3) We look at $\int_0^3 (2x+1)^2 dx$. One way of course is to expand the squared term. If we let $u = 2x + 1$, then we get $du = 2dx$ and the new bounds are 1 and 7. Thus,

$$\int_0^3 (2x+1)^2 dx = \int_1^7 u^2 \cdot \frac{1}{2} du = \frac{1}{6} u^3|_1^7 = \frac{342}{6} = 57.$$

(4) We look at $\int_{-a}^{a} f(x) dx$ for an even function f and $a > 0$. For the negative part from $-a$ to 0, after taking $u = -x$, we get

$$\int_{-a}^0 f(x) dx = \int_{-a}^0 f(-x) dx = \int_a^0 f(u) \cdot (-1) du = \int_0^a f(u) du,$$

and hence

$$\int_{-a}^a f(x) dx = \int_0^a f(u) du + \int_0^a f(x) dx = 2 \int_0^a f(x) dx.$$

Thus, we only have to calculate one part of it.

(5) We look at $\int_{-a}^{a} f(x) dx$ for an odd function f and $a > 0$. For the negative part from $-a$ to 0, after taking $u = -x$, we get

$$\int_{-a}^0 f(x) dx = \int_{-a}^0 -f(-x) dx = \int_a^0 f(u) du = -\int_0^a f(u) du,$$

and hence

$$\int_{-a}^a f(x) dx = -\int_0^a f(u) du + \int_0^a f(x) dx.$$

Thus, the two parts cancel.

5.6. **Integration by parts.** We learn the u-substitution from the chain rule. Next, we mention that from the product rule (or the Leibniz rule), we can also learn something. From

$$(fg)'(x) = f'(x)g(x) + f(x)g'(x),$$

we can integrate and get

$$f(x)g(x) + C = \int f'(x)g(x)dx + \int f(x)g'(x)dx.$$

We can rewrite it as

$$\int f'(x)g(x)dx = f(x)g(x) - \int f(x)g'(x)dx.$$

This doesn't seem very amazing, but the point is, if one could integrate f first, moving the derivative from f to g may simplify the integrand a lot. This technique is called **integration by parts**, and it is one of the most important techniques in mathematics.

Example 5.25. We see some examples.

(1) We look at $\int \log x\, dx$. We view it as $1 \cdot \log x$ and integrate 1 first. This leads to

$$\begin{aligned}\int \log x\, dx &= \int 1 \cdot \log x\, dx \\ &= x \log x - \int x \cdot \frac{1}{x} dx \\ &= x \log x - \int dx = x \log x - x + C.\end{aligned}$$

(2) We look at $\int x \sin x\, dx$. Integrating x first is not very helpful, since $\sin x$ will become $\cos x$. Thus, we try the other way and get

$$\begin{aligned}\int x \sin x\, dx &= -x \cos x + \int \cos x\, dx \\ &= -x \cos x + \sin x + C.\end{aligned}$$

(3) We look at $\int x^2 e^x dx$. Again, integrating x^2 is not helpful, so we integrate e^x (twice) and get

$$\begin{aligned}\int x^2 e^x dx &= e^x x^2 - \int e^x \cdot 2x\, dx \\ &= e^x x^2 - 2e^x x + \int 2e^x dx \\ &= e^x x^2 - 2e^x x + 2e^x + C.\end{aligned}$$

(4) We look at $\int e^x \sin x\, dx$. Starting from either of them is the same. If we start from integrating e^x, we get

$$\begin{aligned}\int e^x \sin x\, dx &= e^x \sin x - \int e^x \cos x\, dx \\ &= e^x \sin x - e^x \cos x - \int e^x \sin x\, dx.\end{aligned}$$

Note that we get the same term at the end. Thus, rearranging and dividing them by 2, we get $\displaystyle\int e^x \sin x\, dx = \frac{1}{2} (e^x \sin x - e^x \cos x) + C$.

5.7. **Areas between curves.** The integral technique to find the area under a curve can be generalized to find the area between two curves. In fact, a definite integral can be viewed as the "signed" area between the curve and the x-axis.

In general, if f and g are two functions on $[a, b]$ with $f \geq g$, then the area between the graphs of f and g is

$$\int_a^b (f(x) - g(x))\, dx.$$

If, more generally, $f - g$ is not non-negative, then if one wants to calculate the actual area (instead of the signed area) between the two graphs, then one evaluates

$$\int_a^b |f(x) - g(x)|\, dx.$$

Example 5.26. We see some examples.

(1) We look at the region bounded between the graphs of e^x and x over $[0, 1]$. On this interval, $e^x > x$, so the area of the region is

$$\int_0^1 (e^x - x)\, dx = \left(e^x - \frac{1}{2}x^2\right)|_0^1 = \left(e - \frac{1}{2}\right) - 1 = e - \frac{3}{2}.$$

(2) We look at the region between the graphs of x^2 and $2x$. First, we find their intersection points by solving $x^2 = 2x$, which implies $x = 0$ or 2. Next, we find that $x^2 \leq 2x$ on $[0, 2]$. Thus, the area of the region between them is

$$\int_0^2 (2x - x^2)\, dx = \left(x^2 - \frac{1}{3}x^3\right)|_0^2 = \frac{4}{3}.$$

(3) We look at the region between $\sin x$ and $\cos x$ over $[0, \pi]$. Since $\sin x - \cos x$ does not have a sign, this area should be

$$\int_0^\pi |\sin x - \cos x|\, dx$$

and we have to determine the sign of the difference on different subintervals. We first solve $\sin x = \cos x$ to see where they intersect. This happens when $x = \pi/4$. For $x \in [0, \pi/4]$, $\sin x \leq \cos x$, while for $x \in [\pi/4, \pi]$, $\sin x \geq \cos x$. Thus, the area between them is

$$\begin{aligned}
&\int_0^{\pi/4} (\cos x - \sin x)\, dx + \int_{\pi/4}^{\pi} (\sin x - \cos x)\, dx \\
&= (\sin x + \cos x)\,|_0^{\pi/4} + (-\cos x - \sin x)\,|_{\pi/4}^{\pi} \\
&= \left(\sqrt{2} - 1\right) + \left(1 + \sqrt{2}\right) = 2\sqrt{2}.
\end{aligned}$$

The same technique can also be used to look at more general region. For example, if the curves are the graphs of functions in y, then we can integrate in the y-direction.

(4) We look at the region between the two curves $y = x-1$ and $y^2 = 2x+6$. We can't write both of them as functions in x, but they are both functions in y. First, we find their intersections, which are solutions to

$$(x - 1)^2 = 2x + 6.$$

This is equivalent to $x^2 - 4x - 5 = 0$, so $x = 5$ or -1. Hence the curves intersect at $(5,4)$ and $(-1,-2)$. We now write them as the graphs of $f_1(y) = y + 1$ and $f_2(y) = \frac{1}{2}(y^2 - 6)$. For $y \in [-2,4]$, we have

$$y + 1 \geq \frac{1}{2}\left(y^2 - 6\right),$$

so the area of the region between them is

$$\int_{-2}^{4} \left(y + 1 - \frac{1}{2}\left(y^2 - 6\right)\right) dy = \left(\frac{1}{2}y^2 + 4y - \frac{1}{6}y^3\right) \Big|_{-2}^{4} = \frac{40}{3} - \left(-\frac{14}{3}\right) = 18.$$

We can also consider more than one curve.

(5) We consider the region bounded among the three curves $y = 1/x$, $y = x$, and $y = x/4$ in the first quadrant. We first find out their intersections. For $y = 1/x$ and $y = x$, we solve $1/x = x$, which means

$$x^2 = 1,$$

so $x = \pm 1$. For $y = 1/x$ and $y = x/4$, we solve $1/x = x/4$, which means

$$x^2 = 4,$$

so $x = \pm 2$. For $y = x$ and $y = x/4$, we solve $x = x/4$, so $x = 0$. Thus, in the first quadrant, any two of them intersect at $(1,1), (2,2)$, and $(0,0)$ respectively.

It is helpful to draw a picture. From $x = 0$ to $x = 1$, we see the two curves $y = x$ and $y = x/4$ with $x/4 \leq x$. From $x = 1$ to $x = 2$, we see the two curves $y = 1/x$ and $y = x/4$ with $x/4 \leq 1/x$. Thus, the region consists of these two parts and the area is

$$\int_0^1 \left(x - \frac{x}{4}\right) dx + \int_1^2 \left(\frac{1}{x} - \frac{x}{4}\right) dx = \frac{3}{8}x^2\Big|_0^1 + \left(\log x - \frac{1}{8}x^2\right)\Big|_1^2 = \frac{3}{8} + \log 2 - \frac{3}{8} = \log 2.$$

The techniques involved in these examples in fact allow us to calculate/estimate any reasonably regular regions. We can usually divide a given region into some smaller ones whose boundaries are given by the graphs of some functions. With a computer, we can then estimate the area of such a region up to an error which could be as small as we want.

5.8. **Average.** We mention an immediate application of integration. This is inspired from taking the average of finitely many numbers. If we are given data that consist of some numbers $a_1, \cdots, a_n$, then the average of them is

$$\frac{1}{n}\left(a_1 + \cdots + a_n\right) = \frac{1}{n}\sum_{i=1}^{n} a_i. \tag{5.27}$$

We can apply this notion to a "continuous set of data." In the case when the data are given by a function, we could use the integral of it to replace the previous notion of taking the sum. We then have the following definition.

Definition 5.28. Let $f\colon [a,b] \to \mathbb{R}$ be an integrable function. Then the **average (value)** of f on $[a,b]$ is defined to be

$$f_{\text{avg}} := \frac{1}{b-a}\int_a^b f(x)\, dx.$$

We remark that, in fact, the traditional way of taking average in (5.27) can also be interpreted using integration. One just has to use a different way to "measure the area of a set."

Example 5.29. We see some examples.

(1) Let $f(x) = \sin x$. Then the average of f on $[0, \pi]$ is
$$\frac{1}{\pi}\int_0^\pi \sin x\, dx = \frac{1}{\pi}\left(-\cos x\right)|_0^\pi = \frac{1}{\pi}\left(1-(-1)\right) = \frac{2}{\pi}.$$

(2) Let $f(x) = \sin^2 x$. Then the average of f on $[0, \pi]$ is
$$\frac{1}{\pi}\int_0^\pi \sin^2 x\, dx = \frac{1}{\pi}\int_0^\pi \frac{1-\cos 2x}{2} dx = \frac{1}{2} - \frac{1}{\pi}\cdot\frac{1}{4}\sin 2x|_0^\pi = \frac{1}{2}.$$

(3) If $f\colon [a, b] \to \mathbb{R}$ is a continuous function, then one can use the fundamental theorem of calculus and the mean value theorem to deduce that there exists $c \in (a, b)$ such that
$$f(c) = \frac{1}{b-a}\int_a^b f(x)\, dx = f_{\text{avg}}.$$

5.9. **Volumes of solids of revolution.** The final application that we want to see is how to use the integration technique to look at a shape with a rotational symmetry. That is, we look at a function $f\colon [a, b] \to \mathbb{R}$ and "rotate" its graph about the x-axis. It results in a "two-dimensional" surface, and we can use integration to calculate the volume of the region it encloses. Here, we use finitely many flat cylinders to approximate the region, and hence for any $n \in \mathbb{N}$, we get a Riemann sum
$$\sum_{i=1}^n \pi\cdot f^2\left(a + i\cdot\frac{b-a}{n}\right)\cdot\frac{b-a}{n}.$$
Here, $f\left(a+i\cdot\frac{b-a}{n}\right)$ is the radius of the cylinder and $\frac{b-a}{n}$ is its height. Therefore, if f is a continuous function, the limit of such Riemann sums is
$$\int_a^b \pi\cdot f(x)^2\, dx.$$

Example 5.30. We look at two examples.

(1) Consider the sphere of radius r centered at the origin. It bounds a solid ball. The sphere can be obtained by rotating the graph of
$$f(x) = \sqrt{r^2 - x^2}$$
for $x \in [-r, r]$. Thus, the volume of the ball is
$$\int_{-r}^r \pi\cdot\sqrt{r^2-x^2}^2\, dx = \pi\int_{-r}^r \left(r^2 - x^2\right) dx = \pi\left(r^2x - \frac{1}{3}x^3\right)|_{-r}^r = \pi\left(\frac{2}{3}r^3 - \left(-\frac{2}{3}r^3\right)\right) = \frac{4}{3}\pi r^3.$$

(2) Consider a cone with base radius r and height h. Such a cone can be obtained by rotating the graph of
$$f(x) = \frac{r}{h}x$$
for $x \in [0, h]$. Thus, the volume of the (solid) cone is
$$\int_0^h \pi\cdot\left(\frac{r}{h}x\right)^2 dx = \frac{\pi r^2}{h^2}\int_0^h x^2\, dx = \frac{\pi r^2}{h^2}\cdot\frac{h^3}{3} = \frac{1}{3}\pi r^2 h.$$