CALCULUS 1, NOTE
APPLICATION OF DIFFERENTIATION

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4. **Applications**

In the following lectures, we will use differentiation to study different properties of functions and explore some applications.

4.1. **Related rates.** In real-world situations, several quantities change over time and are connected to one another. For example, consider the balloon again. The rate at which the volume V changes depends on the rate at which the radius r changes. In this situation, $\frac{dV}{dt}$ and $\frac{dr}{dt}$ are called **related rates** because the variables V and r are related. To calculate them, the chain rule is often used.

Example 4.1. We will examine several examples of quantities that vary with time and learn how to determine one rate of change when another rate of change is known. We first see some direct examples given by explicit examples, and then we will see some other examples coming from the real world.

(1) Suppose $x(t)$ and $y(t)$ are two differentiable functions related by
$$y = x^2 + 3.$$
Suppose $x'(t) = 5$ when $x = 2$. We want to find $y'(t)$ when $x = 2$. To do this, we differentiate the relation and get
$$y'(t) = 2x(t) \cdot x'(t),$$
so when $x = 2$,
$$y'(t) = 2 \cdot 2 \cdot 5 = 20.$$

(2) Suppose $x(t), y(t)$, and $z(t)$ are differentiable functions related by
$$z^2 = x^2 + y^2.$$
Suppose when $(x, y) = (3, 4)$, we have $z(t) > 0$, $x'(t) = 2$, and $y'(t) = 6$. We want to find $z'(t)$ when $(x, y) = (3, 4)$. First, since $z(t) > 0$, we know $z(t) = 5$. We then differentiate the relation and get
$$2zz' = 2xx' + 2yy',$$
so when $(x, y) = (3, 4)$,
$$2 \cdot 5 \cdot z'(t) = 2 \cdot 3 \cdot 2 + 2 \cdot 4 \cdot 6.$$
This implies $z'(t) = 6$.

We mention a few examples that come from the real world.

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(3) A 10-meter ladder is leaning against a wall. Suppose the top of the ladder slides down the wall at a rate of 4 m/sec. We want to know how fast the bottom is moving along the ground when the bottom of the ladder is 5 meters from the wall. To this end, let the distance between the top and the corner be x meters and the distance between the bottom and the corner be y meters. They are both functions in time, and they are related by

$$x^2 + y^2 = 10^2 = 100.$$

Thus, differentiating it gives

$$2xx' + 2yy' = 0.$$

At this time, we know $x = 5$, $y = 5\sqrt{3}$, and $y' = -4$, so

$$x' = -\frac{yy'}{x} = -\frac{5\sqrt{3}\cdot(-4)}{5} = 4\sqrt{3}.$$

(4) A spherical balloon is being filled with air at the constant rate of 5 cm^3/sec. We want to know how fast the radius is increasing when the radius is 5 cm. Suppose the volume is V and the radius is r. They are both functions in time and they are related by

$$V = \frac{4}{3}\pi r^3.$$

Thus, differentiating it gives

$$V' = 4\pi r^2 r'.$$

At this time, we know $V' = 5$ and $r = 5$, so

$$r' = \frac{V'}{4\pi r^2} = \frac{5}{100\pi} = \frac{1}{20\pi}.$$

4.2. **Linear approximation.** The next thing we will do is to make it rigorous what we mean by saying that a differentiable function can be approximated by its tangent. Recall that a function $f\colon I \to \mathbb{R}$ is differentiable at $x_0 \in I$ if

$$f'(x_0) = \lim_{h\to 0}\frac{f(x_0+h) - f(x_0)}{h}$$

exists. Thus, we can look at the function

$$o(h) := \begin{cases} \frac{f(x_0+h)-f(x_0)}{h} - f'(x_0) & \text{if } h \neq 0 \text{ with } |h| \text{ small} \\ 0 & \text{if } h = 0 \end{cases}.$$

The differentiability implies that $o(h)$ is continuous at $h = 0$. Therefore, we get the relation

$$f(x_0 + h) = f(x_0) + f'(x_0)h + o(h)h. \tag{4.2}$$

This relation in fact characterizes the differentiability of f at x_0. In fact, if we know

$$f(x_0 + h) = f(x_0) + ah + o(h)h$$

for a constant a and a continuous function $o(h)$ with $o(0) = 0$, then we can get

$$f'(x_0) = \lim_{h\to 0}\frac{f(x_0+h) - f(x_0)}{h} = \lim_{h\to 0}(a + o(h)) = a.$$

In practice, with (4.2), we can approximate $f(x_0 + h)$ by the linear polynomial $f(x_0) + f'(x_0)h$, since the error $o(h)h$ is very small. That is, when $h \approx 0$, we get

$$f(x_0 + h) \approx f(x_0) + f'(x_0)h.$$

This is the most common form we would use when we want to approximate a function by its tangent.

Example 4.3. We see a few examples of linear approximation.

(1) If we know $\sqrt{x}$, then we can approximate $\sqrt{x+1}$ especially when x is much larger than 1. If we let $f(x) = \sqrt{x}$, then $f'(x) = 1/\left(2\sqrt{x}\right)$ so
$$\sqrt{x+1} = f(x+1) \approx f(x) + \frac{1}{2\sqrt{x}} \cdot 1 = \sqrt{x} + \frac{1}{2\sqrt{x}}.$$
For example, one can approximate
$$\sqrt{90001} \approx \sqrt{90000} + \frac{1}{2\sqrt{90000}} = 300 + \frac{1}{600}.$$

(2) We can get an approximate value of $\sin 31°$. If we let $f(x) = \sin x$, then $f'(x) = \cos x$ so
$$f(x+h) \approx f(x) + \cos x \cdot h.$$
Thus, we get
$$\sin 31° = \sin\left(\frac{\pi}{6} + \frac{\pi}{180}\right) \approx \sin\frac{\pi}{6} + \cos\frac{\pi}{6} \cdot \frac{\pi}{180} = \frac{1}{2} + \frac{\sqrt{3}}{2} \cdot \frac{\pi}{180}.$$

We end this section by proving the chain rule using linear approximation. Recall that the chain rule says that if $f\colon I \to \mathbb{R}$ and $g\colon J \to \mathbb{R}$ are differentiable with $g(J) \subseteq I$, then
$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x).$$
We will use the characterization of derivatives given by (4.2) for g at x and f at $y = g(x)$. It implies that there are continuous functions o_1 and o_2 such that
$$\begin{aligned} f(y+k) &= f(y) + f'(y)k + o_1(k)k \text{ and} \\ g(x+h) &= g(x) + g'(x)h + o_2(h)h \end{aligned}$$
when h and k are small. We want to estimate $f(g(x+h)) - f(g(x))$, so we will plug in $k = g(x+h) - g(x)$. Remember that $y = g(x)$, so from the linear approximation of f, we get
$$f(g(x+h)) = f(g(x)) + f'(g(x))(g(x+h) - g(x)) + o_1(g(x+h) - g(x))(g(x+h) - g(x)).$$
We then use the linear approximation of g to rewrite it as
$$\begin{aligned} f(g(x+h)) &= f(g(x)) + f'(g(x))\left(g'(x)h + o_2(h)h\right) + o_1\left(g'(x)h + o_2(h)h\right)\left(g'(x)h + o_2(h)h\right) \\ &= f(g(x)) + f'(g(x))\,g'(x)h + \left(f'(g(x))o_2(h) + o_1\left(g'(x)h + o_2(h)h\right)\left(g'(x) + o_2(h)\right)\right)h. \end{aligned}$$
In the remainder term, if we let
$$o(h) := f'(g(x))o_2(h) + o_1\left(g'(x)h + o_2(h)h\right)\left(g'(x) + o_2(h)\right),$$
then we get o is also continuous and
$$o(0) = f'(g(x))o_2(0) + o_1(0+0)\left(g'(x) + o_2(0)\right) = 0.$$
Thus, the relation
$$f(g(x+h)) = f(g(x)) + f'(g(x))\,g'(x)h + o(h)h$$
tells us that the derivative of $f \circ g$ is exactly $f'(g(x))\,g'(x)$.

4.3. **First derivative test.** The most commonly used property about the first derivative of a function is about maximization. We will talk about how to use first derivatives to narrow down the number of points that we have to test.

We first recall the notion of global maxima and minima for a function.

Definition 4.4. Let $f\colon I \to \mathbb{R}$ be a function and $x_0 \in I$.

(1) We say f attains its **(global) maximum** $f(x_0)$ at x_0 if $f(x) \leq f(x_0)$ for all $x \in I$.

(2) We say f attains its **(global) minimum** $f(x_0)$ at x_0 if $f(x) \geq f(x_0)$ for all $x \in I$.

This definition works for any real-valued functions, and hence I can be any subset of $\mathbb{R}$. Unfortunately, information about first derivatives usually only allows us to determine local behavior of a function. Thus, we also need to introduce a local version of these definitions.

Definition 4.5. Let $f\colon I \to \mathbb{R}$ be a function and $x_0 \in I$.

(1) We say f attains a **local maximum** $f(x_0)$ at x_0 if there exists $h > 0$ such that $f(x) \leq f(x_0)$ for all $x \in (x_0 - h, x_0 + h)$.

(2) We say f attains a **local minimum** $f(x_0)$ at x_0 if there exists $h > 0$ such that $f(x) \geq f(x_0)$ for all $x \in (x_0 - h, x_0 + h)$.

An informal way to describe these is that we only require the maximization/minimization for those points "near" the point we care about. Also note that by the definition, if $I = [a, b]$ is a bounded closed interval, then a local maximum/minimum never happens at the endpoints of I (i.e., a and b).

Example 4.6. We see some examples.

(1) Consider $f\colon [-2, 2] \to \mathbb{R}$ given by $f(x) = |x|$. Then the maximum of f is 2 (occuring at $x = \pm 2$) and the minimum of f is 0 (occuring at $x = 0$).

(2) Consider $f\colon \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2 + 5$. Then the minimum of f is $5 = f(0)$ and f does not have a maximum. $f(0)$ is also a local minimum of f near $x = 0$.

(3) Consider $f\colon [0, 1] \to \mathbb{R}$ given by $f(x) = \begin{cases} \frac{1}{x} & \text{if } x \in (0, 1] \\ 2 & \text{if } x = 0 \end{cases}$. Then the maximum of f does not exist, and the minimum of f is $1 = f(1)$.

From the examples, we know that maxima or minima may not exist. However, using more advanced tools in calculus, we can rigorously show that for a continuous function, if its domain is a bounded closed interval, then the function must attain its maximum and minimum. We state it as follows.

Theorem 4.7 (Extreme value theorem)**.** Let $a, b \in \mathbb{R}$ and let $f\colon [a, b] \to \mathbb{R}$ be a continuous function. Then f attains its global maximum and global minimum.

We note that both of the assumptions are necessary. If the function is not continuous, the theorem may fail; see Example 4.6(3). If the function is not defined on a bounded closed interval, the theorem may fail; see Example 4.6(2).

We will use Theorem 4.7 directly, and start to see how to find maxima or minima of a continuous funtion on a bounded closed interval. The first observation is that a local extremum happen at a differentiable point when the first derivative vanishes. An intuitive idea to see this is that at a local extremum point a function does not either go up or go down. We can prove this observation rigorously.

Proposition 4.8 (First derivative test, preliminary version)**.** Suppose $f\colon I \to \mathbb{R}$ attains a local extremum at $x_0 \in I$. If f is differentiable at x_0, then $f'(x_0) = 0$.

This preliminary version of the first derivative test is also referred to as Fermat's Theorem. We will see a more refined version of it later.

Proof. Assume f has a local minimum at x_0. This means that we can find $h > 0$ such that for any $x \in (x_0 - h, x_0 + h)$, $f(x) \geq f(x_0)$. Therefore, we know that for $x \in (x_0 - h, x_0)$,

$$\frac{f(x) - f(x_0)}{x - x_0} \leq 0,$$

so

$$\lim_{x \to x_0^-} \frac{f(x) - f(x_0)}{x - x_0} \leq 0. \tag{4.9}$$

On the other hand, for $x \in (x_0, x_0 + h)$,

$$\frac{f(x) - f(x_0)}{x - x_0} \geq 0,$$

so

$$\lim_{x \to x_0^+} \frac{f(x) - f(x_0)}{x - x_0} \geq 0. \tag{4.10}$$

The differentiability of f at x_0 implies that the two one-sided limits (4.9) and (4.10) are both equal to $f'(x_0)$, so $f'(x_0)$ must be zero. The case when f attains a local maximum is similar. $\square$

The proposition helps us locate local extrema since we then just have to check those points with either vanishing first derivatives or first derivatives not existing. We give this kind of points a name.

Definition 4.11. Let $f\colon I \to \mathbb{R}$ and $x_0 \in I$. We call x_0 a **critical point** of f if either $f'(x_0) = 0$ or $f'(x_0)$ does not exist.

Thus, if we are given a continuous function defined on a bounded closed interval, we only have to check the critical points of the function and the endpoints of the interval to find out the global maximum and minimum of the function.

Example 4.12. We work our two examples.

(1) Let $f\colon [0, 2] \to \mathbb{R}$ be given by $f(x) = x^3 - 2x^2 + 5$. To find its maximum and minimum, we have to look at the endpoints 0 and 2 and the critical points of f. Thus, we solve

$$0 = f'(x) = 3x^2 - 4x,$$

which has solutions $x = 0, 4/3$. 0 is an endpoint so it is already included. Thus, we evaluate f at these points and get

$$f(0) = 5, f(2) = 5, \text{ and} f\left(\frac{4}{3}\right) = \frac{103}{27}.$$

Hence, its maximum is 5 and minimum is 103/27.

(2) Let $f\colon [-2, 2] \to \mathbb{R}$ be given by $f(x) = x^{1/3}$. We look at

$$f'(x) = \frac{1}{3}x^{-2/3}.$$

It does not have a zero, but $f'(0)$ does not exist. Thus, we look at 0 and the endpoints 2 and -2 and get

$$f(0) = 0, f(2) = 2^{1/3}, \text{ and} f(-2) = -2^{1/3.}$$

Hence, its maximum is $2^{1/3}$ and minimum is $-2^{1/3}$. Note that $f(0)$ is NOT a local extremum.

As we see in the example above, a function may not attain a local extremum at a critical point. That is, the converse to Proposition 4.8 is not true.

Now, we state a more complete first derivative test that takes nearby points into account. It allows us to determine whether a function attains a local extremum at a critical point.

Theorem 4.13 (First derivative test)**.** Let $f\colon I \to \mathbb{R}$ be a continuous function and x_0 be a critical point of f. Suppose for some $h > 0$, f is differentiable on both $(x_0 - h, x_0)$ and $(x_0, x_0 + h)$.

(1) If $f' \geq 0$ on $(x_0 - h, x_0)$ and $f' \leq 0$ on $(x_0, x_0 + h)$, then f has a local maximum at x_0.

(2) If $f' \leq 0$ on $(x_0 - h, x_0)$ and $f' \geq 0$ on $(x_0, x_0 + h)$, then f has a local minimum at x_0.

(3) If f' is non-zero and has the same sign on both $(x_0 - h, x_0)$ and $(x_0, x_0 + h)$ (that is, $f' > 0$ on one interval and $f' < 0$ on the other), then f does not have a local extremum at x_0.

Note that the theorem works for any critical points of a function, including those at which the function is not differentiable. The theorem essentially follows from the following lemma that allows us to tell whether a function is increasing from its derivative.

Lemma 4.14. Let $f\colon I \to \mathbb{R}$ be a differentiable function.

(1) If $f' \geq 0$ on I, then f is non-decreasing. i.e. $f(a) \leq f(b)$ for any $a < b$ in I.

(2) If $f' \leq 0$ on I, then f is non-increasing. i.e., $f(a) \geq f(b)$ for any $a < b$ in I.

(3) If $f' > 0$ on I, then f is increasing.

(4) If $f' < 0$ on I, then f is decreasing.

The idea of the lemma basically follows from the definition of derivatives. If $f'(x_0) \geq 0$ for some $x_0 \in I$, then it implies

$$f(x_0 - h) \leq f(x_0) \leq f(x_0 + h)$$

for $h \neq 0$ with $|h|$ small. One can then put this local information together to conclude the lemma. Later, we will use the mean value theorem to give a more direct proof.

Example 4.15. Let $f\colon [0, 2\pi] \to \mathbb{R}$ be given by $x + \cos x$. Then $f'(x) = 1 - \sin x$, and hence its critical point is $\pi/2$. One can see that f' has the same sign on the two sides of $\pi/2$, so the first derivative test tells us that $\pi/2$ is not a local extremum point.

We will see more examples later.

4.4. **Second derivative test.** We learn the first derivative test (Theorem 4.13), which is general and applies to any critical points. The next thing we want to investigate is whether we can get more information or more practical criteria when the function is differentiable at a critical point.

We first look at the first situation in Theorem 4.13. Notice that if we particularly know that $f'(x_0) = 0$, f' is differentiable at x_0, and f' is decreasing on $(x_0 - h, x_0 + h)$, then the assumptions in (1) will be satisfied. In particular, by Lemma 4.14, if we assume f'' exists and is negative on $(x_0 - h, x_0 + h$, we can apply (1). Thus, we learn that the sign of f'' can tell us information that allows us to apply the first derivative test. Functions whose second derivatives have a sign can be interpreted using the secants of its graph. To this end, we consider the following definitions.

Definition 4.16. Let $f\colon I \to \mathbb{R}$ be a function on an open interval.

(1) We say f is **convex** if for any $x_1, x_2 \in I$ and $t \in [0, 1]$, we have

$$f\left(tx_1 + (1-t)x_2\right) \leq t \cdot f(x_1) + (1-t) \cdot f(x_2). \tag{4.17}$$

(2) We say f is **concave** if for any $x_1, x_2 \in I$ and $t \in [0, 1]$, we have

$$f\left(tx_1 + (1-t)x_2\right) \geq t \cdot f(x_1) + (1-t) \cdot f(x_2).$$

The definition looks technical, but it essentially means that for a convex function, the graph lies "below" the secant between any two points.

Example 4.18. We mention two examples. Both of them are clear from their graphs.

(1) The absolute value function $f(x) = |x|$ is convex on $\mathbb{R}$. When two points x_1 and x_2 have the same sign, (4.17) is an equality. If $x_1 < 0 < x_2$, then for $t \in [0, 1]$,

$$|tx_1 + (1-t)x_2| \leq |tx_1| + |(1-t)x_2| = t|x_1| + (1-t)|x_2|,$$

where we use the "triangle inequality" $|a + b| \leq |a| + |b|$.

(2) The function $f(x) = x^2$ is convex on $\mathbb{R}$. One can prove the convexity using (4.17) directly. Since this function is much more regular, we can use the information of its second derivative to determine its convexity as we will see later.

When a function $f\colon I \to \mathbb{R}$ is **twice differentiable**, which means that for any $x \in I$, both $f'(x)$ and $f''(x)$ exist, convexity and concavity can be determined using the sign of f''. We state this in the following lemma.

Lemma 4.19. Suppose $f\colon I \to \mathbb{R}$ is twice differentiable.

(1) If $f'' \geq 0$ on I, then f is convex on I.

(2) If $f'' \leq 0$ on I, then f is concave on I.

The idea of this criterion is that if $f'' \geq 0$, then f' will only go up, so a secant will never have a chance to go below the graph. Later, we will use the mean value theorem to give a proof.

Example 4.20. We see some examples.

(1) Let $f(x) = e^x$. Then $f''(x) = e^x > 0$, so f is convex on $\mathbb{R}$.

(2) Let $f(x) = \log x$. Then $f''(x) = -1/x^2 < 0$, so f is concave on $(0, \infty)$.

(3) Let $f(x) = x^4$. Then $f''(x) = 12x^2 \geq 0$, so f is convex on $\mathbb{R}$.

Example 4.21. A function may have different convexity properties when restricted to different smaller domains. We look at an example given by $f(x) = x^3$ for $x \in \mathbb{R}$. It is twice differentiable and we can find $f''(x) = 6x$. Thus, on $(0, \infty)$, f is convex, while on $(-\infty, 0)$, f is concave.

In Example 4.21, we note that f changes from being concave to being convex at $x = 0$. Such a point at which the convexity property of a function changes is called an **inflection point**.

We can now put things together and state the second derivative test.

Theorem 4.22 (Second derivative test)**.** Let $f\colon I \to \mathbb{R}$ be a twice differentiable function at x_0 be a critical point of f. Suppose f'' is continuous near x_0.

(1) If $f''(x_0) > 0$, then f has a local minimum at x_0.

(2) If $f''(x_0) < 0$, then f has a local maximum at x_0.

Proof. This essentially follows from the first derivative test (Theorem 4.13). We prove the first statement as an example. If $f''(x_0) > 0$, the continuity of f'' near x_0 implies that $f'' > 0$ on $(x_0 - h, x_0 + h)$ for some $h > 0$. Thus, f' is increasing on $(x_0 - h, x_0 + h)$ by Lemma 4.14. Since $f'(x_0) = 0$, we can now apply Theorem 4.13 to conclude that f has a local minimum at x_0. $\square$

Example 4.23. We see a few examples.

(1) First, we notice that when we use the second derivative test, it is necessary that the second derivative at a critical point has a sign. If f'' vanishes, then we cannot conclude anything directly. For example, if we look at $f(x) = x^3$, then 0 is a critical point with $f''(0) = 0$, but it is neither a local maximum nor a local minimum.

(2) Let $f(x) = x^4 - 18x^2$. We first find out its critical points. Since it's differentiable, we solve
$$0 = f'(x) = 4x^3 - 36x.$$
Thus, the critical points are $0, \pm 3$. We know
$$f''(x) = 12x^2 - 36.$$
When $x = 0$, $f''(0) = -36$, so f has a local maximum at 0.
When $x = \pm 3$, $f''(\pm 3) = 72$, so f has local minima at ± 3.

(3) Consider $f\colon [0, 2\pi] \to \mathbb{R}$ given by $f(x) = \sin x + \cos x$. Since $[0, 2\pi]$ is a bounded closed interval, we know that f attains its maximum and minimum on this interval. We first find out local minima and maxima. We look at
$$0 = f'(x) = \cos x - \sin x.$$
This happens when $\tan x = 1$, that is, $x = \pi/4$ or $3\pi/4$. Then we look at
$$f''(x) = -\sin x - \cos x.$$

Since $f''(\pi/4) = -\sqrt{2}$, f has a local maximum at $\pi/4$.
Since $f''(3\pi/4) = \sqrt{2}$, f has a local minimum at $3\pi/4$.
To find the global maximum and minimum, we also have to take the endpoints into account. They are $f(0) = 1$ and $f(2\pi) = 1$. Hence, those local extrema are exactly the global extrema of f respectively.

(4) We look at a special example $f(x) = x^6$ at $x = 0$. Note that 0 is a critical point of f. However, $f''(0) = 0$. Thus, we cannot conclude anything from the second derivative test. On the other hand, since $f'(x) = 6x^5 < 0$ when $x < 0$ and $f'(x) = 6x^5 > 0$ when $x > 0$, the first derivative test does apply and we can conclude that f has a local minimum at $x = 0$.

From Example 4 above, we know that though the second derivative test is convenient (since we only need to look at values at one point), there are situations where the test tells us nothing while the first derivative test is applicable. After all, Theorem 4.22 assumes a stronger condition (which implies that the first derivative never vanishes except at the critical point).

Example 4.24. The information about local extrema and convexity of a function allows us to sketch its graph more precisely.

(1) Consider $f(x) = x^2e^{-x}$. We then derive

$$f'(x) = e^{-x}\left(2x - x^2\right) \text{ and}$$
$$f''(x) = e^{-x}\left(2 - 4x + x^2\right).$$

Solving $f'(x) = 0$, we know that the critical points of f are 0 and 2. Since $f''(0) = 2 > 0$, $f(0)$ is a local minimum; since $f''(2) = -2e^{-2} < 0$, $f(2)$ is a local maximum. Solving $f''(x) = 0$, which gives $x = 2 \pm \sqrt{2}$, we can further know that f is concave on $\left(2 - \sqrt{2}, 2 + \sqrt{2}\right)$, and convex on the rest of the real line. We can then sketch its graph using all this information.

(2) Consider $f(x) = e^{-x^2}$, which is an important function in probability and analysis. We derive

$$f'(x) = e^{-x^2}\left(-2x\right) \text{ and}$$
$$f''(x) = e^{-x^2}\left(4x^2 - 2\right).$$

Thus, 0 is the unique critical point of f and $f''(0) = -2 < 0$, so $f(0) = 1$ is a local maximum. When $x = \pm 1/\sqrt{2}$, $f''(x) = 0$, and we can see that f is concave on $(-1/\sqrt{2}, 1/\sqrt{2})$ and convex on the rest of $\mathbb{R}$.

(3) Consider $f(x) = x^{2/3}(x-4)$. We get

$$f'(x) = \frac{2}{3}x^{-1/3}(x-4) + x^{2/3} \text{ and}$$
$$f''(x) = -\frac{2}{9}x^{-4/3}(x-4) + \frac{4}{3}x^{-1/3}.$$

When $x = 0$, $f'(x)$ does not exist, so 0 is a critical point. When $x \neq 0$, we can solve $f'(x) = 0$ and get $x = 8/5$. Near $x = 0$, for small positive h, we have $f'(-h) > 0 > f'(h)$, so $f(0) = 0$ is a local maximum. At $x = 8/5$, one can verify

$$f''\left(\frac{8}{5}\right) = -\frac{2}{9}\left(\frac{8}{5}\right)^{-4/3} \cdot \left(-\frac{12}{5}\right) + \frac{4}{3}\left(\frac{8}{5}\right)^{-1/3} > 0$$

since both terms are positive. Thus, $f(8/5)$ is a local minimum.

4.5. **Optimization problems.** Using the techniques in Sections 4.3 and 4.4, especially the first and the second derivative tests, we can look at minimization and maximization problems arising from different situations. To solve a general problem, we usually have to do the following steps.

(1) Introduce suitable variables. A picture with related variables labeled may be helpful.

(2) Determine the quantity we want to optimize and the range of the variables.

(3) Apply the differentiation techniques.

We will look at a few examples here.

Example 4.25 (Maximizing the area of a rectangle under a parabola)**.** Find the rectangle of largest area that can be inscribed under the curve $y = 4 - x^2$ and above the x-axis.

(1) Let x be the x-coordinate of the top-right corner of the rectangle. We know $x \in [0, 2]$.

(2) The height of the rectangle is $y = 4 - x^2$ and the width is $2x$. Thus, the area is
$$A(x) = 2x(4 - x^2) = 8x - 2x^3$$
with $x \in [0, 2]$.

(3) We find
$$A'(x) = 8 - 6x^2.$$
Setting $A'(x) = 0$ gives us a critical point $x = \sqrt{\frac{4}{3}}$. Note that when x is an endpoint of $[0, 2]$, we have $A = 0$. Thus, the rectangle of largest area has width $2\sqrt{\frac{4}{3}}$ and height $\frac{8}{3}$.

Example 4.26 (Minimizing the surface area of a cylinder)**.** A cylinder with volume $V = 100$ is to be constructed. Find the radius and height that minimize the surface area.

(1) Let r be the radius and h the height. We know $r > 0$.

(2) The volume constraint gives
$$\pi r^2 h = 100.$$
The surface area is
$$S = 2\pi r^2 + 2\pi rh.$$
From the constraint,
$$h = \frac{100}{\pi r^2},$$
so
$$S(r) = 2\pi r^2 + \frac{200}{r}$$
for $r > 0$.

(3) We find
$$S'(r) = 4\pi r - \frac{200}{r^2}.$$
Setting $S'(r) = 0$ gives us
$$r^3 = \frac{50}{\pi},$$
so we get a critical point $r_0 = (50/\pi)^{1/3}$. Note that $S(r) \to \infty$ as $r \to \infty$ or $r \to 0$, so at r_0, S attains its minimum. When this happens, the radius is r_0 and the height is $100/(\pi r_0^2)$.

Example 4.27 (Minimizing travel time from cabin to island)**.** An island is 1 mile due north of the closest point on a straight shoreline. A visitor is staying at a cabin on the shore that is 3 miles west of that point. The visitor plans to go from the cabin to the island. Suppose the visitor runs at 2 mph along the shore and swims at 1 mph directly to the island. How far should the visitor run along the shore before swimming to minimize the total travel time?

(1) Let x be the distance the visitor runs east along the shore from the cabin toward the island. Then we know $x \in [0, 3]$.

(2) The running distance is x miles at 2 mph, so the running time is

$$t_{\text{run}} = \frac{x}{2}.$$

The swimming distance is $\sqrt{(3-x)^2 + 1^2}$ miles, so the swimming time is

$$t_{\text{swim}} = \frac{\sqrt{(3-x)^2+1}}{1} = \sqrt{(3-x)^2+1}.$$

Hence, the total travel time is

$$T(x) = \frac{x}{2} + \sqrt{(3-x)^2+1}$$

for $x \in [0, 3]$.

(3) We look at

$$T'(x) = \frac{1}{2} - \frac{3-x}{\sqrt{(3-x)^2+1}}.$$

Setting $T'(x) = 0$ gives

$$\frac{3-x}{\sqrt{(3-x)^2+1}} = \frac{1}{2} \implies 2(3-x) = \sqrt{(3-x)^2+1}.$$

Squaring both sides leads to

$$4(3-x)^2 = (3-x)^2 + 1 \implies 3(3-x)^2 = 1 \implies 3 - x = \frac{1}{\sqrt{3}}$$

since $3 - x \geq 0$. Thus, we get a critical point

$$x_0 = 3 - \frac{1}{\sqrt{3}}.$$

For this x_0, the total travel time is

$$T(x_0) = \frac{3}{2} - \frac{1}{2\sqrt{3}} + \frac{2}{\sqrt{3}} = \frac{3}{2} + \frac{\sqrt{3}}{2}.$$

We also check that at the endpoints,

$$T(0) = \sqrt{10} \text{ and}$$
$$T(3) = \frac{5}{2}.$$

Thus, among them, $T(x_0)$ is the smallest, so the visitor should run $3 - \frac{1}{\sqrt{3}}$ miles.

One can find more examples in the recommended textbook or from some online resources.

4.6. **Newton's method.** We will now see another application of differentiation. Newton's method provides an approach that allows us to approximate solutions to a given equation. The idea is as follows. Suppose we want to solve $f(x) = 0$.

(1) We start from an initial guess x_0. This can be done by sketching the graph of f.
(2) Let L_0 be the tangent to f at x_0. We let x_1 be the intersection of L_0 and the x-axis.
(3) Let L_1 be the tangent to f at x_1. We let x_2 be the intersection of L_1 and the x-axis.
(4) We iterate this process and get a sequence $x_0, x_1, x_2, \cdots$.

Let's now look at more specific relations among numbers in this sequence. Suppose we start from x_0. Then the tangent to f at x_0 is given by

$$y = f(x_0) + f'(x_0)(x - x_0).$$

Its intersection with the x-axis occurs when $y = 0$, so we get

$$0 = f(x_0) + f'(x_0)(x_1 - x_0),$$

which implies

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

if $f'(x_0) \neq 0$. Similarly, for any $n \in \mathbb{R}$, we get a recurrence relation

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

if $f'(x_n) \neq 0$.

Example 4.28. We look at a classic example when $f(x) = x^2 - 2$. We know $f'(x) = 2x$. If we start from $x_0 = 2$, then we get

$$\begin{aligned}
x_1 &= x_0 - \frac{f(x_0)}{f'(x_0)} = 2 - \frac{2}{4} = \frac{3}{2}, \\
x_2 &= x_1 - \frac{f(x_1)}{f'(x_1)} = \frac{3}{2} - \frac{1/4}{3} = \frac{17}{12}, \text{ and} \\
x_3 &= x_2 - \frac{f(x_2)}{f'(x_2)} = \frac{17}{12} - \frac{1/144}{17/6} = \frac{577}{408} \approx 1.4142156.
\end{aligned}$$

This is already very close to $\sqrt{2}$.

Newton's method may fail for a few reasons. First, it is possible that $f'(x_n) = 0$ for some n, and then the iteration cannot be continues. Second, when the equation has more than one solution, if the starting point is not close enough to the solution we want, the sequence may approach another solution. Finally, it is possible that the iteration may go back to x_0 and form a loop. We see an example in the following.

Example 4.29. Consider $f(x) = x^3 - 2x + 2$ and $x_0 = 0$. Then $f'(x) = 3x^2 - 2$ so

$$\begin{aligned}
x_1 &= x_0 - \frac{f(x_0)}{f'(x_0)} = 0 - \frac{2}{-2} = 1, \text{ and} \\
x_2 &= x_1 - \frac{f(x_1)}{f'(x_1)} = 1 - \frac{1}{1} = 0.
\end{aligned}$$

Thus, we get a sequence $0, 1, 0, 1, 0, 1, \cdots$ and it does not converge to a single value.

4.7. **Mean value theorem.** We will introduce an important result, the mean value theorem. Based on it, we will justify many criteria we use in the first and second derivative tests.

Theorem 4.30 (Mean value theorem)**.** Let $a, b \in \mathbb{R}$ and let $f\colon [a,b] \to \mathbb{R}$ be a continuous function. If f is differentiable on (a,b), then there exists $c \in (a,b)$ such that

$$f'(c) = \frac{f(b)-f(a)}{b-a}.$$

The theorem says that for such a function, the slope of the secant equals the slope of the tangent at a point in between. That is, over an interval, the average change rate must be realized as the instantaneous change rate at certain time. If one draws a picture, it is intuitive to find such a point.

Proof of Theorem 4.30. We will first prove a special case when $f(a) = f(b)$. Therefore, we want to find a point $c \in (a,b)$ such that $f'(c) = 0$. This reminds us of local extrema. To this end, we apply the extreme value theorem (Theorem 4.7) to find $x_1, x_2 \in [a,b]$ such that

$$f(x_1) = \max_{[a,b]} f \text{ and } f(x_2) = \min_{[a,b]} f.$$

We hope that x_1 or x_2 are not either a or b so that they will be local extremum points. To make this clear, we deal with the following two cases.

Case 1: If $f(x_1) = f(x_2) = f(a) = f(b)$, then f is constant on $[a,b]$, and hence any $c \in (a,b)$ is a critical point so we are done.

Case 2: If $f(x_1)$ or $f(x_2)$ is different from $f(a)$, say $f(x_1) \neq f(a)$, then $x_1 \in (a,b)$ and $f(x_1)$ is a local maximum. Thus, the preliminary first derivative test (Proposition 4.8) tells us that $f'(x_1) = 0$ and we are done. The situation when $f(x_2) \neq f(a)$ is similar.

Next, we deal with the general case. In general, f does not satisfy $f(a) = f(b)$. However, there are many ways to modify it so that the resulting function does. One way is to use the secant that connects $(a, f(a))$ and $(b, f(b))$. To be more specific, this secant is the graph of the function

$$f(a) + \frac{f(b)-f(a)}{b-a}(x-a).$$

Therefore, if we look at the difference of f and this linear function, that is,

$$h(x) := f(x) - \left(f(a) + \frac{f(b)-f(a)}{b-a}(x-a)\right),$$

this new function satisfies

$$\begin{aligned} h(a) &= f(a) - f(a) = 0 \text{ and} \\ h(b) &= f(b) - f(b) = 0. \end{aligned}$$

Moreover, h is continuous on $[a,b]$ and differentiable on (a,b), so we can use the first special case of the mean value theorem (proven in the preceding paragraph) to conclude that there exists $c \in (a,b)$ such that $h'(c) = 0$. That is,

$$0 = h'(c) = f'(c) - \frac{f(b)-f(a)}{b-a}.$$

Thus, we get

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

and the theorem follows. □

The mean value theorem in the special case when $f(a) = f(b)$ is sometimes referred to as Rolle's Theorem. We remark that the differentiability condition is necessary as given by the following example.

Example 4.31. Consider $f\colon [-1,1] \to \mathbb{R}$ given by $f(x) = |x|$. Then there does not exist $x \in (-1,1)$ such that

$$f'(x) = \frac{f(1) - f(-1)}{1 - (-1)} = 0.$$

We are not allowed to apply the mean value theorem since f is not differentiable on $(-1,1)$.

Next, we are going to see a series of consequences of the mean value theorem. The most important ones are summarized in the following. Part of them were mentioned in Lemma 4.14.

Corollary 4.32. Let $f\colon I \to \mathbb{R}$ be a differentiable function on an open interval.

(1) If $f' > 0$ on I, then f is increasing on I.

(2) If $f' < 0$ on I, then f is decreasing on I.

(3) If $f' = 0$ on I, then f is constant on I.

The good thing is that using the mean value theorem, we can prove all of these at the same time.

Proof. We take two points $a < b$ in I. By the mean value theorem, there exists $c \in (a,b)$ such that

$$f(b) - f(a) = f'(c)(b-a).$$

Now we deal with three possibilities.

If $f' > 0$, we get $f'(c) > 0$, so $f(b) > f(a)$ for any two points a and b.

If $f' < 0$, we get $f'(c) < 0$, so $f(b) < f(a)$ for any two points a and b.

If $f' = 0$, we get $f'(c) = 0$, so $f(b) = f(a)$ for any two points a and b. $\square$

For the third case (3), it has the following immediate consequence that characterize when two functions have the same derivatives.

Corollary 4.33. Let f and g be continuous functions on $[a,b]$. If f and g are differentiable on (a,b) and $f' = g'$, then $f - g$ is a constant.

We will see a further consequence of this result in the assignment.

Next, we note that the remaining cases mentioned in Lemma 4.14 can be proven exactly as in Corollary 4.32. We list them here again, and can even combine them with Corollary 4.32.

Corollary 4.34. Let $f\colon I \to \mathbb{R}$ be a differentiable function.

(1) If $f' \geq 0$ on I, then f is non-decreasing on I. i.e. $f(a) \leq f(b)$ for any $a < b$ in I.

(2) If $f' \leq 0$ on I, then f is non-increasing on I. i.e., $f(a) \geq f(b)$ for any $a < b$ in I.

Besides directly using them to prove a function's behavior, we can use these corollaries to prove some inequalities between functions.

Example 4.35. We will show that for $x > 0$, $e^x > 1 + x$. We consider their difference $f(x) = e^x - 1 - x$. Then for $x > 0$,

$$f'(x) = e^x - 1 > 0,$$

so f is increasing. In particular, for $x > 0$, $f(x) > f(0) = 0$, which means $e^x > 1 + x$.

Next, we can improve it to $e^x > 1+x+\frac{1}{2}x^2$ for $x > 0$. To this end, consider $g(x) = e^x-1-x-\frac{1}{2}x^2$. Then for $x > 0$,

$$g'(x) = e^x - 1 - x > 0$$

from the previous case. Thus, g is increasing and hence for $x > 0$, $g(x) > g(0)$, which means $e^x > 1 + x + \frac{1}{2}x^2$. One can iterate this and show that $e^x > 1 + x + \frac{1}{2}x^2 + \frac{1}{3\cdot 2\cdot 1}x^3$ for $x > 0$ and so on.

Finally, we verify Lemma 4.19 about criteria for convexity. We state the lemma here again.

Lemma 4.36. Suppose $f\colon I \to \mathbb{R}$ is twice differentiable.

(1) If $f'' \geq 0$ on I, then f is convex on I.

(2) If $f'' \leq 0$ on I, then f is concave on I.

Proof. We will prove (1), and then (2) can be proven similarly. Let $x_1 < x_2$ be two points in I and $t \in [0, 1]$. We want to understand the value of the function at

$$x_t = tx_1 + (1 - t)x_2.$$

Thus, we apply the mean value theorem to f on $[x_1, x_t]$ and $[x_t, x_2]$. This produces $c_1 \in (x_1, x_t)$ and $c_2 \in (x_t, x_2)$ such that

$$f'(c_1) = \frac{f(x_t) - f(x_1)}{x_t - x_1} \text{ and } f'(c_2) = \frac{f(x_2) - f(x_t)}{x_2 - x_t}.$$

Since $f'' \geq 0$, by Corollary 4.34, we know f' is non-decreasing, so $f'(c_1) \leq f'(c_2)$, which means

$$\frac{f(x_t) - f(x_1)}{x_t - x_1} \leq \frac{f(x_2) - f(x_t)}{x_2 - x_t}.$$

This implies

$$(x_2 - x_t + x_t - x_1)\, f(x_t) \leq (x_2 - x_t)f(x_1) + (x_t - x_1)f(x_2);$$

that is,

$$f(x_t) \leq \frac{x_2 - x_t}{x_2 - x_1}f(x_1) + \frac{x_t - x_1}{x_2 - x_1}f(x_2). \tag{4.37}$$

Note that

$$\frac{x_2 - x_t}{x_2 - x_1} = \frac{tx_2 - tx_1}{x_2 - x_1} = t,$$

and similarly

$$\frac{x_t - x_1}{x_2 - x_1} = \frac{(t - 1)x_1 + (1 - t)x_2}{x_2 - x_1} = 1 - t.$$

Hence, (4.37) tells us that

$$f(x_t) \leq tf(x_1) + (1 - t)f(x_2).$$

This completes the proof. □

4.8. **L'Hôpital's rule (Bernoulli's rule).** The final topic we will talk about differentiation is L'Hôpital's rule, or Bernoulli's rule. The result was first introduced to l'Hôpital Johann Bernoulli. It allows us to evaluate many limits (old or new) in a simpler manner.

Example 4.38. We work on an example to get an idea of the result (even before we write down the most general form of it). If we look at $\lim_{x\to 1}\frac{x^3-1}{x^2-1}$, based on our experience, we can find a common factor $x-1$ for both of them and remove it. Here, the new idea is that we keep it there, and use the limit law to deduce

$$\lim_{x\to 1}\frac{x^3-1}{x^2-1}=\lim_{x\to 1}\frac{\frac{x^3-1}{x-1}}{\frac{x^2-1}{x-1}}=\frac{\lim_{x\to 1}\frac{x^3-1}{x-1}}{\lim_{x\to 1}\frac{x^2-1}{x-1}}=\frac{\left(x^3\right)'|_{x=1}}{(x^2)'|_{x=1}}=\frac{3\cdot 1}{2\cdot 1}=\frac{3}{2}.$$

Here, we use the definition of derivatives, which simplifies the old process that may lead to something like

$$\lim_{x\to 1}\frac{x^3-1}{x^2-1}=\lim_{x\to 1}\frac{(x-1)(x^2+x+1)}{(x-1)(x+1)}.$$

This looks ok here but if one replaces 3 with 10, then it's a different story (for this old method).

Now, we state L'Hôpital's rule in its general form.

Theorem 4.39. Let I be an open interval, $x_0\in I$, and f and g be two differentiable functions on $I\setminus\{x_0\}$ with $g'(x)\neq 0$. Suppose the following two conditions hold.

(1) $\lim_{x\to x_0} f(x)=\lim_{x\to x_0} g(x)=0$ or $\lim_{x\to x_0}|f(x)|=\lim_{x\to x_0}|g(x)|=\infty$.

(2) $\lim_{x\to x_0}\frac{f'(x)}{g'(x)}$ exists or $\lim_{x\to x_0}\frac{f'(x)}{g'(x)}=\pm\infty$.

Then $\lim_{x\to x_0}\frac{f(x)}{g(x)}=\lim_{x\to x_0}\frac{f'(x)}{g'(x)}$. Moreover, if the assumptions are true in the situation of one-sided limits, $\lim_{x\to x_0^\pm}$, or asymptotics at $\pm\infty$, $\lim_{x\to\pm\infty}$, then the rule is still valid.

Note that, of course, it is good if f and g are differentiable functions on the whole I. The theorem says that, as what we learned, the exact behavior of the functions at x_0 is not relevant. On the other hand, it is extremely important to verify all the conditions before applying the rule. All of the assumptions are necessary, as we will see in the following examples.

Remark 4.40. We explain why all the four conditions are necessary.

(1) f and g have to be differentiable in a neighborhood of x_0. Otherwise, the limit $\lim_{x\to x_0}\frac{f'(x)}{g'(x)}$ may not make sense.

(2) g' has to be non-vanishing in a neighborhood of x_0. Otherwise, the limit $\lim_{x\to x_0}\frac{f'(x)}{g'(x)}$ may not make sense.

(3) f/g has to be of the form $0/0$ or $\pm\infty/\infty$ as $x\to x_0$. Otherwise, for example, one can see $\lim_{x\to 0}\frac{x+2}{2x+3}\neq\lim_{x\to 0}\frac{1}{2}$.

(4) $\lim_{x \to x_0} \frac{f'(x)}{g'(x)}$ has to exist or has a definite asymptotics for us to get something from the rule. Otherwise, anything is possible, and the original limit may even exist. For example,

$$\lim_{x \to \infty} \frac{x + \sin x}{x} = 1,$$

but

$$\lim_{x \to \infty} \frac{1 + \cos x}{1} = \lim_{x \to \infty} (1 + \cos x)$$

does not exist. From this, we cannot conclude anything about the original limit.

Example 4.41. We will see a few examples. Some of them were studied before with some effort.

(1) $\lim_{x \to 0} \frac{\sin x}{x} = \lim_{x \to 0} \frac{\cos x}{1} = 1$. $\lim_{x \to 0} \frac{1 - \cos x}{x} = \lim_{x \to 0} \frac{\sin x}{1} = 0$.

(2) $\lim_{x \to 1} \frac{\sin \pi x}{\log x} = \lim_{x \to 1} \frac{\pi \cos \pi x}{1/x} = \frac{-\pi}{1} = -\pi$.

(3) $\lim_{x \to \infty} \frac{e^x}{x^2} = \lim_{x \to \infty} \frac{e^x}{2x} = \lim_{x \to \infty} \frac{e^x}{2} = \infty$, so the original limit does not exist.

(4) $\lim_{x \to 0^+} \frac{\tan x}{\log(1 + x)} = \lim_{x \to 0^+} \frac{\sec^2 x}{1/(1 + x)} = 1$.

We remark that when we write down those equalities using L'Hôpital's rule, we have to first observe that the resulting limit exists or is $\pm\infty$ and then we can conclude that the equality holds.

Another remark we would like to bring up is that all of these limits are in fact "comparing" the convergence rates of these functions either to 0 or to ∞. For example, (2) is saying that $\sin \pi x$ and $\log x$ tend to 0 in a comparable rate; while (3) is saying that e^x blows up faster than x^2 does as $x \to \infty$.

Example 4.42. On the other hand, when we know how to evaluate a limit directly, we should just do it. Sometimes even when L'Hôpital's rule applies, it may not be helpful. We mention two examples.

(1) We look at $\lim_{x \to \infty} \frac{x}{\sqrt{x^2 - 4}}$. If one directly uses L'Hôpital's rule, one will get

$$\lim_{x \to \infty} \frac{x}{\sqrt{x^2 - 4}} = \lim_{x \to \infty} \frac{1}{\frac{x}{\sqrt{x^2-4}}} = \lim_{x \to \infty} \frac{\sqrt{x^2 - 4}}{x} = \lim_{x \to \infty} \frac{\frac{x}{\sqrt{x^2-4}}}{1} = \lim_{x \to \infty} \frac{x}{\sqrt{x^2 - 4}}.$$

Thus, the rule does not help.

(2) We look at an extreme example given by $\lim_{x \to 0} \frac{e^{-1/x^2}}{e^{-1/x^2}}$. In this case, the rule also does not help since any high derivative of e^{-1/x^2} tends to 0 as $x \to 0$.

In both cases, we should just evaluate the limits directly.

L'Hôpital's rule can also be applied to some situations in which the limits do not look like $0/0$ or ∞/∞ directly. These include $0 \cdot \infty, \infty - \infty$, and so on. All of these are called **indeterminate forms**, for which we are not able to determine the existence of limits directly.

Example 4.43. We look at examples of other indeterminate forms.

(1) We look at $\lim_{x\to 0^+} x\log x$. This is not in a form for which we can apply the rule. However, if we view it by $x\log x = \dfrac{\log x}{1/x}$, then we are in a good position and can get

$$\lim_{x\to 0^+} x\log x = \lim_{x\to 0^+} \frac{\log x}{1/x} = \lim_{x\to 0^+} \frac{1/x}{-1/x^2} = \lim_{x\to 0^+} -x = 0.$$

This is saying that x is tending to 0 faster than $\log x$ tends to ∞ as $x \to 0^+$.

(2) We look at $\lim_{x\to 0^+} \left(\dfrac{1}{x^2} - \dfrac{1}{\tan x}\right)$. We first combine the two terms and get

$$\lim_{x\to 0^+} \left(\frac{1}{x^2} - \frac{1}{\tan x}\right) = \lim_{x\to 0^+} \frac{\tan x - x^2}{x^2\tan x},$$

for which now we can apply the rule. Thus, we get

$$\lim_{x\to 0^+} \left(\frac{1}{x^2} - \frac{1}{\tan x}\right) = \lim_{x\to 0^+} \frac{\tan x - x^2}{x^2\tan x} = \lim_{x\to 0^+} \frac{\sec^2 x - 2x}{2x\tan x + x^2\sec^2 x} = \frac{1-0}{0+0} = \infty,$$

so the original limit does not exist.

(3) We look at $\lim_{x\to\infty} x^{1/x}$. Here, we use an idea we applied in the logarithmic differentiation. That is, $x^{1/x} = e^{\log x^{1/x}} = e^{(\log x)/x}$. Since e^y is a continuous function, we can first evaluate the limit in the exponent and then put it back to e^y. Thus, we evaluate

$$\lim_{x\to\infty} \frac{\log x}{x} = \lim_{x\to\infty} \frac{1/x}{1} = 0,$$

and we can use it to get

$$\lim_{x\to\infty} x^{1/x} = \lim_{x\to\infty} e^{\frac{\log x}{x}} = e^{\lim_{x\to\infty}\frac{\log x}{x}} = e^0 = 1.$$

Example 4.44. We end with more examples about the convergence rates/growth rates of functions.

(1) As $x\to\infty$, we know both e^x and x^3 blow up. We can evaluate

$$\lim_{x\to\infty} \frac{e^x}{x^3} = \lim_{x\to\infty} \frac{e^x}{3x^2} = \lim_{x\to\infty} \frac{e^x}{6x} = \lim_{x\to\infty} \frac{e^x}{6} = \infty$$

to see that e^x blows up faster. In fact, e^x blows up faster than any polynomials.

(2) One can compare the blow-up rates between any two polynomials and find that it only depends on their degrees. If a polynomial has a higher degree, then it blows up faster.

(3) We can compare the blow-up rates of $\log x$ and x by looking at

$$\lim_{x\to\infty} \frac{\log x}{x} = \lim_{x\to\infty} \frac{1/x}{1} = 0.$$

Thus, x blows up faster than $\log x$. Therefore, although $\log x\to\infty$ as $x\to\infty$, we now know that it does so in a strictly slower manner.

(4) One can also compare $\log x$ and $\log\log x$, and get

$$\lim_{x\to\infty} \frac{\log\log x}{\log x} = \lim_{x\to\infty} \frac{1/\log x\cdot 1/x}{1/x} = \lim_{x\to\infty} \frac{1}{\log x} = 0.$$

One then sees that the order is: exponential $>$ polynomial $>\log x > \log\log x$ and so on.