CALCULUS 1, NOTE
LIMITS

TANG-KAI LEE

2. **Limits**

Next, we are going to talk about limiting behavior of real-valued functions.

2.1. **Motivation.** The notion of limits can be inspired by the question of finding the **tangent** of the graph of a function. The idea is that if the graph of a function looks continuous, then at a point P, we can approximate the tangent of the graph by the **secant** passing through P and another different point that is very close to P.

To be more precise, given a real-valued function f on an open interval I, suppose we want to find the tangent of its graph at $(x_0, f(x_0))$ for some $x_0 \in I$. The tangent is a line, and since it passes through $(x_0, f(x_0))$, we only need to find out its slope[1] m. Once we know what m is, the tangent is then given by the line

$$y - f(x_0) = m\,(x - x_0)\,.$$

For any $x \in I$ that is close to but different from x_0, the secant passing through $(x, f(x))$ and $(x_0, f(x_0))$ has slope

$$\frac{f(x) - f(x_0)}{x - x_0}.$$

Thus, what we want is this value as x is arbitrarily close to x_0. When this value exists, we denote it by

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}.$$

Example 2.1. We consider $f(x) = x^2$. We try to find the tangent of its graph at $P = (1, 1)$. We find out the slope of the secant that passes through P and a nearby point:

- The slope of the secant passing through P and $(1.1, f(1.1))$ is
$$\frac{1.1^2 - 1^2}{1.1 - 1} = \frac{0.21}{0.1} = 2.1.$$
- The slope of the secant passing through P and $(1.01, f(1.01))$ is
$$\frac{1.01^2 - 1^2}{1.01 - 1} = \frac{0.0201}{0.01} = 2.01.$$
- The slope of the secant passing through P and $(1.001, f(1.001))$ is
$$\frac{1.001^2 - 1^2}{1.001 - 1} = \frac{0.002001}{0.001} = 2.001.$$

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[1] Given any two different points (x_1, y_1) and (x_2, y_2) on a line, the slope of the line is $\frac{y_1 - y_2}{x_1 - x_2}$.

One can then reasonably guess that this slope will be closer and closer to 2, so the limit we expect should be

$$\lim_{x\to 1}\frac{x^2-1^2}{x-1}=2.$$

Many problems in science are modeled on this tangent problem. We list a few of them here.

(1) Instantaneous velocity.

(2) Instantaneous acceleration.

(3) Growth rate of population.

(4) Reaction rate.

(5) Elasticity.

Basically, if the problem is related to taking an average ration (corresponding to secants), its instantaneous counterpart corresponds to the tangent problem.

2.2. **Formal definition.** We now talk about the definition of the process of taking limits. That is, we wan to know what it really means to say

$$\lim_{x\to x_0} f(x).$$

That is, given a function f on an interval I which either contains x_0 or has x_0 as a boundary point, what do we mean when we say "f has a limit when x tends to x_0?"

The central idea is: if such a limit is L, then we hope that $f(x)$ can be arbitrarily close to L when we make x very close to (but different from) x_0. Thus, we can note that here are two quantities. The first one is the variation you want to restrict for $f(x)$ from L, and the second is the resulting range of x (measured from x_0). We can now write it down clearly. Suppose $f\colon I\to\mathbb{R}$ is a function where $I\subseteq\mathbb{R}$. We say

$$\lim_{x\to x_0} f(x)=L$$

for some $L\in\mathbb{R}$ if given any (restriction) $R>0$, there exists (a range) $A>0$ such that

$$\text{if } 0<|x-x_0|<A, \text{ then } |f(x)-L|<R. \tag{2.2}$$

Note that x_0 does not necessarily lie in the domain of f.

We won't be focusing on checking this formal definition for those common functions. For most of them, we will accept the fact that their function values are the same as their limits. That is,

$$\lim_{x\to x_0} f(x)=f(x_0)$$

when x_0 is in their domains. This is a crucial feature called **continuity**, which we will come back to discuss later.

Example 2.3. Though we won't use this formal definition later, we test it with some simple examples.

(1) Let $f\colon\mathbb{R}\to\mathbb{R}$ be $f(x)=2x$. We want to see its limit at $x_0=2$. A reasonable guess for its limit is $f(x_0)=4$. We now check whether this is correct. Given any restriction $R>0$, we hope to find a range $A>0$ so that for $x\in(2-A,2+A)$ (but not necessarily for $x=2$), we get

$$|4-2x|<R.$$

Note that this happens if and only if

$$|2 - x| < \frac{R}{2},$$

so we can directly choose $A = R/2$.

(2) (optional) Let $f\colon (1,5) \to \mathbb{R}$ be $f(x) = x^2$. We look at $x_0 = 2$, which does not lie in $(2,5)$. Then given any restriction $R > 0$, we hope

$$|4 - x^2| < R$$

for x close to 2. Note that

$$|4 - x^2| = |(x-2)(x+2)|,$$

so we only need

$$|x - 2| < \frac{R}{|x+2|}.$$

Since $x \in (2,5)$, the minimum of $R/|x+2|$ is $R/7$, so we can take $A = R/7$.
To chekc whether this is valid, if we assume $x \in (2,5)$ satisfies $|x-2| < R/7$, we get

$$|x^2 - 4| = |x+2| \cdot |x-2| < 7 \cdot \frac{R}{7} = R,$$

so it is correct that the limit is 4.

(3) We can also sometimes find out limits using the graphs of functions. Suppose

$$f(x) = \begin{cases} 2x+2 & \text{if } x > 2 \\ 5 & \text{if } x = 2 \\ x+4 & \text{if } x < 2 \end{cases}.$$

Then as x gets closer to $x_0 = 2$, from the graph, we know that $f(x)$ gets closer to 6, which is $2 \cdot 2 + 2$ and $x + 4$. Thus, $\lim\limits_{x \to 2} f(x) = 6$.

(4) As $x \to 0$, $\sin \frac{1}{x}$ does not have a limit. It oscillates and admits each values in $[-1,1]$ infinitely many times.

Note that from example (3), we see that the limit is unrelated to the function value in general. This can be seen from the definition (2.2), which says nothing about $x = x_0$.

2.3. **One-sided limits.** It is possible that the limit exists only when one approaches a point from one direction. It is similar to (2.2), but now the the range A only applies to one side of x_0.

We can similarly write them down clearly. Suppose $f\colon I \to \mathbb{R}$ is a function where $I \subseteq \mathbb{R}$. We say

$$\lim_{x \to x_0^+} f(x) = L$$

for some $L \in \mathbb{R}$ if given any (restriction) $R > 0$, there exists (a range) $A > 0$ such that

$$\text{if } 0 < x - x_0 < A, \text{ then } |f(x) - L| < R.$$

In this case, we say L is the **right-sided limit** of $f(x)$ at x_0. Likewise, we say

$$\lim_{x \to x_0^-} f(x) = L$$

for some $L \in \mathbb{R}$ if given any (restriction) $R > 0$, there exists (a range) $A > 0$ such that

$$\text{if } 0 < x_0 - x < A, \text{ then } |f(x) - L| < R.$$

In this case, we say L is the **left-sided limit** of $f(x)$ at x_0.

By the definitions, we have the following characterization of limits.

Theorem 2.4. Let $f\colon I \to \mathbb{R}$ be a function and $x_0, L \in \mathbb{R}$. Then

$$\lim_{x\to x_0} f(x) = L \text{ if and only if } \lim_{x\to x_0^+} f(x) = \lim_{x\to x_0^-} f(x) = L.$$

Example 2.5. We consider

$$f(x) = \begin{cases} 2x+2 & \text{if } x > 2 \\ 5 & \text{if } x = 2 \\ x+2 & \text{if } x < 2 \end{cases}.$$

Then

$$\lim_{x\to 2^+} f(x) = 6 \text{ and } \lim_{x\to 2^-} f(x) = 4.$$

As a consequence, $\lim_{x\to 2} f(x)$ does not exist.

2.4. **Infinity.** A special case when the limit does not exist is when the function diverges to infinity. This means that the function becomes arbitrarily large as x tends to a point.

We write down the definition for one of the cases, and all the other are similar. Suppose $f\colon I \to \mathbb{R}$ is a function where $I \subseteq \mathbb{R}$. We say

$$\lim_{x\to x_0^+} f(x) = \infty$$

if given any (restriction) $R > 0$, there exists (a range) $A > 0$ such that

$$\text{if } 0 < x - x_0 < A, \text{ then } f(x) > R. \tag{2.6}$$

Similarly, we say

$$\lim_{x\to x_0^+} f(x) = -\infty$$

if given any (restriction) $R > 0$, there exists (a range) $A > 0$ such that

$$\text{if } 0 < x - x_0 < A, \text{ then } f(x) < -R.$$

For example,

$$\lim_{x\to 0^+} \frac{1}{x} = \infty \text{ and } \lim_{x\to 0^-} \frac{1}{x} = -\infty.$$

We emphasize that this particularly means that the limit of $f(x)$ at 0 does NOT exist and we just have a special notation to indicate this special situation.

For a function $f(x)$, if $\lim_{x\to x_0^+} f(x) = \pm\infty$ or $\lim_{x\to x_0^-} f(x) = \pm\infty$, then we say $x = x_0$ is a vertical asymptote of the graph of $f(x)$.

Example 2.7. $\log(x-1)$ as $x \to 1^+$ and $\tan x$ as $x \to \pi/2^+$.

2.5. **Limit laws.** We introduce a few laws that will allow us to calculate the limits of many functions.

Proposition 2.8. Let f and g be real-valued functions on an interval I. Suppose $\lim_{x \to x_0} f(x) = L_1$ and $\lim_{x \to x_0} g(x) = L_2$ exist for some $x_0 \in \mathbb{R}$.

(1) $\lim_{x \to x_0} (f(x) + c \cdot g(x)) = L_1 + cL_2$ for $c \in \mathbb{R}$.

(2) $\lim_{x \to x_0} (f(x) \cdot g(x)) = L_1 \cdot L_2$.

(3) $\lim_{x \to x_0} \frac{f(x)}{g(x)} = \frac{L_1}{L_2}$ if $L_2 \neq 0$.

(4) $\lim_{x \to x_0} (f(x))^n = L_1^n$ for $n \in \mathbb{N}$.

(5) $\lim_{x \to x_0} \sqrt[n]{f(x)} = \sqrt[n]{L_1}$ for $n \in \mathbb{N}$ when $L_1 \geq 0$.

These are for the basic operations. We also have the laws for some basic functions.

Proposition 2.9. Let $x_0 \in \mathbb{R}$.

(1) $\lim_{x \to x_0} P(x) = P(x_0)$ for a polynomial P.

(2) $\lim_{x \to x_0} \sin x = \sin x_0$, $\lim_{x \to x_0} \cos x = \cos x_0$, and if $\cos x_0 \neq 0$, $\lim_{x \to x_0} \tan x = \tan x_0$.

(3) For $a > 0$, $\lim_{x \to x_0} a^x = a^{x_0}$ and if $x_0 > 0$, $\lim_{x \to x_0} \log_a x = \log_a x_0$.

Combining Propositions 2.8 and 2.9, we will be able to evaluate a large class of limits.

Example 2.10. We calculate some eamples.

(1) Basic examples come from polynomials and rational function at points in their domains.

(2) We look at $f(x) = \frac{x^2-4}{x-2}$ $(x \neq 2)$ and $x_0 = 2$. Note that $f(x)$ is not defined at $x = x_0$. However, if we care about the limit of $f(x)$ when x tends to x_0, we mainly care about the behavior of the function "near but not at x_0." Especially, we care about those x's that are NOT x_0, so we get

$$\lim_{x \to 2} \frac{x^2 - 4}{x - 2} = \lim_{x \to 2} \frac{(x+2)(x-2)}{x - 2} = \lim_{x \to 2} (x + 2) = 4.$$

We stress again that the second equality is valid because $x \neq 2$ in the limiting process. One can compare this with Example 2.3(3).

(3) We look at

$$\lim_{x \to 0} \frac{\sqrt{x^2 + 4} - 2}{x^2}.$$

Both the numerator and the denominator vanish when $x = 0$, so we don't apply the quotient rule. Here, we can rationalize the numerator and get

$$\begin{aligned}
\lim_{x\to 0} \frac{\sqrt{x^2+4}-2}{x^2} &= \lim_{x\to 0} \frac{\left(\sqrt{x^2+4}-2\right)\left(\sqrt{x^2+4}+2\right)}{x^2\left(\sqrt{x^2+4}+2\right)} \\
&= \lim_{x\to 0} \frac{\left(x^2+4\right)-4}{x^2\left(\sqrt{x^2+4}+2\right)} \\
&= \lim_{x\to 0} \frac{x^2}{x^2\left(\sqrt{x^2+4}+2\right)} \\
&= \lim_{x\to 0} \frac{1}{\left(\sqrt{x^2+4}+2\right)} = \frac{1}{4}.
\end{aligned}$$

We emphasize again that we can cancel the x^2 terms because away from $x = 0$, x^2 does not vanish.

The rules in Propositions 2.8 and 2.9 are also valid for one-sided limits. Note that by Theorem 2.4, we can determine the existence of limits or calculate them using the limits from both sides.

(4) We look at

$$\lim_{x\to 0} |x|.$$

Note that when $x > 0$, $|x| = x$, so we get

$$\lim_{x\to 0^+} |x| = \lim_{x\to 0^+} x = 0.$$

Similarly, using $|x| = -x$ when $x < 0$, we get

$$\lim_{x\to 0^-} |x| = \lim_{x\to 0^-} (-x) = 0.$$

Thus, by Theorem 2.4, $\lim\limits_{x\to 0} |x|$ exists and is 0.

(5) We look at

$$\lim_{x\to 0} \frac{x}{|x|}.$$

We can similarly calculate

$$\lim_{x\to 0^+} \frac{x}{|x|} = \lim_{x\to 0^+} \frac{x}{x} = 1$$

and

$$\lim_{x\to 0^-} \frac{x}{|x|} = \lim_{x\to 0^-} \frac{x}{-x} = -1.$$

Thus, the limit $\lim\limits_{x\to 0} \frac{x}{|x|}$ does not exist.

(6) Let $f\colon \mathbb{R} \to \mathbb{Z}$ be $x \mapsto \lfloor x \rfloor$, the largest integer that is less than or equal to x. Then if we look at $\lim\limits_{x\to 5} f(x)$, we note that $f(x) = 5$ for $x \in (5, 6)$, so

$$\lim_{x\to 5^+} f(x) = 5.$$

On the other hand, $f(x) = 4$ for $x \in (4,5)$, so

$$\lim_{x \to 5^-} f(x) = 4.$$

Thus, $\lim_{x \to 5} \lfloor x \rfloor$ does not exist.

We mention one more powerful tool allowing us to calculate more limits.

Theorem 2.11 (Squeeze theorem)**.** Let I be an open interval, $x_0 \in I$, and f, g, and h be real-valued functions on I. Suppose $f(x) \leq g(x) \leq h(x)$ for all $x \in I \setminus \{x_0\}$. If for some $L \in \mathbb{R}$,

$$\lim_{x \to x_0} f(x) = \lim_{x \to x_0} h(x) = L,$$

then lim $\lim_{x \to x_0} g(x) = L$.

We recall that $I \setminus \{x_0\} = \{x \in I : x \neq x_0\}$. That is, as before, we only care about the information close to but not at x_0.

The theorem is particularly helpful when we are not able to directly apply those limit laws, as we will see in the following example.

Example 2.12. A typical example is when we look at the product of a function decaying fast enough and another with stranger limiting behavior. For example, we look at

$$\lim_{x \to 0} x\, e^{\cos \frac{1}{x}}.$$

Note that the limit of $\cos \frac{1}{x}$ does not exist when x tends to 0. However, since $\cos \frac{1}{x} \in [-1, 1]$ for all $x \neq 0$, in particular, we have $e^{-1} \leq e^{\cos \frac{1}{x}} \leq e$. Thus, we know

$$x\, e^{-1} \leq x\, e^{\cos \frac{1}{x}} \leq x\, e$$

for $x > 0$, and similarly,

$$x\, e^{-1} \geq x\, e^{\cos \frac{1}{x}} \geq x\, e$$

for $x < 0$. Therefore, combining them, we get

$$-|x|e \leq x\, e^{\cos \frac{1}{x}} \leq |x|e$$

for all $x \neq 0$. The squeeze theorem then implies

$$\lim_{x \to 0} x\, e^{\cos \frac{1}{x}} = 0.$$

Example 2.13. We will use the squeeze theorem to evaluate

$$\lim_{x \to 0} \frac{\sin x}{x}.$$

Note that since the denominator tends to 0 as $x \to 0$, we are not able to use the limit law to calculate it. Instead, we will derive it from the inequality

$$\cos x \leq \frac{\sin x}{x} \leq 1 \tag{2.14}$$

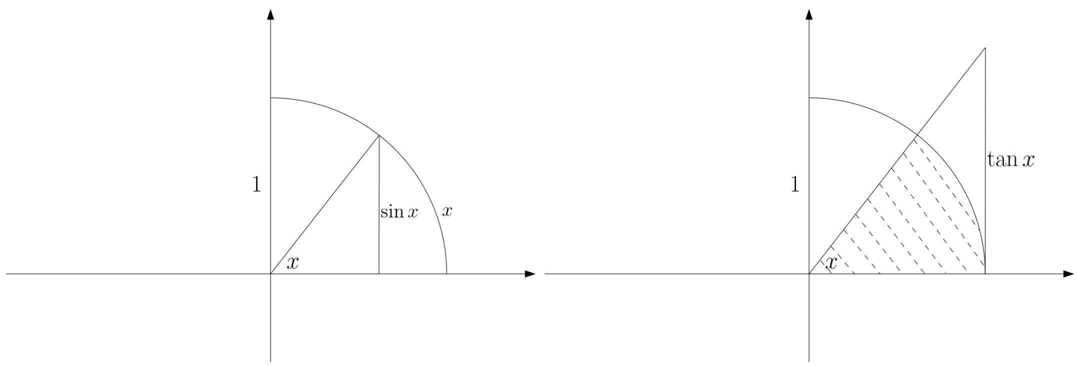
for $x \in (-\pi/2, \pi/2)$. In fact, if (2.14) is known, then using

$$\lim_{x \to 0} \cos x = \cos 0 = 1,$$

the squeeze theorem implies

$$\lim_{x\to 0} \frac{\sin x}{x} = 1.$$

It remains to check (2.14). Since all the functions involved are even, we only need to look at the case when $x \in (0, \pi/2)$. There are two inequalities: $x \le \tan x$ and $\sin x \le x$. They can be taken care of by the following pictures. The left one shows $\sin x \le x$. The right one compares the areas of the sector and the larger triangle and shows $\pi \cdot \frac{x}{2\pi} \le \frac{1}{2}\tan x$.



2.6. **Continuous functions.** We know that any polynomial $P(x)$ satisfies that

$$\lim_{x\to x_0} P(x) = P(x_0)$$

for any $x_0 \in \mathbb{R}$. Pictorially, this means that the graph of $P(x)$ is a "curve without a break," or continuous.

In general, given a function $f\colon I \to \mathbb{R}$ on an open interval I and $x_0 \in I$, we say f is **continuous** at x_0 if

$$\lim_{x\to x_0} f(x) = f(x_0).$$

If f is continuous at any $x \in I$, we say f is a **continuous function** on I.

Example 2.15. Most of the functions we encountered are continuous on their domains.

(1) Let $P\colon \mathbb{R} \to \mathbb{R}$ be a polynomial. Then P is continuous.

(2) $\sin, \cos$, and tan are continuous functions on their domains.

(3) For any $a > 0$, a^x and $\log_a x$ are continuous functions on their domains.

(4) The function

$$f(x) = \begin{cases} x-3 & \text{if } x \le 3 \\ (x-3)^2 & \text{if } x > 3 \end{cases}$$

is continuous on $\mathbb{R}$.

(5) The function

$$f(x) = \begin{cases} \frac{1}{x} & \text{if } x \ne 0 \\ 2 & \text{if } x = 0 \end{cases}$$

is not continuous at $x = 0$.

We quickly mention some properties of continuous functions. Most of them follow directly from the limit laws we learned.

Proposition 2.16. Let f and g be continuous functions on an interval I.

(1) $f + cg$ is continuous for $c \in \mathbb{R}$.

(2) $f \cdot g$ is continuous.

(3) If $g(x_0) \neq 0$ at $x_0 \in I$, then $\dfrac{f}{g}$ is continuous at x_0.

(4) If $f \circ g$ is well-defined, i.e., if $g(I) \subseteq I$, then $f \circ g$ is continuous.

The results above also work for one-sided limits. In general, we say f is **continuous from the right** at x_0 if

$$\lim_{x \to x_0^+} f(x) = f(x_0),$$

and f is **continuous from the left** at x_0 if

$$\lim_{x \to x_0^-} f(x) = f(x_0).$$

Then the laws in Proposition 2.16 also hold for those functions that are continuous from one side.

On a closed interval $[a, b]$, we say $f\colon [a, b] \to \mathbb{R}$ is a continuous function if for $x_0 \in (a, b)$, we have

$$\lim_{x \to x_0} f(x) = f(x_0)$$

and moreover,

$$\lim_{x \to a^+} f(x) = f(a) \text{ and } \lim_{x \to b^-} f(x) = f(b).$$

An important result for a continuous function on a closed interval is the Intermidiate value theorem.

Theorem 2.17. Let f be a continuous function on a closed interval $[a, b]$. If $f(a) \neq f(b)$, then given any number R between $f(a)$ and $f(b)$, there exists $c \in [a, b]$ such that $f(c) = R$.

This allows us to locate roots of equations coming from continuous functions.

Example 2.18. $2^x - x^2 = 0$ has at least two roots by seeing $x = 1, 3, 5$.

2.7. **Asymptotics at infinity.** When a function f is defined on an open interval of the form (a, ∞) for some $a \in \mathbb{R}$, we can look at the asymptotic behavior of f as x is larger and larger. This allows us to define

$$\lim_{x \to \infty} f(x) = L$$

for some $L \in \mathbb{R}$ similarly to the usual way. That is, we say $f(x)$ has limit L as $x \to \infty$ if given any restriction $R > 0$, there exists a range $A > 0$ such that

$$\text{if } x \geq A, \text{then } |f(x) - L| < R.$$

One can see that this is based on the same ideas as (2.2) and (2.6).

Example 2.19. The limit laws in Proposition 2.8 also holds in this case. We see some examples.

(1) The basic example is
$$\lim_{x\to\infty}\frac{1}{x}=0.$$
This can be verified by the definition directly. Given any $R>0$, we can simply take $A=2/R$, since if $x\geq 2/R$, we get
$$\left|\frac{1}{x}-0\right|=\frac{1}{x}\leq\frac{R}{2}<R.$$

(2) From example (1), we can calculate most limits of rational functions. For example,
$$\lim_{x\to\infty}\frac{2x-1}{x^2+1}=\lim_{x\to\infty}\frac{(2x-1)/x^2}{(x^2+1)/x^2}=\lim_{x\to\infty}\frac{\frac{2}{x}-\frac{1}{x^2}}{1+\frac{1}{x^2}}=\frac{0-0}{1+0}=0.$$

(3) When such an asymptotic limit does not exist, it is also possible to determine its blow-up behavior as in Section 2.4. For example,
$$\lim_{x\to\infty}2x^2=\infty \text{ and } \lim_{x\to\infty}-x^3=-\infty.$$

(4) We can use the trick of rationalization here also. For example,
$$\begin{aligned}\lim_{x\to\infty}\left(\sqrt{x^2+4}-x\right)&=\lim_{x\to\infty}\frac{\left(\sqrt{x^2+4}-x\right)\left(\sqrt{x^2+4}+x\right)}{\sqrt{x^2+4}+x}\\&=\lim_{x\to\infty}\frac{4}{\sqrt{x^2+4}+x}=0.\end{aligned}$$
Note that we can't take the difference of the two terms directly since none of these two limits exists.

(5) Other examples include
$$\begin{aligned}\lim_{x\to\infty}\left(\frac{1}{2}\right)^x&=0,\\ \lim_{x\to\infty}\arctan x&=\frac{\pi}{2}, \text{ and}\\ \lim_{x\to-\infty}\arctan x&=-\frac{\pi}{2},\end{aligned}$$
where the asymptotic at $-\infty$ can be defined rigorously in the same way.

The squeeze theorem also holds in this situation.

Theorem 2.20 (Squeeze theorem)**.** Let $a\in\mathbb{R}$ and f,g, and h be real-valued functions on (a,∞). Suppose $f(x)\leq g(x)\leq h(x)$ for all $x\in(a,\infty)$. If for some $L\in\mathbb{R}$,
$$\lim_{x\to\infty}f(x)=\lim_{x\to\infty}h(x)=L,$$
then lim $\lim_{x\to\infty}g(x)=L$.

Example 2.21. $\lim_{x\to\infty}\frac{\sin x}{x}=0.$