CALCULUS 1, NOTE
FUNCTIONS

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The subject of the course is calculus on the (one-dimensional) real line $\mathbb{R}$. Thus, we want to look at functions defined on parts (or all) of the real line. We start by introducing the notions of sets and functions on the real line.

1. **Functions**

Functions are rules between sets, so we start from talking about what a set is.

1.1. **Sets.** A **set** is a collection of objects that we care about.

Example 1.1. We mention some sets that we will use many times later.

(1) The set of positive integers will be denoted by
$$\mathbb{N} = \{1, 2, 3, \cdots\}.$$

(2) The set of integers will be denoted by
$$\mathbb{Z} = \{0, \pm 1, \pm 2, \cdots\}.$$

(3) The set of real numbers will be denoted by $\mathbb{R}$, containing $3, -\sqrt{2}, \pi$, and (infinitely) many other numbers.

We usually use $A, B, C, ...$ to denote a set. When an object a is an element of A, we write $a \in A$.

Example 1.2. There are a few descriptions to explain how a set is made of, especially when we are working on the real line.

(1) In general, we use the bracket
$$\{a \in A : \text{the property that you want to prescribe}\}$$
to collect elements in a known set A that satisfy certain properties.

(2) For example, we use
$$[2, 3] := \{x \in \mathbb{R} : 2 \leq x \leq 3\}$$
to denote the **closed interval** between 2 and 3 on the real line.

(3) Similarly, we use
$$(2, 3) := \{x \in \mathbb{R} : 2 < x < 3\}$$
to denote the **open interval** between 2 and 3 on the real line.

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Given two sets A and B, if all the elements of A belong to B, then we say A is a subset of B and write $A \subseteq B$. For example, $(2,3) \subseteq [2,3] \subseteq \mathbb{R}$.

In this class, we mostly work on subsets of the real line $\mathbb{R}$.

1.2. **Functions.** Given two subsets A and B of $\mathbb{R}$, a **function** f from A to B is a rule to assign to any element $a \in A$ a unique element $f(a) \in B$. We often write $f \colon A \to B$. For this function, A is the **domain** of f and B is the **codomain** of f.

Example 1.3. There are examples based on the basic operations we know.

(1) $f(x) = 2x + 3$ is a function from $\mathbb{R}$ to $\mathbb{R}$.

(2) Given $x \in \mathbb{R}$, we let $\lfloor x \rfloor$ be the largest integer that is at most x. Thus, we can define a function $g \colon \mathbb{R} \to \mathbb{Z}$ by letting $g(x) = \lfloor x \rfloor$. e.g., $g(2) = 2$ and $g(3.4) = 3$.

(3) $h(x) = x^3 + 2x^2 - x - 3$ is a function from $\mathbb{R}$ to $\mathbb{R}$. It is a polynomial function.

It is possible that a function is not defined on the whole real line. We may need to remove some parts of $\mathbb{R}$. In general, given two subsets $A, B \subseteq \mathbb{R}$, we use

$$A \setminus B = \{a \in A : a \notin B\}$$

to denote the difference of A and B.

(4) Quotients of polynomials are also common functions. For example,

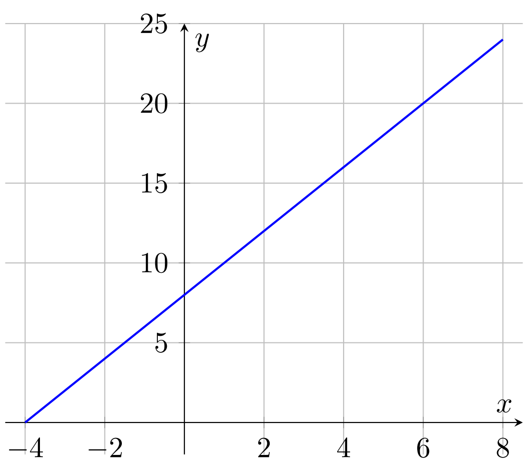
$$x \mapsto \frac{x^2 - 3}{x + 1}$$

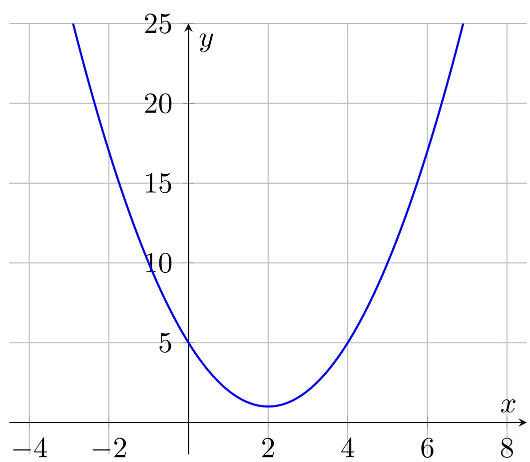
defines a function from $\mathbb{R} \setminus \{-1\}$ to $\mathbb{R}$.

1.3. **Graphs.** An efficient way to present a function is through its **graph**. Given a function $f \colon A \to B$, the graph of f is the set that collects all the points of the form $(a, f(a))$ for $a \in A$. When A and B are subsets of $\mathbb{R}$, we can draw the graph of f on the two-dimensional plane $\mathbb{R}^2$.

Example 1.4. We draw the graphs of some functions.

(1) Let $f(x) = 2x + 8$. Then the graph of f on $[-4, 8]$ is as below.





(2) Let $f(x) = x^2 - 4x + 5$. Then the graph of f on $[-4, 8]$ is as above.

1.4. **Common properties of real-valued functions.** We introduce different properties of functions.

Definition 1.5. Let $f\colon I \to \mathbb{R}$ be a function on an interval I in $\mathbb{R}$.

(1) We say f is **increasing** if $f(a) < f(b)$ for any $a, b \in I$ with $a < b$. E.g., $f(x) = 2x$.

(2) We say f is **decreasing** if $f(a) < f(b)$ for any $a, b \in I$ with $a > b$. E.g., $f(x) = -x - 2$.

For the following two properties, we further assume that I is symmetric with respect to the origin. A common example is when $I = \mathbb{R}$.

(3) We say f is **even** if $f(x) = f(-x)$ for any $x \in I$. E.g., $f(x) = x^2 + 3$.

(4) We say f is **odd** if $f(x) = -f(-x)$ for any $x \in I$. E.g., $f(x) = x^3 + x$.

After we review more types of functions, we can determine whether they satisfy these properties or not.

1.5. **Different types of functions.** We mention some common functions that we will often deal with in the rest of the course.

1.5.1. *Polynomials.* A **polynomial** in variable x is of the form

$$f(x) = a_n x^n + \cdots + a_1 x + a_0.$$

For example, a quadratic polynomial is of the form $ax^2 + bx + c$.

There are three important problems commonly associated with polynomial functions.

(1) Finding extrema: When we are given a quadratic polynomial, we can find its global maximum/minimum by completing the square. For example,

$$\begin{aligned} f(x) &= 2x^2 + 4x + 6 \\ &= 2(x+1)^2 + 4, \end{aligned}$$

so $f(x)$ has the global minimum 4 at $x = -1$.

(2) Solving equations: We know how to solve a quadratic polynomial equation $ax^2 + bx + c = 0$ by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

when $a \neq 0$. This sometimes allows us to solve higher degree polynomial equation. For example, for $x^3 + 3x^2 + 2x = 0$, we can first factor

$$x^3 + 3x^2 + 2x = x(x^2 + 3x + 2),$$

and using $x^2 + 3x + 2 = (x+1)(x+2)$, we can solve the equation by $x = 0, -1, -2$.

(3) Finding a polynomial function passing through certain points: Usually, there exists a unique degree n polynomial whose graph passes through given $(n+1)$ points. For example, given three points (a_1, b_1), (a_2, b_2), and (a_3, b_3), we can plug them into $f(x) = ax^2 + bx + c$ to solve a, b, and c.

1.5.2. *Rational functions.* Quotients of polynomials are called **rational functions**. They are defined at those values at which the denominator does not vanish, i.e., away from the roots of the denominator. These points are not in the domain of the rational function.

Example 1.6. Let $f(x) = \frac{x+2}{2x^2-3x+1}$. Then the denominator is $(2x-1)(x-1)$, so f is not defined at $x = 1$ or $x = 1/2$.

It is important to know how to simplify a given expression consisting of rational functions. This is like calculations of fractions.

1.5.3. *Trigonometric functions.* We recall the general definition of trigonometric fucntions. Remember that we mostly use the radian measure (unless otherwise indicated). That is, the amount π means $180°$.

For a point on the unit circle, if the segment from the point to the origian has angle x, then the coordinate of the point is $(\cos x, \sin x)$. Based on this definition, we know that sin and cos are **periodic** function, in the sense that for any $x \in \mathbb{R}$,

$$\sin(x + 2\pi) = \sin x \text{ and } \cos(x + 2\pi) = \cos x.$$

When $x \in (0, \pi/2)$, i.e., x is an acute angle, we can use a right triangle to determine $\sin x$ and $\cos x$.

When $\cos x \neq 0$, we also commonly use $\tan x = \frac{\sin x}{\cos x}$. Here are some commonly used trigonometric ratios for an acute angle.

$$\begin{aligned} &\sin\frac{\pi}{6} = \frac{1}{2}, && \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}, && \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}, \\ &\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}, && \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}, && \cos\frac{\pi}{3} = \frac{1}{2}, \\ &\tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}, && \tan\frac{\pi}{4} = 1, && \tan\frac{\pi}{3} = \sqrt{3}. \end{aligned}$$

For a general angle, one can first determine its parity based on the quadrant, and then its value using the angel with the x-axis. For example, if we are given the angle $2\pi/3 = 120°$, it is in the second quadrant and has angle $\pi/3$ with the x-axis, so

$$\sin\frac{2\pi}{3} = \frac{\sqrt{3}}{2}, \cos\frac{2\pi}{3} = -\frac{1}{2}, \text{ and } \tan\frac{2\pi}{3} = -\sqrt{3}.$$

An important relation is

$$\sin^2 x + \cos^2 x = 1 \tag{1.7}$$

for any $x \in \mathbb{R}$. In particular, we know that both $\cos x$ and $\sin x$ are valued in $[-1, 1]$. Moreover, using this, we can determine the magnitude of all trigonometric functions when given one.

Example 1.8. Suppose $x \in [\pi, 3\pi/2]$ and $\sin x = -1/3$. We can then determine $\cos x$ and $\tan x$. In fact, by the relation (1.7), we know $\cos^2 x = 8/9$. Since $\cos x$ should be non-positive, we get

$$\cos x = -\frac{2\sqrt{2}}{3} \text{ and hence } \tan x = \frac{1}{2\sqrt{2}}.$$

We recall some other useful relations for $\sin x$ and $\cos x$.

(1) $\sin(x + y) = \sin x \cos y + \sin y \cos x$.

(2) $\cos(x+y) = \cos x \cos y - \sin x \sin y$.

One can first determine whether $\sin x$ and $\cos x$ are even or odd functions, and use this to derive formulas for $\sin(x-y)$ and $\cos(x-y)$.

Example 1.9. With those relations in hand, we can solve equations involving trigonometric functions. For example, given

$$1 + \cos 2x = \cos x$$

with $x \in [0, \pi]$, using $\cos 2x = \cos^2 x - \sin^2 x = 2\cos^2 x - 1$, we obtain

$$2\cos^2 x = \cos x,$$

so $\cos x = 0$ or $1/2$. Hence, $x = \pi/3$ or $\pi/2$.

1.5.4. *Exponential fucntion and logarithm.* Exponential functions form another important class of functions that we will encounter a lot. They are (simplified) models for many real-world problems.

For $a > 0$, the function $f(x) = a^x$ is the **exponential function with base** a. This definition can be clarified in a few steps.

(1) When $x \in \mathbb{N}$, we define it iteratively. That is, $a^1 = a$, $a^2 = a \cdot a$, and so on.

(2) When $x = 1/n$ for some $n \in \mathbb{N}$, we define $a^{1/n} = \sqrt[n]{a}$. It is the positive number such that

$$\left(a^{1/n}\right)^n = \left(\sqrt[n]{a}\right)^n = a.$$

(3) When $x = -1$, we let $a^{-1} = 1/a$.

Using these and the laws of exponents:

$$a^x \cdot a^y = a^{x+y} \text{ and } a^{xy} = (a^x)^y\,, \tag{1.10}$$

we can define a^x for all $x \in \mathbb{Q}$, the set of **rational numbers** or fractions. Any real number can be approximated by rational numbers, so we can define its exponential accordingly.

(4) For $x \in \mathbb{R}$, we choose a sequence x_n that is closer and closer to x and define a^x to be the number that is the "limit" of a^{x_n}.

For example, if we want to calculate the value of 2^π, we can look at the sequence

$$2^3, 2^{3.1}, 2^{3.14}, 2^{3.141}, 2^{3.1415}, 2^{3.14159}, \cdots$$

and see what it converges to. By its construction, one can see that a^x is always positive for $a > 0$ and $x \in \mathbb{R}$.

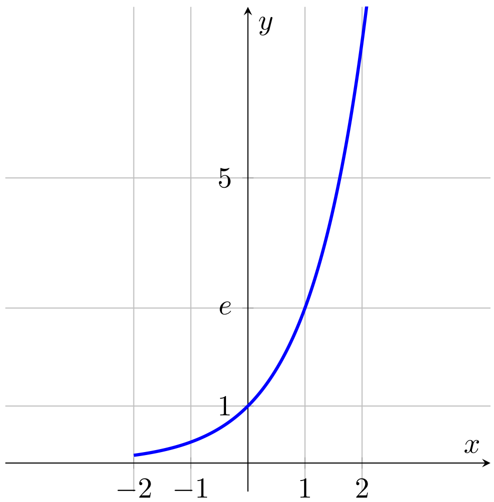
Example 1.11. It is important to know how to use the laws of exponents to simplify expression involving unknowns. For example,

$$\frac{\left(2x^{5/3}\right)^3}{\left(4x^{-1/3}\right)^2} = \frac{8x^5}{16x^{-2/3}} = \frac{1}{2}x^{17/3}.$$

There is a special choice of the base of exponential functions. One way to define it is by

$$e := \lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n.$$

Its value is about 2.71828. It naturally arises in many applications, e.g., compounding interest. Another way to define it is that e is the unique positive number a such that the slope of the tangent of $f(x) = a^x$ at $x = 0$ is 1. Its graph looks like the following.



We will see this in more detail later when we learn how to calculate the slope of a tangent.

Logarithm is the inverse operation of taking exponential. For $a > 0$, the function $\log_a$ is defined on the set of positive real number. Given $x > 0$, $\log_a x$ is defined to be the unique number such that

$$a^{\log_a x} = x = \log_a a^x.$$

This is based on the fact that the exponential function a^x is "surjective" onto the set of positive real numbers. Similarly, it satisfies the laws of logarithms, corresponding to (1.10):

$$\log_a xy = \log_a x + \log_a y \text{ and } \log_a x^r = r \log_a x.$$

When the base is e, we will simply write $\log_e = \log = \ln$. Note that we can always change any base to e by

$$\log_a x = \frac{\log x}{\log a}.$$

Example 1.12. Using logarithms is a valid way to express solutions to equations involving exponetial functions. For example, given the equation $e^{4x-2} = 6$, we can take the logarithm to get $4x - 2 = \log 6$, so

$$x = \frac{1}{4}(2 + \log 6).$$

1.5.5. *Piecewise function.* We can also define a function by different formulas on different parts of $\mathbb{R}$. For example the function

$$f(x) = \begin{cases} 2x - 3 & \text{if } x \in (2, 3] \\ 3x^2 & \text{if } x \in [0, 2] \end{cases}$$

is a function on $[0, 3]$. An important example is the **absolute value** function, given by

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x \leq 0 \end{cases}.$$

It measures the distance from x to the origin, so it is always non-negative.

1.6. **Special types of functions.** (optional) We talk about some commonly used types of functions here.

Definition 1.13. Let $f\colon A \to B$ be a functions between two sets.

(1) The set A is called the **domain** of f.

(2) The set $\{f(a) \in B : a \in A\}$ is called the **image** of f.

(3) If $f(a_1) \neq f(a_2)$ for any $a_1 \neq a_2$ in A, then we say f is **injective**.

(4) If for any $b \in B$, there exists $a \in A$ such that $b = f(a)$, then we say f is **surjective**.

(5) If f is both injective and surjective, then we say f is **bijective**.

We already know what the domain of a function is. A subtle point is that the image is not always equal to the codomain. When they are, the function is surjective. On the other hand, the injectivity is related to the possibility of inverting a function. These are directly illustrated by the graph of a function.

Example 1.14. We emphasize that the injectivity, surjectivity, and bijectivity all depends on the domain and codomain of a function. We see some examples.

(1) Consider $f, g\colon [0, \pi] \to [0, 1]$ defined by $f(x) = \sin x$ and $g(x) = \cos x$. Then f is not injective but g is injective. They are both surjective, so g is bijective.

(2) If we view $\sin\colon \mathbb{R} \to \mathbb{R}$ as a function mapped into $\mathbb{R}$, then it is not surjective anymore.

(3) The exponential function e^x is a bijection from the real line to the positive real line. Note that it is surjective since given any $y > 0$, we can always find $x = \log y$ such that $e^x = y$.

1.7. **Inverse and composition of functions.** Example 3 above implicitly tells us that a function is bijective when we can kind of "invert" the function. We will make this relation clear. We know that logarithms are inverse operations to exponential functions. This is a general operation that we can do for other functions. To make this concrete, we first introduce an operation.

Definition 1.15. Let $f\colon A \to B$ and $g\colon B \to C$ be two functions. Their **composition** is defined by

$$g \circ f(a) := g\left(f(a)\right).$$

That is, $g \circ f\colon A \to C$ is a function from A to C.

Example 1.16. Let $f(x) = \sin x$ and $g(x) = x^2 - 2$. Then

$$f \circ g(x) = \sin\left(x^2 - 2\right) \text{ and } g \circ f(x) = \sin^2 x - 2.$$

Example 1.17. Rational functions can be viewed as the product of a polynomial and the composition of $\frac{1}{y}$ and another polynomial.

One should be careful that $f \circ g$ and $g \circ f$ may not make sense at the same time. For example, if $f\colon \mathbb{N} \to \mathbb{R}$ is defined by $f(x) = 1/x$ and $g\colon \mathbb{N} \to \mathbb{N}$ is defined by $g(x) = 2x$, then $f \circ g$ is valid but $g \circ f$ is not. It is important to notice the domains and the codomains of the functions.

Now, we can talk about the general notion of inverse functions. We first define the **identity function** for any set A to be

$$\mathrm{id}_A\colon A \to A, a \mapsto a.$$

For example, for any subset A of the real line, $\mathrm{id}_A(x) = x$.

Definition 1.18. Let $f\colon A \to B$ be a function. If $g\colon B \to A$ is a function such that

$$f \circ g = \mathrm{id}_B \text{ and } g \circ f = \mathrm{id}_A,$$

then g is the **inverse function** of f.

One may notice that we say "the inverse function." This is because such an inverse function must be unique if it exists. When it does, we say both f and g are **invertible**. Note that being invertible places strong restriction on f, including its domain and codomain. Thus, similar to bijectivity and the other properties, invertibility highly depends on the domain and codomain of a function. In fact, invertibility and bijectivity are equivalent.

Theorem 1.19. Let $f\colon A \to B$ be a function between two sets. Then f is invertible if and only if f is bijective.

That is, a function $f\colon A \to B$ is invertible if and only if given any $b \in B$, there exists a unique element $a \in A$ such that $f(a) = b$.

Example 1.20. We mention some common examples of inverse functions. We let $\mathbb{R}_{>0} := \{x \in \mathbb{R} : x > 0\}$ be the set of positive real numbers.

(1) Let $f\colon \mathbb{R} \to \mathbb{R}_{>0}$ be the exponential function $f(x) = e^x$. Then f is bijective and its inverse is the logarithm $g\colon \mathbb{R}_{>0} \to \mathbb{R}$ given by $g(x) = \log x$.

(2) Let $\sin\colon [-\pi/2, \pi/2] \to [-1, 1]$ be the sine function restricted to smaller intervals on which it is bijective. Then its inverse is denoted by

$$\arcsin\colon [-1, 1] \to [-\pi/2, \pi/2].$$ [1]

For example, $\arcsin 1/2 = \pi/6$.

(3) Similarly, consider $\cos\colon [0, \pi] \to [-1, 1]$ and $\tan\colon (-\pi/2, \pi/2) \to \mathbb{R}$. Then their inverses are denoted by

$$\arccos\colon [-1, 1] \to [0, \pi] \text{ and } \arctan\colon \mathbb{R} \to \left(-\frac{\pi}{2}, \frac{\pi}{2}\right).$$

Note that there are many ways to find an inverse trigonometric functions if we restrict them to different intervals of $\mathbb{R}$. The definitions we make above are choices that are commonly used.

Example 1.21. $\arcsin \frac{1}{2} = \frac{\pi}{6}$, $\arccos \frac{1}{2} = \frac{\pi}{3}$, and $\arctan 1 = \frac{\pi}{4}$.

Example 1.22. In general, given a function, it is also possible to find its inverse function on a smaller domain. For example, given $f(x) = x^2 + 2$, if its inverse $g = g(y)$ exists, it will satisfy

$$y = f\left(g(y)\right) = g(y)^2 + 2,$$

so $g(y) = \pm\sqrt{y - 2}$. We know that these are well-defined only when we restrict to $y \geq 2$. Thus, we can find such a g when we restrict f to $[0, \infty)$ or $(-\infty, 0]$.

[1] Another common notation for the inverse of sin is $\sin^{-1} x$, which may be confusing with the reciprocal of $\sin x$.