5. Introduction to Riemann surfaces

In the second part of this semester, we will talk about topics related to different aspects of Riemann surfaces. It will mainly be about harmonic functions, conformal mapping, and the uniformization problem. All of these topics require techniques of taking limits of a sequence of holomorphic or meromorphic functions, and hence we will start from this.

5.1. Normal families in function spaces.

5.1.1. *Normal families of continuous functions.* We will talk about compactness properties of a family of holomorphic functions on an open set U. This question already appears in the study of real-valued functions (for example, the Arzelà–Ascoli theorem), and, as expected, will have many interesting applications. We will see again that because of many nice properties of holomorphic functions, compactness may hold under much weaker conditions.

We first recall some properties for families of continuous functions. They can be discussed in a very general situation, but we just focus on functions on an open set U in $\mathbb{C}$. For simplicity, we will let $\mathcal{C}(U)$ be the set of (real-valued) continuous functions on U.

Definition 5.1. Let $\mathcal{F}$ be a family of continuous functions on U; that is, $\mathcal{F} \subseteq \mathcal{C}(U)$.

(1) We say $\mathcal{F}$ is **equicontinuous** on a set $S \subseteq U$ if given any $\varepsilon > 0$, there exists $\delta > 0$ such that for any $f \in \mathcal{F}$ and $z, w \in S$ with $|z - w| < \delta$, we have $|f(z) - f(w)| < \delta$.

(2) We say $\mathcal{F}$ is **normal** if any sequence $f_n \in \mathcal{F}$ has a subsequence which converges locally uniformly on U.[12]

This may remind us of the sequential compactness properties for metric spaces. Remember it says that in a metric space X, a subset K is compact if and only if it is sequentially compact.[ex] Thus, a natural question is whether we can find a "reasonable metric" on the space $\mathcal{C}(U)$. If so, then it may give us good criteria to check property (2).

There is a natural norm that one can put on the space $\mathcal{C}(K)$ when K is a compact subset of U. That is, the **sup norm**, defined by

$$\|f\|_{\mathcal{C}(K)} := \sup_K |f| \tag{5.2}$$

for $f \in \mathcal{C}(K)$. This is a well-defined complete norm when K is compact, and we know that convergence with respect to $\| \cdot \|_{\mathcal{C}(K)}$ means uniform convergence on K.[ex] However, we would like to work with $\mathcal{C}(U)$ where U is an open set, so not necessarily (and mostly not) compact. An idea we can use is to try to write U as a union of compact sets K_j's. Then try to connect the convergence property we want to a combination of metrics on these K_j's. We divide this idea into three steps.

(a) **Exhaustion.** We suppose there exist compact sets K_j's such that

$$U = \bigcup_{j=1}^{\infty} K_j$$

with $K_j \subseteq \mathrm{int} K_{j+1}$ for all $j \in \mathbb{N}$. This then tells us that knowing how to control a family $\mathcal{F}$ in each K_j is enough for us to control it in an arbitrary compact set $K \subseteq U$. This is because given a

[12] Note that the limit does not have to lie in $\mathcal{F}$.

compact set $K \subseteq U$, the assumptions above then allow us to find some $j \in \mathbb{N}$ such that $K \subseteq K_j$.[ex] Such an exhaustion always exists, and one possible construction have been used before; see (4.23) in the proof of the Mittag-Leffler theorem.

(b) **Normalized norms.** For $f, g \in \mathcal{C}(U)$, we can naturally view them as elements in $\mathcal{C}(K_j)$ for any j by restricting them to $K_j \subseteq U$. Thus, as in (5.2), we can consider

$$d_j(f,g) := \sup_{K_j} |f-g|,$$

well-defined on $\mathcal{C}(U)$ (though not a metric yet). Since we are going to combine all information from K_j's, we first normalize a single d_j by considering

$$\delta_j(f,g) := \frac{d_j(f,g)}{1+d_j(f,g)}.$$

A few simple consequences of this normalization are listed below.

(1) $\delta_j(f,g) < 1$ and $\delta_j(f,g) \le d_j(f,g)$.

(2) $d_j(f,g) \le 1$ if and only if $\frac{1}{2} d_j(f,g) \le \delta_j(f,g) \le \frac{1}{2}$.

(c) **Locally uniform convergence property.** For $f, g \in \mathcal{C}(U)$, we now combine all $\delta_j(f,g)$'s by taking

$$\rho_U(f,g) = \rho(f,g) := \sum_{j=1}^{\infty} \frac{1}{2^j} \delta_j(f,g),$$

which is always finite by property (1) above. One can show that ρ defines a metric on $\mathcal{C}(U)$.[ex] Our desired convergence property is summarized in the following lemma.

Lemma 5.3. Let U be an open set in $\mathbb{C}$ and f_n's and f be elements in $\mathcal{C}(U)$. Then the following are equivalent.

(1) In the metric space $(\mathcal{C}(U), \rho_U)$, $f_n \to f$ as $n \to \infty$.

(2) On any compact subset of U, f_n converges to f uniformly.

Proof. First, we prove (1)⇒(2). Given a compact set $K \subseteq U$, we can take j large enough such that $K \subseteq K_j$. Since $\rho(f_n, f) \to 0$, in particular, we have $\delta_j(f_n, f) \to 0$. This implies $d_j(f_n, f) \to 0$. Thus, f_n converges to f uniformly on K_j, and hence, on K.

Next, we prove (2)⇒(1). By the assumption, we know that f_n converges to f on every K_j. To show that $\rho(f_n, f) \to 0$, we are going to bound it in two parts as follows. For some $J \in \mathbb{N}$ to be chosen, we write

$$\begin{aligned}\rho(f_n, f) &= \sum_{j=1}^{\infty} \frac{1}{2^j} \delta_j(f_n, f) \\ &\le \sum_{j=1}^{J} \frac{1}{2^j} \delta_j(f_n, f) + \sum_{j=J+1}^{\infty} \frac{1}{2^j}\end{aligned}$$

using $\delta_j < 1$ in property (1) of δ_j's. To this end, given any $\varepsilon > 0$, we find J so large that $2^{-J} < \varepsilon/2$. After fixing this J, using the uniform convergence on K_J, we can find N such that for any $n \ge N$,

$\delta_J(f_n, f) < \varepsilon/(2J)$. In particular, $\delta_j(f_n, f) < \varepsilon/(2J)$ for all $j \leq J$ and $n \geq N$. Putting these into the decomposition above, we have

$$\begin{aligned}\rho(f_n, f) &\leq \sum_{j=1}^{J} \frac{1}{2^j}\delta_j(f_n, f) + \sum_{j=J+1}^{\infty} \frac{1}{2^j} \\ &\leq \sum_{j=1}^{J} \frac{\varepsilon}{2J} + \frac{\varepsilon}{2}\sum_{j'=1}^{\infty} \frac{1}{2^{j'}} \\ &= \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon\end{aligned}$$

for $n \geq N$. This proves $\rho(f_n, f) \to 0$. □

This characterization also tells us that the metric ρ_U is a complete metric. This follows directly from Lemma 5.3 and the the completeness of $\left(\mathcal{C}(K), \|\cdot\|_{\mathcal{C}(K)}\right)$ for a compact set K.

Corollary 5.4. Given an open set U in $\mathbb{C}$, the metric space $(\mathcal{C}(U), \rho_U)$ is complete.

Now we can rephrase the definition in (2). That is, a family $\mathcal{F} \subseteq \mathcal{C}(U)$ is normal if and only if $\overline{\mathcal{F}}$ is compact in $\mathcal{C}(U)$. What we are looking for is "good description" that guarantees normality. We mention two equivalent conditions here. The first one is through the notion of total boundedness.

Definition 5.5. A subset S in a metric space (X, d) is called **totally bounded** if given any $\varepsilon > 0$, S can be covered by finitely many open balls of radius ε. That is, there exist $x_1, \cdots, x_N \in X$ for some $N \in \mathbb{N}$ such that

$$S \subseteq \bigcup_{n=1}^{N} B_\varepsilon(x_n).$$

Recall that a metric space is compact if and only if it is complete and totally bounded. As in Lemma 5.3, we may expect that we can translate this equivalence to the local setting, that is, a description on a compact set.

Proposition 5.6. Given an open set U in $\mathbb{C}$ and a family $\mathcal{F} \subseteq \mathcal{C}(U)$, the following are equivalent.

(1) $\mathcal{F}$ is normal.

(2) $\mathcal{F}$ is totally bounded in $(\mathcal{C}(U), \rho_U)$.

(3) For any compact $K \subseteq U$ and $\varepsilon > 0$, there exist finitely many $f_1, \cdots, f_N \in \mathcal{F}$ such that for any $f \in \mathcal{F}$, there exists f_ℓ with

$$\sup_K |f - f_\ell| < \varepsilon.$$

Note that (3) may appear to be a straightforward reformulation of (2). As in Lemma 5.3, however, the connection is more subtle due to the openness of U. We leave the proof, which uses the compactness of $\overline{\mathcal{F}}$ ((1)⇔(2)) and the same idea as in Lemma 5.3 ((2)⇔(3)), to Assignment 6.

One consequence of Proposition 5.6 is the famous Arzelà–Ascoli theorem.

Theorem 5.7 (Arzelà–Ascoli)**.** Let U be an open set in $\mathbb{C}$ and $\mathcal{F} \subseteq \mathcal{C}(U)$. Then $\mathcal{F}$ is a normal family if and only if the following conditions are true.

(1) $\mathcal{F}$ is equicontinuous on any compact set $K \subseteq U$.

(2) For any $z \in U$, the set $\{f(z) : f \in \mathcal{F}\}$ is bounded.

Proof. First we show that the normality of $\mathcal{F}$ implies (1) and (2). For (1), given a compact set $K \subseteq U$ and a positive number ε, we can take $f_1, \cdots, f_N$ such that for any $f \in \mathcal{F}$, there exists f_ℓ such that

$$\sup_K |f - f_\ell| < \frac{\varepsilon}{3} \tag{5.8}$$

by Proposition 5.6 (3). Now, since each of f_n's is uniformly continuous on K, we can find $\delta > 0$ such that for any $\ell = 1, \cdots, N$ and $z, w \in K$ with $|z - w| < \delta$, we have

$$|f_\ell(z) - f_\ell(w)| < \frac{\varepsilon}{3}. \tag{5.9}$$

Combining (5.8) and (5.9), we get (1), since for any $f \in \mathcal{F}$, we first find f_ℓ with (5.8), and hence

$$|f(z) - f(w)| \leq |f(z) - f_\ell(z)| + |f_\ell(z) - f_\ell(w)| + |f_\ell(w) - f(w)| \leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon,$$

so (1) follows.

For (2), given $z \in U$, one can apply Proposition 5.6 (3) again to a small compact ball around z and a positive number ε, to get finitely many $f_1, \cdots, f_\ell$. Then for any $f \in \mathcal{F}$, we have

$$|f(z)| \leq \max \{|f_n(z)| + \varepsilon : n = 1, \cdots, \ell\}.$$

This shows the boundedness of the set.

In the rest of the proof, we show that (1) and (2) imply $\mathcal{F}$ is normal. Let Q be the set of rational points in U, and hence Q is a countable dense set. We number these rational points by

$$Q = \{q_k : k \in \mathbb{N}\}.$$

Now, suppose we are given a sequence f_n in $\mathcal{F}$. We look at q_1. By condition (2), we know that $f_n(q_1)$ is a bounded sequence in $\mathbb{C}$, so the Bolzano–Weierstrass theorem implies that it admits a convergent subsequence. This gives a subsequence f_{n1} of f_n. Next, we look at this new subsequence at q_2, that is, the sequence $f_{n1}(q_2)$. This is again a bounded sequence, so we can find a further subsequence f_{n2} of f_{n1} such that $f_{n2}(q_2)$ converges. Iteratively, for each k, we can find a sequence f_{nk}, a subsequence of f_n such that f_{nk} is a subsequence of $f_{n(k-1)}$ and $f_{nk}(q_k)$ is a convergent sequence.

We take the "diagonal subsequence" f_{nn}, and we claim that this is a subsequence we desire. Let $K \subseteq U$ be a compact set. Given $\varepsilon > 0$, by condition (1), we can find $\delta > 0$ such that for any $f \in \mathcal{F}$ and $z, w \in K$ with $|z - w| < \delta$,

$$|f(z) - f(w)| < \frac{\varepsilon}{3}. \tag{5.10}$$

Since K is compact, we can cover it with finitely many balls $B_\delta(x_1), \cdots, B_\delta(x_N)$ with radius δ. For eacu x_j, we can choose a rational point $q_{k_j} \in B_\delta(x_j)$. We know that the sequences $f_{nn}\left(q_{k_j}\right)$ are convergent, and hence Cauchy, for each j. Thus, we can find $n_0 \in \mathbb{N}$ such that

$$\left|f_{nn}(q_{k_j}) - f_{mm}(q_{k_j})\right| \leq \frac{\varepsilon}{3} \tag{5.11}$$

for any $j = 1, \cdots, N$ and $n, m \geq n_0$. Combining (5.10) and (5.11), we can conclude the proof. In fact, using the uniform number n_0 above (only depending on K and ε), given any $z \in K$, we can find some $j = 1, \cdots, N$ such that $z \in B_\delta(x_j)$, and hence

$$\begin{aligned}|f_{nn}(z) - f_{mm}(z)| &\leq |f_{nn}(z) - f_{nn}(x_j)| + |f_{nn}(x_j) - f_{mm}(x_j)| + |f_{mm}(x_j) - f_{mm}(z)| \\ &\leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon,\end{aligned}$$

for $n, m \geq n_0$. This shows that f_{nn} is a Cauchy sequence in $\mathcal{C}(K)$ and hence finishes the proof. □

5.1.2. *Normal families of holomorphic functions.* Now, as promised, we will look at families of holomorphic functions. To this extend, we let $\mathcal{H}(U)$ be the set of holomorphic functions on U.

Because holomorphic functions behave much more rigidly than continuous functions, one should expect a (pre)compactness result with a weaker condition than the Arzelà–Ascoli theorem. The condition we will look at is the following.

Definition 5.12. Let $\mathcal{F} \subseteq \mathcal{H}(U)$. We say $\mathcal{F}$ is **locally bounded** if given any compact set $K \subseteq U$, there exists $C_K < \infty$ such that $\sup_K |f| \leq C_K$ for any $f \in \mathcal{F}$.

Note that if a family $\mathcal{F}$ of functions satisfies (1) and (2) in Theorem 5.7, then $\mathcal{F}$ is locally bounded.[ex] This is a much weaker condition then (1) and (2), especially for families of continuous functions. However, we will see that it is sufficient to guarantee normality for families in $\mathcal{H}(U)$.

Theorem 5.13 (Montel)**.** Let U be an open set in $\mathbb{C}$ and $\mathcal{F} \subseteq \mathcal{H}(U)$. Then $\mathcal{F}$ is normal if and only if $\mathcal{F}$ is locally bounded.

We remark that once we have a normal family $\mathcal{F} \subseteq \mathcal{H}(U)$, then given a sequence in $\mathcal{F}$, after taking a subsequence that converges locally uniformly on U, Weierstrass's theorem (Corollary 2.23) implies that the derivatives of this subsequence also converges locally uniformly on U.

Proof. The only if part is based on Theorem 5.7 and the exercise above that (2) in Theorem 5.7 imply local boundedness. In the rest, we assume $\mathcal{F} \subseteq \mathcal{H}(U)$ is locally bounded.

By Theorem 5.7, it suffices to show that $\mathcal{F}$ is equicontinuous on any compact $K \subseteq U$, which is equivalent to show that $\mathcal{F}$ is equicontinuous on any small open ball.[ex] At a point $z_0 \in U$, take $\delta > 0$ so small that $\overline{B}_{3\delta}(z_0) \subseteq U$. Since $\mathcal{F}$ is locally bounded, we can find $M < \infty$ such that $\sup_{\overline{B}_{3\delta}(z_0)} |f| \leq M$ for all $f \in \mathcal{F}$. Now, for any $f \in \mathcal{F}$ and $z_1, z_2 \in B_\delta(z_0)$, we can then estimate

$$\begin{aligned}|f(z_1) - f(z_2)| = \left|\int_{\partial B_{2\delta}(z_0)} \left(\frac{f(z)}{z - z_1} - \frac{f(z)}{z - z_2}\right) dz\right| &\leq M \int_{\partial B_{2\delta}(z_0)} \frac{|z_1 - z_2|}{|z - z_1| \cdot |z - z_2|} dz \\ &\leq M \cdot \frac{|z_1 - z_2|}{\delta^2} \cdot 2\pi \cdot 2\delta.\end{aligned}$$

This implies the equicontinuity, and the proof is done. □

5.1.3. *Normal families of meromorphic functions.* Montel's theorem has many applications, as we will see later. The first application we will see is its application after being extended to the case of meromorphic functions.

Recall that given an open set $U \subseteq \mathbb{C}$, we let $\mathcal{M}(U)$ be the set of meromorphic functions. It collects functions that are holomorphic away from a discrete set of points, at which the functions have poles. We now introduce a new notion for looking at them, serving as a precursor to our introduction to Riemann surfaces. We consider a new space

$$\hat{\mathbb{C}} := \mathbb{C} \dot{\cup} \{\infty\}$$

by "adding an infinity point" to the complex plane. We have seen this construction in Section 1.2.2, and now we can give it more structures. This space is called the **extended complex plane** or the **Riemann sphere**, and by viewing it as the one-point compactification of $\mathbb{C}$, we get an induced topological structure on it (cf. Exercise 6 in Assignment 2).

A natural way to look at the space is to use the stereographic projection from the unit two-sphere S^2. That is, we look at the map $P\colon S^2 \to \hat{\mathbb{C}}$ defined by

$$P(x, y, w) := \frac{x}{1-w} + \frac{y}{1-w} i$$

where we view S^2 as a unit sphere centered at the origin in $\mathbb{R}^3$. Note that when we look at the north pole (i.e., when $w = 1$), we naturally get $P(0, 0, 1) = \infty \in \hat{\mathbb{C}}$. One can show that this map is a homeomorphism and hence $\hat{\mathbb{C}}$ is homeomorphic to the two-sphere S^2.$^{\text{ex}}$

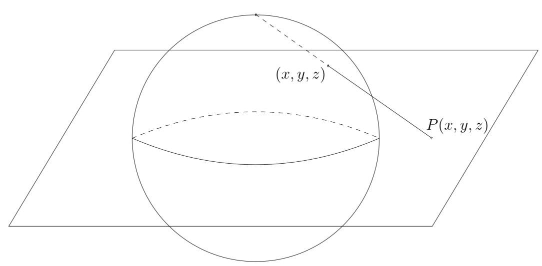

FIGURE 11. A homeomorphism between S^2 and $\hat{\mathbb{C}}$ maps $(0, 0, 1)$ to ∞.

We want to view a meromorphic function $f \in \mathcal{M}(U)$ as a "usual map" into $\hat{\mathbb{C}}$. To this end, we pull make the metric structure on S^2 to $\hat{\mathbb{C}}$. Note that given $z \in \hat{\mathbb{C}}$, one can compute$^{\text{ex}}$

$$P^{-1}(z) = \left(\frac{2z}{1 + |z|^2}, \frac{|z|^2 - 1}{|z|^2 + 1} \right) \in \mathbb{C} \times \mathbb{R} \simeq \mathbb{R}^3. \tag{5.14}$$

Therefore, if we define $d_{\hat{\mathbb{C}}}(z_1, z_2)$ by $d_{\text{Euc}}\left(P^{-1}(z_1), P^{-1}(z_2)\right)$, we can finally get$^{\text{ex}}$

$$d_{\hat{\mathbb{C}}}(z_1, z_2) = \frac{2|z_1 - z_2|}{\sqrt{1 + |z_1|^2} \cdot \sqrt{1 + |z_2|^2}}$$

and we have a well-defined metric structure that is compatible with the topology given by the one-point compactification on $\hat{\mathbb{C}}$. The metric $d_{\hat{\mathbb{C}}}$ is called the **spherical metric** on $\hat{\mathbb{C}}$.

Before studying families of meromorphic functions, as in the holomorphic case, we first look at the space $\mathcal{C}(U; \hat{\mathbb{C}})$ of continuous functions from an open set U to $\hat{\mathbb{C}}$. The discussion before

Theorem 5.7, as we mentioned, holds for functions valued in a metric space, and therefore, we can again endow a metric $\hat{\rho}_U$ on the space $\mathcal{C}(U;\hat{\mathbb{C}})$ by the same process, and have the same conclusion of Theorem 5.7 and all the results before it. Moreover, notice that condition (2) is now vacuous since $\left(\hat{\mathbb{C}}, d_{\hat{\mathbb{C}}}\right)$ is a bounded metric space. With this, we summarize them in the following corollary.

Corollary 5.15. Let U be an open set in $\mathbb{C}$ and $\mathcal{F} \subseteq \mathcal{C}(U;\hat{\mathbb{C}})$. Then $\mathcal{F}$ is a normal family (in the sense of Definition 5.1(2)) if and only if $\mathcal{F}$ is equicontinuous on any compact set $K \subseteq U$.

The problem now is that the space $\mathcal{C}(U;\hat{\mathbb{C}})$ contains too many functions. Given a function $f \in \mathcal{C}(U;\hat{\mathbb{C}})$, $f^{-1}(\infty)$ may be very large. Thus, to get structural results for families in $\mathcal{M}(U)$, we have to understand possible behavior near infinity along a convergent sequence first. That is done in the following lemma.

Lemma 5.16. Let U be a connected open set in $\mathbb{C}$ and f_n be a sequence in $\mathcal{M}(U)$. If f_n converges to a function f locally uniformly, then f is either meromorphic on U or identically ∞. For the first case, if $f_n \in \mathcal{H}(U)$, then f is holomorphic.

Note that the locally uniform convergence is equivalent to the convergence in the metric $\hat{\rho}_U$. Also, note that $\mathcal{M}(U) \subseteq \mathcal{C}(U;\hat{\mathbb{C}})$ by the blow-up characterization of poles. The lemma tells us that the situation is not too bad, since the only bad case when one has too many singularities is exactly when the limit is identically infinity.

Proof. First, we know that f is a continuous function in the spherical metric $d_{\hat{\mathbb{C}}}$. Take $z_0 \in U$. We deal with two cases.

If $f(z_0) \neq \infty$, then in a neighborhood of z_0, $f_n \neq \infty$ for n large. Hence, Weierstrass's theorem (Corollary 2.23) implies that f is holomorphic in such a neighborhood of z_0.

If $f(z_0) = \infty$, then in a neighborhood of z_0, $f_n \neq 0$ for n large. The sequence of analytic functions $1/f_n$ then converges to $1/f$ uniformly in a slightly smaller neighborhood (when using the spherical metric, and hence when using the usual Euclidean metric), and hence Weierstrass's theorem implies $1/f$ is holomorphic. If z_0 is and isolated zero of $1/f$, then f is meromorphic at z_0. Otherwise, we know $1/f \equiv 0$ and hence $f \equiv \infty$.[13]

For the last conclusion, if $f_n \in \mathcal{H}(U)$ but $f(z_0) = \infty$, then in the neighborhood chosen in the preceding paragraph, f_n cannot be non-vanishing by Hurwitz's theorem (Corollary 3.38). This implies f_n is not holomorphic, a contradiction. □

In view of Lemma 5.16, it may be reasonable to allow the function ∞ when considering normal families of meromorphic functions. Thus, we relax the condition in Definition 5.1(2) and propose the following definition for meromorphic functions.

Definition 5.17. Let $\mathcal{F} \subseteq \mathcal{M}(U) \subseteq \mathcal{C}(U;\hat{\mathbb{C}})$ be a family of meromorphic functions on a connected open set U. We say $\mathcal{F}$ is **normal** if any sequence $f_n \in \mathcal{F}$ has a subsequence which, on any compact subset of U, either converges uniformly to a meromorphic function or converges uniformly to ∞.

[13]Note that based on Hurwitz's theorem (Corollary 3.38), the second case necessarily happens when f_n's are holomorphic.