To make notations simpler, we let

$$E_N(z) := (1-z)e^{\sum_{k=1}^N \frac{1}{k}z^k}.$$

Thus, from (4.31), we obtain an entire function of the form

$$z^m \cdot \prod_{n=m+1}^{\infty} E_n\left(\frac{z}{z_n}\right).$$

In fact, since given an entire function $g(z)$, its exponential $e^g(z)$ is always an non-vanishing entire function, we obtain a lot of functions of the form

$$e^{g(z)} \cdot z^m \cdot \prod_{n=m+1}^{\infty} E_n\left(\frac{z}{z_n}\right) \tag{4.33}$$

satisfying Theorem 4.28. It is good that it has exactly the zero set that we want. A problem is that we do not have a controlled growth of this function. In the expression (4.33), the growth is essentially controlled by the growth of g and the n in E_n. For the part of E_n, from the estimate in (4.32), one can first see that if the growth of the zeros is controlled, then we can first get a uniform E_d for all n. We formulate it as a corollary of the Weierstrass factorization theorem here.[ex]

Corollary 4.34. Let $Z = \{z_n\}$ be a countable set in $\mathbb{C}$. Suppose Z satisfies that $z_n = 0$ if and only if $n \le m$ and

$$\sum_{i=m+1}^{\infty} \frac{1}{|z_n|^{\alpha+1}} < \infty. \tag{4.35}$$

If we let $d := \lfloor \alpha \rfloor$, then for any polynomial g of degree at most d, the function

$$f(z) := e^{g(z)} \cdot z^m \cdot \prod_{n=m+1}^{\infty} E_d\left(\frac{z}{z_n}\right)$$

satisfies that $Z(f) = Z$ and is of order at most α. As before, we allow multiplicities in both the sets Z and $Z(f)$.

The converse of this corollary is a characterization for entire function of finite order. It is the Hadamard factorization theorem.

Theorem 4.36 (Hadamard factorization theorem)**.** Let $\alpha \ge 0$ and f an entire function of order at most α. Suppose $Z = \{z_n\}$ is the zero set of f with $z_n = 0$ if and only if $n \le m$. If we let $d := \lfloor \alpha \rfloor$, then there exists a polynomial g of degree at most d such that

$$f(z) := e^{g(z)} \cdot z^m \cdot \prod_{n=m+1}^{\infty} E_d\left(\frac{z}{z_n}\right).$$

4.3. **Gamma function and Zeta function.**

4.3.1. *Gamma function.* We look at one example for the Hadamard factorization theorem (Theorem 4.36). Take $f(z) = \sin \pi z$. We know that

$$Z(f) = \mathbb{Z} = \{0, \pm 1, \pm 2, \cdots\}$$

and all the zeros have order one. Moreover, we have

$$|\sin \pi z| = \frac{1}{2}\left|e^{i\pi z} - e^{-i\pi z}\right| \leq e^{\pi|z|},$$

and hence the order of f is at most 1. (Recall how this is defined in (4.15).) Thus, Theorem 4.36 implies that there is a polynomial $g(z)$ of degree at most one such that

$$\sin \pi z = e^{g(z)} \cdot z \cdot \prod_{n \neq 0} E_1\left(\frac{z}{n}\right).$$

Since we can switch the order of the terms in the product, we can first calculate

$$E_1\left(\frac{z}{n}\right) \cdot E_1\left(-\frac{z}{n}\right) = e^{\frac{z}{n}}\left(1 - \frac{z}{n}\right) \cdot e^{-\frac{z}{n}}\left(1 + \frac{z}{n}\right) = 1 - \frac{z^2}{n^2}.$$

That is,

$$\sin \pi z = e^{g(z)} \cdot z \cdot \prod_{n=1}^{\infty}\left(1 - \frac{z^2}{n^2}\right). \tag{4.37}$$

To find out g, we look at

$$\log \frac{\sin \pi z}{z} = g(z) + \sum_{n=1}^{\infty} \log\left(1 - \frac{z^2}{n^2}\right),$$

where the logarithm on the left hand side is a suitable branch. We can look at the Taylor series of both of the functions near 0. Recall again that we have (4.7). Thus, we know

$$\sum_{n=1}^{\infty} \log\left(1 - \frac{z^2}{n^2}\right) = \sum_{n=1}^{\infty}\left(-\frac{z^2}{n^2} + \cdots\right) = -\left(\sum_{n=1}^{\infty} \frac{1}{n^2}\right) z^2 + \cdots, \tag{4.38}$$

where we do not need an explicit expression of the second-order term.[10] We keep track of the terms up to second order since we know that the degree of g is at most one. Similarly, we have

$$\log \frac{\sin \pi z}{\pi z} = \log\left(1 - \frac{\pi^2 z^2}{3!} + \cdots\right) = k \cdot 2\pi i - \frac{\pi^2 z^2}{6} + \cdots, \tag{4.39}$$

where the multiple of $2\pi i$ comes from the possibility of different branches. Thus, comparing (4.38) and (4.39) and noting that $\deg g \leq 1$, we know that $g(z) = \log \pi + k \cdot 2\pi i$ for some k, and hence we know that (4.37) can be now explicitly written as

$$\sin \pi z = \pi z \cdot \prod_{n=1}^{\infty}\left(1 - \frac{z^2}{n^2}\right), \tag{4.40}$$

the famous Euler sine product formula. Note that comparing the coefficients of z^2 leads to another proof of $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.

[10] We could have used the formula $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$ to get the coefficient.

Another consequence of (4.40) is that any entire function f of order at most one and with $Z(f) = \mathbb{Z}$ is of the form

$$f(z) = e^{az+b} \cdot \sin \pi z$$

for some $a, b \in \mathbb{C}$. One then can ask what happens if we would like to find f with $Z(f) = \mathbb{N}$.

To follow the tradition, we assume $Z(f)$ to be the set of non-positive integers as simple zeros. That is,

$$Z(f) = \mathbb{Z}_{\leq 0} = \{0, -1, -2, -3, \cdots\}.$$

If we require, as above, that the order of f is at most one, Theorem 4.36 then implies

$$f(z) = e^{az+b} \cdot z \cdot \prod_{n=1}^{\infty} E_1\left(-\frac{z}{n}\right) = e^{az+b} \cdot z \cdot \prod_{n=1}^{\infty} e^{-z/n}\left(1 + \frac{z}{n}\right)$$

for some $a, b \in \mathbb{C}$. We first choose $b = 0$ for simplicity, and we want to find a so that the resulting f is an "interesting function."

Since the zero set is almost translation-invariant (at least in one direction), we first look at how $f(z)$ and $f(z+1)$ are related. Recall that we've chosen $b = 0$, and then note that

$$\begin{aligned} f(z+1) &= e^{az+a}(z+1) \cdot \prod_{n=1}^{\infty} e^{-\frac{z+1}{n}}\left(1 + \frac{z+1}{n}\right) \\ &= e^{az+a}(z+1) \cdot \prod_{n=2}^{\infty} e^{-\frac{z+1}{n-1}}\left(1 + \frac{z+1}{n-1}\right) \\ &= e^{az+a}(1+z) \cdot \prod_{n=2}^{\infty} e^{-\frac{1}{n-1}}\left(1 + \frac{1}{n-1}\right) e^{-\frac{z}{n-1}}\left(1 + \frac{z}{n}\right) \end{aligned}$$

after a change of variables $n \mapsto n-1$. We compare this with

$$f(z) = e^{az} z \cdot \prod_{n=1}^{\infty} e^{-\frac{z}{n}}\left(1 + \frac{z}{n}\right) = e^{az} z \cdot e^{-z}(1+z) \cdot \prod_{n=2}^{\infty} e^{-\frac{z}{n}}\left(1 + \frac{z}{n}\right),$$

where we single out the first term in the product. Notice that most of the terms can be compared directly, and for the exponential terms, we have

$$\prod_{n=2}^{\infty} e^{-\frac{z}{n-1}} \Big/ \prod_{n=2}^{\infty} e^{-\frac{z}{n}} = e^{-z\sum_{n=2}^{\infty}\left(\frac{1}{n-1} - \frac{1}{n}\right)} = e^{-z}.$$

Thus, we can write

$$\begin{aligned} z \cdot f(z+1) &= f(z) \cdot e^{a} \cdot \prod_{n=2}^{\infty} e^{-\frac{1}{n-1}}\left(1 + \frac{1}{n-1}\right) \\ &= f(z) \cdot e^{a} \cdot \prod_{n=1}^{\infty} e^{-\frac{1}{n}}\left(1 + \frac{1}{n}\right). \end{aligned}$$

Thus, we choose $a = \gamma \in \mathbb{R}$ such that

$$e^{\gamma} \cdot \prod_{n=1}^{\infty} e^{-\frac{1}{n}} \left(1 + \frac{1}{n}\right) = 1,$$

and such a number γ is called the Euler—Mascheroni constant. A familiar form is

$$\gamma = \sum_{n=1}^{\infty} \left(\frac{1}{n} - \log\left(1 + \frac{1}{n}\right)\right) = \lim_{N \to \infty} \left(1 + \frac{1}{2} + \cdots + \frac{1}{N} - \log(N+1)\right) = 0.577 \cdots .$$

With this choice of $a = \gamma$, we know that f satisfies $zf(z+1) = f(z)$. This f then has (simple) zero set exactly consisting of non-positive integers. The **Gamma function** is then defined by its reciprocal; that is,

$$\Gamma(z) := f(z)^{-1} = e^{-\gamma z} \cdot z^{-1} \cdot \prod_{n=1}^{\infty} e^{\frac{z}{n}} \left(1 + \frac{z}{n}\right)^{-1}.$$

We summarize some properties of Γ.

(1) Γ has simple poles at $z \in \mathbb{Z}_{\leq 0}$ and is holomorphic and non-vanishing elsewhere.

(2) The relation of f implies

$$\Gamma(z+1) = z\Gamma(z) \tag{4.41}$$

for $z \notin \mathbb{Z}_{\leq 0}$.

(3) The recurrence relation (4.41) implies

$$\Gamma(1) = \lim_{z \to 0} z\Gamma(z) = \lim_{z \to 0} e^{-\gamma z} \cdot \prod_{n=1}^{\infty} e^{\frac{z}{n}} \left(1 + \frac{z}{n}\right)^{-1} = e^0 \cdot \prod_{n=1}^{\infty} e^0 \cdot 1 = 1.$$

Hence, applying the relation (4.41) again leads to

$$\Gamma(n) = (n-1)!$$

for $n \in \mathbb{N}$. That is, $\Gamma(z)$ can be regarded as an extension of the factorial function to the complex plane.

4.3.2. *Zeta function and prime number theorem.* We now talk about the function that we've mentioned in the introduction. That is, the **Riemann zeta function**

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s},$$

which converges locally uniformly in the half plane[11]

$$H_{>1} := \{s \in \mathbb{C} : \mathrm{Re}(s) > 1\}.$$

In particular, ζ is a holomorphic function on $H_{>1}$. This function is one of the most important functions in math. We know its value when $s = 2$ is

$$\zeta(2) = \frac{\pi^2}{6},$$

which has already appeared before.

[11]We use the principal branch of the logarithm to define $n^s := e^{s \log n}$ for $s \in H_{>1}$.

In this section, we will relate the function ζ to a theorem about the distribution of prime numbers. The theorem says that the counting function

$$\pi(x) := \text{the number of primes} \leq x$$

grows like $x/\log x$. That is, the "density" of primes grows like $1/\log x$.

Theorem 4.42 (Prime number theorem)**.** As $x \to \infty$, $\frac{\pi(x)}{x/\log x} \to 1$.

We will mention a heuristic why such a result holds. See the discussion below Corollary 4.45. First, we can notice a relation between ζ and the set P of prime numbers, encoded in the following Euler's product identity.

Lemma 4.43 (Euler product identity)**.** For $s \in H_{>1}$, we have

$$\zeta(s) = \prod_{p \in P} \frac{1}{1 - p^{-s}}. \tag{4.44}$$

In particular, we know that $\zeta\colon H_{>1} \to \mathbb{C}$ is a non-vanishing holomorphic function.

A priori, we have to fix an order in P before knowing the convergence of the product. We can simply write it in the increasing order; that is, $P = \{p_j : j \in \mathbb{N}\}$ with $p_j < p_{j+1}$.

Proof of Lemma 4.43. The proof is basically based on the fundamental theorem of arithmetic. Recall that it says that every positive integer greater than one can be written as a product of prime numbers, and the expression is unique up to the order of the factors. That is, for any $n \in \mathbb{N}$, we can write

$$n = \prod_{p \in P} p^{a_{n,p}}$$

where $a_{n,p} = 0$ for all but finitely many $p \in P$. Thus, for $x \geq 1$ and $N \in \mathbb{N}$, we can write

$$\prod_{p \leq x} \sum_{a=0}^{N} \frac{1}{p^{as}} = \sum_{n} \frac{1}{n^s},$$

where the sum is running over all n with $a_{n,p} \leq N$ for $p \in x$ and $a_{n,p} = 0$ for $p > x$. When s is real with $s > 1$, letting N and x tend to infinity, we see that the right hand side is monotonically converging to $\zeta(s)$ and hence

$$\zeta(s) = \prod_{p \in P} \sum_{a=0}^{\infty} \frac{1}{p^{as}} = \prod_{p \in P} \frac{1}{1 - p^{-s}}.$$

For a complex s with $\mathrm{Re}(s) > 1$, one can use

$$|\zeta(s)| \leq \sum_{n=1}^{\infty} \frac{1}{n^{\mathrm{Re}(s)}}$$

and the dominated convergence to conclude the same formula. $\square$

By looking at the formula (4.44) as $s \to 1$, one can get the following consequence which is a well-known fact first proven by Euclid.[ex] This is a sign that the study of ζ can help us understand the distribution of prime numbers, and this is a reason why we intentionally avoid using $|P| = \infty$ in the proof of Lemma 4.43.

Corollary 4.45. $|P| = \infty$. That is, there are infinitely many prime numbers. In particular, we have $\pi(x) \to \infty$ as $x \to \infty$.

Let's now briefly mention why we should believe the asymptotics in Theorem 4.42, as now we know that $\pi(x) \to \infty$. This is not a formal argument, and the symbol $\approx$ means nothing in this paragraph. A very rough idea is that by the idea of removing composite numbers, we have

$$\pi(x) \approx x \cdot \prod_{p \leq x} \left(1 - \frac{1}{p}\right).$$

The reciprocal of this is, similar to above, then

$$\frac{x}{\pi(x)} \approx \prod_{p \leq x} \sum_{a=0}^{\infty} \frac{1}{p^a} = \sum_{n} \frac{1}{n}$$

where the sum is running over n with $a_{n,p} \leq x$. When x is large, in the sum, one can imagine that those terms with $n > x$ will contribute (relatively) very little. If so, then we get

$$\frac{x}{\pi(x)} \approx \sum_{n=1}^{x} \frac{1}{n} \approx \log x.$$

To make this rigorous, one crucial step is to prove that ζ has no zeros on the line $1 + i\mathbb{R} = H_{=1}$. This is also related to the infamous conjecture of Riemann (see Riemann hypothesis).