4.2. **Approximating meromorphic functions.** We understand how the number of the zeros of an entire function may grow. We next can ask for refined information, such as patterns or distribution of those zeros. For example, given a set of points, can we find an entire function whose zero set is exactly the given set? From Proposition 3.1, we know that there are some restrictions. That is, if f is an entire function and its zero set $Z(f)$ is not the whole complex plane $\mathbb{C}$, then $Z(f)$ only contains discrete points. Thus, we have a necessary condition for a set to be the zero set of an entire function. We will see that the condition is also sufficient, due to Weierstrass, at the end of the seciotn.

We will start by Runge's approximation theorem. Using it, we will be able to derive two results about prescribing vanishing or singular behavior of a holomorphic functions by Mittag-Leffler and Weierstrass. For the Weierstrass theorem, we will in fact use a more direct method to prove it.

4.2.1. *Runge's approximation theorem.* The question Runge asked is the possibility of approximating a holomorphic function by rational functions or polynomial functions. This reminds us of the Weierstrass approximation theorem, which says that every continuous function defined on a closed interval can be uniformly approximated as closely as desired by a polynomial function.

For holomorphic functions, recall that we know they are analytic (Theorem 2.17), so it is true that they can locally be approximated by polynomial functions. In general, we would like to understand what condition is necessary or sufficient to guarantee such an approximation on a compact set. One can see that using polynomials is not always enough by looking at $1/z$ on the compact set ∂B_1, along which every polynomial has trivial line integral. Restrictions come from non-trivial topology of the set K, as in the case of ∂B_1, its complement is not connected. If we allow rational functions, that it works for all compact sets.

Theorem 4.16 (Runge)**.** Let K be a compact set in $\mathbb{C}$ and let f be a holomorphic function defined on a neighborhood of K.

(1) f can be approximated uniformly on K by rational functions whose poles are away from K.

(2) f can be approximated uniformly on K by polynomial functions if $\mathbb{C} \setminus K$ is connected.

We will first prove the first part, that is, Theorem 4.16(1). After that, we will discuss how to approximate a rational function by another with prescribed poles.

Proof of Theorem 4.16(1)*.* Suppose f is a holomorphic function on U which is a bounded open neighborhood of K. We will first divide the boundary of a (slightly smaller) neighborhood of K.

Let $d := d_{\text{Euc}}(K, \partial U)/100$. Consider a grid formed by closed solid squares with sides parallel to the axes and of length d. Let $\mathcal{R}$ be the set $\{R_1, \cdots, R_N\}$ that collects all squares that intersect K, and let $\gamma_1, \cdots, \gamma_M$ be those sides of squares in $\mathcal{R}$ that do not belong to two adjacent squares. By the construction, each $\gamma_m \subseteq U \setminus K$.

We let

$$\partial\mathcal{R} := \bigcup_{n=1}^{N} \partial R_n$$

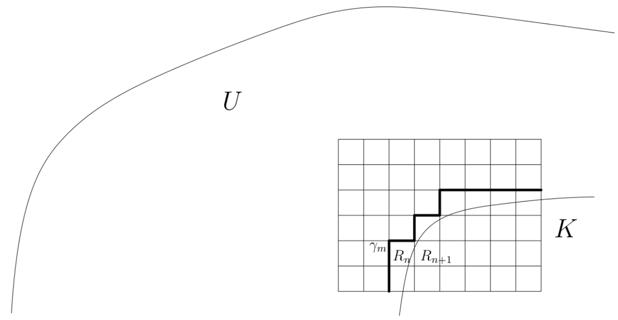

FIGURE 10. A possible choice of $\mathcal{R}$.

be the union of the boundaries of R_n's. For eah $z \in K \setminus \partial\mathcal{R}$, we can find $R_{n_z} \in \mathcal{R}$ such that $z \in \text{int}R_{n_z}$, and hence

$$\frac{1}{2\pi i}\int_{\partial R_n} \frac{f(w)}{w-z}dw = \begin{cases} f(z) & \text{if } n = n_z \\ 0 & \text{otherwise} \end{cases}.$$

Hence, summing over $n = 1, \cdots, N$ so that the contribution from adjacent sides cancel, we get

$$\sum_{m=1}^{M} \frac{1}{2\pi i}\int_{\gamma_m} \frac{f(w)}{w-z}dw = f(z) \tag{4.17}$$

for $z \in K \setminus \partial\mathcal{R}$. For $z \in K \cap \partial\mathcal{R}$, since $\gamma_m \subseteq U \setminus K$ for each m, the left hand side of (4.17) still makes sense and defines a holomorphic function (Proposition 2.4). The continuity then implies that (4.17) is true for all $z \in K$.

Now we use the expression (4.17) to get an approximation. For each γ_m, since it is a compact segment that is disjoint from K, we can find a closed ball $\overline{B}_{r_m}(z_m)$ that contains γ_m and is still disjoint from K. Then the function

$$f_m(z) := \frac{1}{2\pi i}\int_{\gamma_m} \frac{f(w)}{w-z}dw$$

is holomorphic on $\mathbb{C} \setminus \overline{B}_{r_m}(z_m)$, and Theorem 3.9 implies that we have a Laurent series expansion

$$f_m(z) = \sum_{n=-\infty}^{\infty} a_n^m (z - z_m)^n$$

where the convergence is locally uniform in $\mathbb{C} \setminus \overline{B}_{r_m}(z_m)$. Therefore, the sequence

$$\sum_{m=1}^{M} \sum_{n=-j}^{j} a_j^m (z - z_m)^j \tag{4.18}$$

is a sequence of rational functions that converges uniformly to f on K (since the sum of $f_1, \cdots, f_M$ is f on K). Note that all the poles z_n's are in $\mathbb{C} \setminus K$. □

Based on the construction (4.18) in the proof, to achieve the second part of the theorem, one would like to move the poles of rational functions to infinity. We will do this in a general way in the following result.

Proposition 4.19. Let K be a compact set in $\mathbb{C}$ and let U be a connected open set in $\hat{\mathbb{C}}$ such that $U \cap K = \emptyset$. Given $z_0 \in U$, a rational function with poles in U can be approximated uniformly on K by rational functions with exactly a pole at z_0.

Proof. We let $\mathcal{R}(z_0)$ be the set of rational functions with exactly a pole at z_0, that is,

$$\mathcal{R}(z_0) := \left\{ \sum_{j=0}^{N} \frac{p_j(z)}{(z-z_0)^j} : \text{each } p_j \text{ is a polynomial} \right\}$$

when $z_0 \in \mathbb{C}$, and $\mathcal{R}(z_0)$ is the set of polynomials when $z_0 = \infty$. Consider the subset

$$V := \left\{ w \in U : \frac{1}{z-w} \text{ can be approximated uniformly on } K \text{ by rational functions in } \mathcal{R}(z_0) \right\}.$$

Since a rational functional can be expressed by partial fractions, it suffices to show that $V = U$. By its definition, V contains z_0 automatically, and we will show that it is both closed and open.

The closedness is more direct. If $z_n \in V$ is a sequence and $z_n \to z_\infty \in U$ as $n \to \infty$, then $\frac{1}{z-z_n}$ converges uniformly to $\frac{1}{z-z_\infty}$ on K as $n \to \infty$. Thus, given $\varepsilon > 0$, we can find n large such that

$$\left| \frac{1}{z-z_\infty} - \frac{1}{z-z_n} \right| < \frac{\varepsilon}{2}$$

for $z \in K$, and we can find $R \in \mathcal{R}(z_0)$ with

$$\left| R(z) - \frac{1}{z-z_n} \right| < \frac{\varepsilon}{2}$$

for $z \in K$. Combining this, we see that $1/(z-z_\infty)$ can be approximated uniformly on K by functions in $\mathcal{R}(z_0)$, and hence $z_\infty \in V$. This proves that V is closed in U.

The difficulty mainly lies in proving the openness of V. Suppose $w_0 \in V$. We want to approximate $1/(z-w)$ by functions in $\mathcal{R}(w_0)$ for w close enough to w_0. We first deal with the case when $w_0 = \infty$, which means $\mathcal{R}(w_0)$ consists of polynomials. First, we take $R < \infty$ large such that $K \subseteq B_R$. For w close to w_0, in the sense that $|w|$ is large, say $|w| \geq 2R$, we get

$$\frac{1}{z-w} = -\frac{1}{w} \frac{1}{1-z/w} = -\frac{1}{w} \sum_{n=0}^{\infty} \frac{z^n}{w^n}$$

with $|z/w|^n \leq (1/2)^n$. Thus, Weierstrass' M-test implies that the series converges uniformly on K, and hence we get $w \in \mathcal{R}(\infty)$.

The case when $w_0 \in \mathbb{C}$ is similar, using the technique in the proof of Proposition 2.4.[9] That is, we write

$$\frac{1}{z-w} = \frac{1}{z-w_0} \cdot \frac{z-w_0}{z-w} = \frac{1}{z-w_0} \cdot \frac{1}{1-\frac{w-w_0}{z-w_0}}.$$

For such a $w_0 \in V \subseteq U$, take a positive $\varepsilon < d_{\mathrm{Euc}}(w_0, K)$. Then for $w \in B_{\varepsilon/2}(w_0)$, we have

$$\frac{1}{z-w} = \frac{1}{z-w_0} \sum_{n=0}^{\infty} \left(\frac{w-w_0}{z-w_0} \right)^n$$

[9]In fact, it is exactly the same proof after a change of variables $z \mapsto 1/(z_0 - w_0)$.

with $\left|\frac{w-w_0}{z-w_0}\right|^n < (1/2)^n$. Weierstrass' M-test again implies that the series converges uniformly on K, and hence we get $w \in \mathcal{R}(w_0)$. This finishes the proof of the openness of V, and hence the proposition follows. □

As a corollary, we can now prove the second part of Theorem 4.16.

Proof of Thoemre 4.16(2). Given f, from Thoemre 4.16(1), f can be approximated by rational functions with poles away from K. That is, given $\varepsilon > 0$, f can be $\varepsilon/2$-approximated on K by $R_1 + \cdots + R_N$, where R_n has exactly one pole at $z_n \in \mathbb{C} \setminus K$. By Proposition 4.19, each R_n can be $\varepsilon/(2N)$-approximated on K by a function in $\mathcal{R}(\infty)$, the set of polynomials. Combining these, we complete the proof. □

In fact, the same argument leads to a slightly stronger result. We record it here and leave the (repeated) arguments to exercises.

Corollary 4.20. Let K be a compact set in $\mathbb{C}$ and let f be a holomorphic function defined on a neighborhood of K. If S is a subset of $\hat{\mathbb{C}} \setminus K$ such that each component of $\hat{\mathbb{C}} \setminus K$ intersects S, then f can be approximated uniformly on K by rational functions whose poles are in S.

All of these results will be referred to as Runge's approximation theorems. Runge's theorems are very useful when one tries to construct holomorphic and meromorphic functions. We will see one example in the next section, and before that, we first show that it provides an analytic way to characterize simple connectivity (cf. Assignment 5).

Corollary 4.21. Let U be a bounded connected open set in $\mathbb{C}$. If $\mathbb{C} \setminus U$ is connected, then U is "holomorphically simply connected," in the sense that given any holomorphic function f on U and a closed path $\gamma \subseteq U$, $\int_\gamma f dz = 0$.

We know from Proposition 4.4 that a simply connected set is holomorphically simply connected. The converse of this can be obtained easily after we learn the Riemann mapping theorem later.

Proof of Corollary 4.21. Let f be a holomorphic function on U and γ be a closed path in U. Consider the compact set

$$K := \{z \in U : d_{\mathrm{Euc}}(z, \mathbb{C} \setminus U) \geq \varepsilon\}$$

for small $\varepsilon > 0$. We want to show that $\mathbb{C} \setminus K$ is connected so that we can apply Runge's theorem. This is a purely topological fact, and we can do it directly.

To this end, suppose for a contradiction that we can write $\mathbb{C} \setminus K = V_1 \cup V_2$ where V_1 and V_2 are non-empty disjoint open sets. With this, consider

$$C_i := V_i \setminus U$$

for $i = 1, 2$. We claim that C_1 and C_2 are non-empty disjoint closed sets. If this is done, then we get a contradiction since $\mathbb{C} \setminus U = C_1 \cup C_2$ while $\mathbb{C} \setminus U$ is connected.

For the claim, it is clear that C_1 and C_2 are disjoint. To show that they are closed, suppose $z_n \in C_1$ is a sequence that converges to a point z_0. Then since U is open, we know $z_0 \in \mathbb{C} \setminus U$, so

$z_0 \notin K$ by the definition of K. Hence $z_0 \in V_1 \cup V_2$, and the disjoint property implies $z_0 \in V_1$. This proves that C_1 is closed, and similarly so is C_2.

Finally, we show that C_1 is non-empty (and similarly so is C_2). If $C_1 = \emptyset$, then we would get $V_1 \subseteq U$. For any $w \in V_1$, since $w \notin K$, there exists $z \in \mathbb{C} \setminus U$ such that $|w - z| < \varepsilon$ and $\overline{wz} \subseteq \mathbb{C} \setminus K$. The idea of the rest of the proof is that because $z \in V_2$ (since $V_1 \subseteq U$), a limiting point on $\overline{zw}$ would make a contradiction. In practice, one can consider

$$\bar{t} := \sup \{t \in [0,1] : (1-t)z + tw \in V_2\}$$

and see that it cannot belong to either V_1 or V_2. This finishes the proof of the claim.

Now, by Runge's theorem (Thoemre 4.16(2)), we can approximate f uniformly on K by polynomials. Since γ is compact, by choosing ε small, we can assume $\gamma \subseteq K$. This then finishes the proof by the Cauchy theorem. □

4.2.2. *Mittag-Leffler theorem.* Recall that given a meromorphic function f and one of its poles z_0, its principal part is a rational function P, which is a polynomial in $1/(z - z_0)$ such that $f - P$ has a removable singularity at z_0 (and hence can be viewed as holomorphic near z_0), see Proposition 3.15. Using Runge's theorem 4.16, we will see that we can prescribe poles and even their principal parts for a meromorphic function.

Theorem 4.22 (Mittag-Leffler)**.** Let $Z = \{z_n\}$ be a countable set of distinct points in $U \subseteq \mathbb{C}$. Suppose Z does not have accumulation points in U. Given a sequence of rational functions P_n's which are polynomials in $1/(z - z_n)$, there exists a meromorphic function $f \in \mathcal{M}(U)$ whose poles are exactly z_n's with each $f - P_n$ holomorphic near z_n for all n.

The idea is as follows. A naive way to get such a meromorphic function is to just take the sum of all the principal parts, which works when there are only finitely many z_n's. When there are infinitely many poles, there is no reason to believe that the sum of P_n's converges. We will then use Runge's theorem to approximate P_n by rational functions, and hence their differences can form a convergent series.

Proof. For $j \in \mathbb{N}$, consider the compact set

$$K_j := \left\{ z \in U : z \in \overline{B}_j \text{ and } d_{\mathrm{Euc}}(z, \partial U) \geq 1/j \right\}.$$

When $\partial U = \emptyset$, e.g., when $U = \mathbb{C}$, $d_{\mathrm{Euc}}(z, \partial U)$ is always considered to be ∞. Note that each component of $\hat{\mathbb{C}} \setminus K_j$ intersects $\hat{\mathbb{C}} \setminus U$.

For each $j \in \mathbb{N}$, let I_j be the index set of those n's such that $z_n \in K_{j+1} \setminus K_j$. This implies that I_j is a finite set, and we consider

$$f_j(z) := \sum_{n \in I_j} P_n(z),$$

which is a holomorphic function on a neighborhood of K_j. By Runge's theorem (Corollary 4.20), there exists a rational function g_j with poles in $\hat{\mathbb{C}} \setminus U$ such that

$$\sup_{K_j} |f_j - g_j| \leq \frac{1}{2^j}.$$

Since $K_j \subseteq K_{j+1}$ for all j, by Weierstrass' M-test, the function

$$\sum_{j=m}^{\infty} (f_j - g_j)$$

converges uniformly and hence is holomorphic on K_m. On the other hand,

$$\sum_{j=1}^{n}(f_j - g_j)$$

has poles at z_n for $n \in I_1 \cup \cdots \cup I_j$ with the prescribed principal parts. Thus, the function

$$\sum_{j=1}^{\infty}(f_j - g_j)$$

satisfies all the conditions we list. $\square$

Example 4.23. We mention an example of meromorphic function that has simple poles at positive integers with residues all equal to 1. The corresponding principal part is $\frac{1}{z-n}$ for $z_n = n \in \mathbb{N}$. Clearly, the sum (consisting of f_j in the proof)

$$\sum_{n=1}^{\infty} \frac{1}{z-n}$$

does not converge. To find out suitable g_j's, we note that the zero-order term in the power series expansion of $1/(z-n)$ at 0 is $-1/n$, so we consider

$$\sum_{n=0}^{\infty} \left(\frac{1}{z-n} + \frac{1}{n} \right).$$

Note that for $z \in \overline{B}_R$ and $n > 2R$, we have $|z-n| > n/2$, so

$$\left| \frac{1}{z-n} + \frac{1}{n} \right| = z \left| \frac{1}{(z-n)n} \right| \leq z \left| \frac{1}{n/2 \cdot n} \right|.$$

Hence, the series converges locally uniformly, and hence defines a meromorphic function with the prescribed data.

Remark 4.24. We remark that the idea of substracting zero-order parts from a power series is a commonly used technique in analysis. For example, it can be used to construct doubly periodic meromorphic functions with double poles. We will not talk about it in detail but just mention that given two linearly independent complex numbers ω_1 and ω_2, the Weierstrass $\wp$-function associated to them is

$$\wp(z) := \frac{1}{z^2} + \sum_{(m,n) \neq (0,0)} \left(\frac{1}{(z - m\omega_1 - n\omega_2)^2} - \frac{1}{(m\omega_1 + n\omega_2)^2} \right).$$

Weierstrass $\wp$-functions transform the study of doubly periodic meromorphic functions into a study of elliptic curves via a geometric relationship, providing elegant geometric proofs for the addition formula of elliptic functions and creating a link between these two fields of mathematics.

4.2.3. *Weierstrass factorization theorem.* In this section, we will see another consequence of Runge's theorem. It is parallel to the Mittag-Leffler theorem, but know we want to prescribes zeros and poles with given orders.

Different from prescribing principal parts, which involves taking infinite sums, prescribing zeros will require taking infinite products. Thus, we start with a brief discussion on this.

Let a_n ($n \in \mathbb{N}$) be a sequence of complex numbers. We say the infinite product

$$\prod_{n=1}^{\infty} a_n$$

converges non-degenerately if the limit

$$\lim_{N\to\infty} \prod_{n=n_0}^{N} a_n$$

exists and is a non-zero complex number for some $n_0 \in \mathbb{N}$. As expected, this non-degenerate convergence is related to the infinite sum of the logarithm series. Note that if $\prod\limits_{n=1}^{\infty} a_n$ converges non-degenerately, the sequence a_n necessarily converges to 1 as $n \to \infty$. Thus, for large n, we have $a_n \in \mathbb{C} \setminus (-\infty, 0]$, on which we define the principal branch of the logarithm, which we will keep using the notation $\log$.

By the continuity of the exponential function, we get the following criterion of the convergence of infinite product.

Lemma 4.25. Let a_n ($n \in \mathbb{N}$) be a complex sequence such that $\{n : a_n = 0\}$ is finite. If there exists $n_0 \in \mathbb{N}$ such that $\sum\limits_{n=n_0}^{\infty} \log a_n$ converges, then $\prod\limits_{n=1}^{\infty} a_n$ converges non-degenerately.

In fact, the two conditions are equivalent. We will mainly use this direction, so we do not discuss the other direction here. When the sum converges uniformly on a set, the continuity of the exponential function implies the product also converges uniformly. This will be used when proving Weierstrass' theorem.

Example 4.26. We mention some examples here.

(1) The infinite product

$$\prod_{n=1}^{\infty} \left(1 + \frac{(-1)^{k+1}}{k}\right) = (1+1)\left(1 - \frac{1}{2}\right)\left(1 + \frac{1}{3}\right)\cdots$$

converges to 1. This follows from

$$\prod_{n=1}^{m} \left(1 + \frac{(-1)^{k+1}}{k}\right) = \begin{cases} 1 & \text{if } m \text{ is even} \\ 1 + \frac{1}{m} & \text{if } m \text{ is odd} \end{cases}.$$

(2) The product $\prod\limits_{n=1}^{\infty} \left(1 + \frac{i}{n}\right)$ does not converge, but $\prod\limits_{n=1}^{\infty} \left|1 + \frac{i}{n}\right|$ does converge. Thus, the convergence of the two products are not related as in the sum case.

Now, we state Weierstrass' theorem. A proof of the theorem is parallel to that of Mittag-Leffler's theorem (Theorem 4.22).

Theorem 4.27 (Weierstrass)**.** Let $Z = \{z_n\}$ be a countable set of distinct points in $U \subseteq \mathbb{C}$. Suppose Z does not have accumulation points in U. Given a sequence of non-zero integers N_n's, there exists a meromorphic function $f \in \mathcal{M}(U)$ whose zeros and poles are exactly z_n's with orders N_n's.

In the statement, we use the convention that if z_n is a pole, then its order is a **negative** integer. This is different from our definition in Section 3.2.2, but we use this convention here since the theorem can be stated in this way so that one can determine whether z_n is a zero or a pole by the parity of N_n.

We are not going to talk about the proof using the Runge theorem. A drawback of using Runge's theorem is that it does not produce an explicit function directly. Instead, we look at the following special case, for which we can write down a holomorphic function explicitly.

Theorem 4.28 (Weierstrass factorization theorem)**.** Let $Z = \{z_n\}$ be a countable set in $\mathbb{C}$, and we allow multiplicities in Z. Suppose Z is either finite or diverging to infinity, in the sense that

$$\lim_{n\to\infty} |z_n| = \infty. \tag{4.29}$$

Then there is an entire function f such that $Z(f) = Z$.

Different from Theorem 4.27, we only talk about zeros, and we do not use a separate index to record their multiplicities but just list them out.

Before giving a proof, we first observe that it is reasonable to consider the product

$$\prod_{n=1}^{\infty} \left(1 - \frac{z}{z_n}\right).$$

This doesn't make sense when $z_n = 0$ for some n, which can be fixed by simply taking all the zero points out and get

$$z^m \cdot \prod_{n=m+1}^{\infty} \left(1 - \frac{z}{z_n}\right) \tag{4.30}$$

if we assume $z_n = 0$ if and only if $n \le m$. Conceptually, at least pointwise at a point z, what we need is the convergence of the sum

$$\sum_{n=m+1}^{\infty} \left|\frac{z}{z_n}\right|,$$

which, since it should be true for all z, is basically equivalent to the convergence of

$$\sum_{n=m+1}^{\infty} \frac{1}{|z_n|}.$$

This is definitely not true in general, so we need to adjust more. Let's complete it in the proof.

Proof of Theorem 4.28. To make each term in (4.30) decay fast enough, we consider

$$z^m \cdot \prod_{n=m+1}^{\infty} \left(1 - \frac{z}{z_n}\right) e^{P_n(z)} \tag{4.31}$$

where each P_n is a polynomial and we hope that they will make

$$\sum_{n=m+1}^{\infty} \log\left(1 - \frac{z}{z_n}\right) e^{P_n(z)} = \sum_{n=m+1}^{\infty} \left(\log\left(1 - \frac{z}{z_n}\right) + P_n(z)\right)$$

converge so that we can apply Lemma 4.25. To this end, first, we observe that for a fixed radius R, given any $z \in B_R$, we have $|z/z_n| < 1/2$ for n large enough by the assumption $z_n \to \infty$. Thus, from the expansion, we have

$$\log\left(1 - \frac{z}{z_n}\right) = -\frac{z}{z_n} - \frac{1}{2}\left(\frac{z}{z_n}\right)^2 - \frac{1}{3}\left(\frac{z}{z_n}\right)^3 - \cdots .$$

Therefore, we consider

$$P_n(z) = \sum_{k=1}^{m_n} \frac{1}{k}\left(\frac{z}{z_n}\right)^k$$

for some m_n to be chosen. For $z \in B_R$ and n large as above, we can estimate

$$\begin{aligned} \left|\log\left(1 - \frac{z}{z_n}\right) + P_n(z)\right| = \left|\sum_{k=m_n+1}^{\infty} \frac{1}{k}\left(\frac{z}{z_n}\right)^k\right| &\le \frac{1}{m_n+1}\left(\frac{|z|}{|z_n|}\right)^{m_n+1} \cdot \left(1 - \frac{|z|}{|z_n|}\right)^{-1} \\ &\le \frac{1}{m_n+1} \cdot \left(\frac{1}{2}\right)^{m_n+1} \cdot 2. \end{aligned} \tag{4.32}$$

Thus, if we choose, for example, $m_n = n$ for each P_n, then the series

$$\sum_{n=m+1}^{\infty} \left(\log\left(1 - \frac{z}{z_n}\right) + P_n(z)\right)$$

converges locally uniformly. This tells us that (4.31) defines an entire function with those prescribed zeros. ∎