3.3. **Meromorphic functions and residue theory.** Let U be an open set in $\mathbb{C}$. We say f is a **meromorphic function** on U if there is a discrete subset E of points in U such that

(1) f is holomorphic in $U \setminus E$, and
(2) f has a pole at each $p \in E$.

The set of all meromorphic functions on U will be denoted by $\mathcal{M}(U)$.

We've seen some examples of meromorphic functions. They may have singularities, but by allowing them, the set of all meromorphic functions has a nicer structure. In fact, if U is connected, then $\mathcal{M}(U)$ is a **field**, in the sense that given $f, g \in \mathcal{M}(U)$, we have

- $f \pm g$ and $f \cdot g$ are both in $\mathcal{M}(U)$.
- $f/g \in \mathcal{M}(U)$ if $g \in \mathcal{M}(U) \setminus \{0\}$.

The identity element of this field is the constant function 1. Moreover, one can show that the order of poles at a given point is a **valuation**, in terms of the language of commutative algebra.[ex] We will see more structures or consequences about meromorphic functions in this section.

A useful consequence about the theory of meromorphic functions comes from calculating their resides. Suppose a meromorphic function f on U has a pole at $z_0 \in U$, say

$$f(z) = \sum_{n \geq -N} a_n (z - z_0)^n$$

near z_0. The coefficient a_{-1} is called the **residue** of f at the pole z_0. We will write

$$\mathrm{Res}_f(z_0) := a_{-1}.$$

We can first see why this term will stand out from the Laurent series by looking at the integral $\int_{\partial B_r(z_0)} f dz$. First, we take $r > 0$ such that $B_r(z_0) \subseteq U$ and f is holomorphic in $\hat{B}_r(z_0)$. Then by the Cauchy theorem, the integral of the holomorphic part vanishes, and by parametrizing $\partial B_r(z_0)$ with $\gamma(t) = z_0 + re^{it}$ ($t \in [0, 2\pi]$), we can calculate

$$\begin{aligned}
\frac{1}{2\pi i} \int_{\partial B_r(z_0)} f dz &= \frac{1}{2\pi i} \sum_{n=-N}^{-1} \int_{\partial B_r(z_0)} a_n (z - z_0)^n dz \\
&= \frac{1}{2\pi i} \sum_{n=-N}^{-1} \int_0^{2\pi} a_n \left(re^{it}\right)^n \cdot ire^{it} dt \\
&= \frac{1}{2\pi} \sum_{n=-N}^{-1} \int_0^{2\pi} a_n \left(re^{it}\right)^{n+1} dt = a_{-1};
\end{aligned}$$

that is,

$$\frac{1}{2\pi i} \int_{\partial B_r(z_0)} f dz = \mathrm{Res}_f(z_0). \tag{3.21}$$

Thus, the line integral only detects the term $a_{-1}(z - z_0)^{-1}$. As in the holomorphic case, this fact is generally true when we take the index of a cycle into account. This is recorded in the following residue theorem, whose proof will make use of the model case (3.21).

Theorem 3.22. Let f be a meromorphic function on U and let E be the set of the poles of f. If γ is a cycle in $U \setminus E$ such that

$$\mathrm{Ind}_\gamma(z) = 0 \text{ for all } z \in \mathbb{C} \setminus U, \tag{3.23}$$

then we have

$$\frac{1}{2\pi i} \int_\gamma f dz = \sum_{p \in E} \mathrm{Res}_f(p) \cdot \mathrm{Ind}_\gamma(p). \tag{3.24}$$

We remark that the sum in (3.24) is essentially finite, which we will discuss in the proof. Another remark for the condition (3.23) is that a similar condition appears in the Cauchy theorem, see (2.39) in Theorem 2.38. Since every holomorphic function is particularly meromorphic, Theorem 3.22 can be viewed as a generalization of Theorem 2.38 (1). In fact, we will use Theorem 2.38 in the proof. The idea is that we can apply the Cauchy theorem to the difference of the cycle γ and the sum of some circles around the poles whose indices (of γ) are non-zero. See Figure 6.

Proof. We consider

$$E_\gamma := \{p \in E : \mathrm{Ind}_\gamma(p) \neq 0\},$$

the set of the poles that we care about. Note that if we let U_γ be the unbounded connected component of $\mathbb{C} \setminus \gamma$, we have $E_\gamma \subseteq \mathbb{C} \setminus U_\gamma$, a compact set. Since E is a discrete set, in particular, this implies that E_γ is finite, say $E_\gamma = \{p_n : n = 1, \cdots, N\}$.

For each p_n, since $p_n \notin \gamma$, we can find $r_n > 0$ such that $\overline{B}_{r_n}(p_n) \subseteq U \setminus \gamma$. On the open set

$$U' := U \setminus \left(E \cup \bigcup_{n=1}^{N} \overline{B}_{r_n/2}(p_n) \right),$$

f is a holomorphic function. We want to apply the Cauchy theorem 2.38 on this open set, so we look at the cycle

$$\Gamma := \gamma - \sum_{n=1}^{N} \mathrm{Ind}_\gamma(p_n) \cdot \partial B_{r_n}(p_n),$$

where we recall that each $\partial B_{r_n}(p_n)$ is oriented counterclockwise. For any $z \in \mathbb{C} \setminus U'$, there are three situations.

(1) $z \in \mathbb{C} \setminus U$,

(2) $z \in E \setminus E_\gamma$, or

(3) $z \in \overline{B}_{r_j/2}(p_j)$ for some j.

If $z \in \mathbb{C} \setminus U$, then obviously $\mathrm{Ind}_\Gamma(z) = 0$. If $z \in E$, we have

$$\mathrm{Ind}_\Gamma(z) = \mathrm{Ind}_\gamma(z) - \sum_{n=1}^{N} \mathrm{Ind}_\gamma(p_n) \cdot \mathrm{Ind}_{\partial B_{r_n}(p_n)}(z) = 0 - 0 = 0,$$

where the second zero comes from the fact that z and p_n's are in different components of $U \setminus \gamma$, so $z \notin \overline{B}_{r_n/2}(p_n)$ for all n. If $z \in \overline{B}_{r_j/2}(p_j)$ for some j, then

$$\mathrm{Ind}_\Gamma(z) = \mathrm{Ind}_\gamma(z) - \sum_{n=1}^{N} \mathrm{Ind}_\gamma(p_n) \cdot \mathrm{Ind}_{\partial B_{r_n}(p_n)}(z) = \mathrm{Ind}_\gamma(p_j) - \mathrm{Ind}_\gamma(p_j) \cdot 1 = 0.$$

In conclusion, $\mathrm{Ind}_\Gamma(z) = 0$ for all $\mathbb{C} \setminus U'$.

Now we can use the Cauchy theorem to finish the proof. Since f is holomorphic on U' and $\mathrm{Ind}_\Gamma(z) = 0$ for all $\mathbb{C} \setminus U'$, Theorem 2.38 implies $\int_\Gamma f dz = 0$. Based on the definition of Γ, this means

$$\frac{1}{2\pi i} \int_\gamma f dz = \frac{1}{2\pi i} \sum_{n=1}^{N} \mathrm{Ind}_\gamma(p_n) \int_{\partial B_{r_n}(p_n)} f dz.$$

For each n, the model case (3.21) implies $\frac{1}{2\pi i} \int_{\partial B_{r_n}(p_n)} f dz = \mathrm{Res}_f(p_n)$. Combining these then leads to (3.24). $\square$

The main applications of the residue theorem (Theorem 3.22) usually include two parts. The first one is its usage of calculating some definite integrals; the second is about counting zeros and poles. We leave the latter to the next section, and see some examples of integrals in the rest of this section.

We start from a simple observation about calculating residues. When f has a simple pole at p, then we know

$$f(z) = \frac{a_{-1}}{z-p} + h_1(z) = \frac{1}{h_2(z)}$$

in some $\hat{B}_r(p)$, where h_1 and h_2 are holomorphic in $B_r(p)$ and we know that h_2 has a simple zero at p with $h_2'(p) \neq 0$. Thus, we get two ways to calculate a_{-1}. That is,

$$\mathrm{Res}_f(p) = \lim_{z \to p} (z-p) f(z) = \frac{1}{h_2'(p)}.$$

One can derive similar formulas when the order of a pole is higher, and we record it as a lemma.$^{\mathrm{ex}}$

Lemma 3.25. Suppose f has a pole of order N at z_0, say

$$f(z) = \frac{h(z)}{(z-z_0)^N}$$

in $\hat{B}_r(z_0)$ for some $r > 0$ where h is a holomorphic function on $B_r(z_0)$. Then

$$\mathrm{Res}_f(z_0) = \frac{1}{(N-1)!} h^{(N-1)}(z_0).$$

Now we look at some examples. We first look at integrals of the type

$$\int_{-\infty}^{\infty} f(x) dx. \tag{3.26}$$

Complex analysis can help if the function f can extend to a meromorphic function with suitable decay at infinity.

Example 3.27. We look at an example, which asks the value of

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^4}.$$

The idea is to approximate the real line by the boundary of a large half disc. Here we use the half disc in the upper half plane. Note that the function $f(z) := 1/(1+z^4)$ has four poles, two of which are in the upper half plane. They are $p_1 = e^{i\pi/4}$ and $p_2 = e^{3i\pi/4}$. They are both simple poles, so we can calculate

$$\mathrm{Res}_f(p_1) = \lim_{z\to p_1} \frac{1}{(1+z^4)'} = \frac{1}{4p_1^3} = \frac{1}{4}e^{-3i\pi/4}$$

and similarly

$$\mathrm{Res}_f(p_2) = \lim_{z\to p_2} \frac{1}{(1+z^4)'} = \frac{1}{4p_2^3} = \frac{1}{4}e^{-i\pi/4}.$$

Hence, if we let the upper circular arc be S_R, the residue theorem (Theorem 3.22) implies

$$\int_{-R}^{R} f(x)dx + \int_{S_R} f dz = 2\pi i \left(\frac{1}{4}e^{-3i\pi/4} + \frac{1}{4}e^{-i\pi/4}\right). \tag{3.28}$$

The key is to deal with the additional term $\int_{S_R} f dz$ when $R \to \infty$. In this case, note that

$$\left|\int_{S_R} f dz\right| \le 2\pi R \cdot \frac{1}{R^4 - 1},$$

which tends to 0 as $R \to \infty$. Thus, (3.28) implies

$$\int_{-\infty}^{\infty} f(x)dx = 2\pi i \left(\frac{1}{4}e^{-3i\pi/4} + \frac{1}{4}e^{-i\pi/4}\right) = \frac{\pi}{\sqrt{2}}.$$

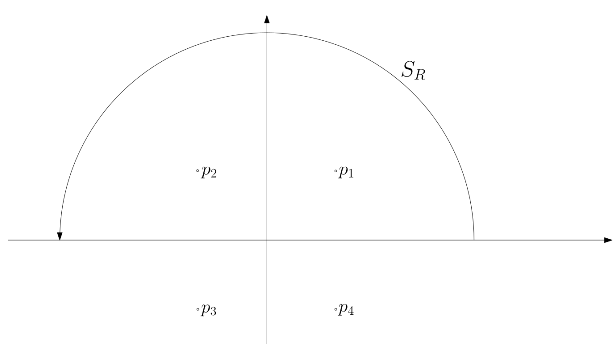

FIGURE 8. A semicircle enclosing two poles.

Observe that the path we choose in Example 3.27 works when the function f satisfies a decaying estimate[ex]

$$f(z) = o\left(\frac{1}{z}\right) \text{ as } z \to \infty. \tag{3.29}$$

Example 3.30 (Fourier transform)**.** We look at the integral

$$\int_{-\infty}^{\infty} f(x)e^{ix}dx, \tag{3.31}$$

which is still of the form (3.26), but it is related to the Fourier transform of the function f. We calculate an example in Example 2.31, and now we can look at more general functions. When f can be extended to a meromorphic function with finitely many poles away from the real line, we can calculate (3.31) in terms of the residues of $f(z)e^{iaz}$ when

$$f(z) = O\left(\frac{1}{z}\right) \tag{3.32}$$

as $z \to \infty$. Note that this is a weaker condition compared with (3.29). To this end, we choose a different path. Consider the square of side length L, with vertices L, $L + 2Li$, $-L + 2Li$, and $-L$ in $\mathbb{C}$. We estimate the top side and the right side. This is similar to the path in Figure 5, but the difference is that all the sides are growing in the case here.

For the top side from $L + 2Li$ to $-L + 2Li$, we can parametrize it by $\gamma_{\text{top}}(t) = -tL + 2Li$ for $t \in [-1, 1]$. Then for L large enough,

$$\left|\int_{\gamma_{\text{top}}} f(z)e^{iz}dz\right| = \left|\int_{-1}^{1} f\left(-tL + 2Li\right) e^{i(-tL+2Li)} \cdot (-L)\, dt\right| \leq 2 \cdot \frac{C}{2L} \cdot e^{-2L} \cdot L, \tag{3.33}$$

where we use (3.32) to get some $C < \infty$ such that $|f(z)| \leq C/|z|$ when $|z|$ is large.

For the right side from L to $L + 2Li$, we can parametrize it by $\gamma_{\text{right}}(t) = L + tLi$ for $t \in [0, 2]$. Then for L large enough,

$$\left|\int_{\gamma_{\text{right}}} f(z)e^{iz}dz\right| = \left|\int_{0}^{2} f\left(L + tLi\right) e^{i(L+tLi)} \cdot (Li)\, dt\right| \leq \frac{C}{L}\int_0^2 e^{-tL} L dt = \frac{C}{L}\left(1 - e^{2L}\right). \tag{3.34}$$

Note that both (3.33) and (3.34) tend to 0 as $L \to \infty$. One can estimate the left side similarly like (3.34). Thus, the residue theorem (Theorem 3.22) implies

$$\int_{-\infty}^{\infty} f(x)e^{ix}dx = 2\pi i \cdot \sum_{\mathrm{Im}(p)>0} \mathrm{Res}_h(p)$$

where $h(z) := f(z)e^{iaz}$.

3.4. **Argument principle and other applications.** We now talk about other applications of the residue theorem. The first one is called the argument principle. It counts the number of zeros and poles inside a closed path.

In the following, we talk about the simplest case when the curve γ is a simple closed path. That is, it is a closed embedded curve which is piecewise smooth. Thus, by Theorem 2.7, we know that $\mathbb{C} \setminus \gamma$ has two components, one of which is bounded, called Ω_γ. We assume the curve γ is positively oriented so that for any $z \in \Omega_\gamma$, we have $\mathrm{Ind}_\gamma(z) = 1$, and for $z \in \mathbb{C} \setminus (\gamma \cup \Omega_\gamma)$, $\mathrm{Ind}_\gamma(z) = 0$. In this setting, we can count the number of the zeros and the poles of a meromorphic function using a line integral.

Theorem 3.35 (Argument principle)**.** Let γ be a simple closed path such that both γ and the bounded component Ω_γ it bounds are contained in an open set U. Let f be a meromorphic function on U such that f does not have a pole or a zero on γ. If we write

$$\begin{aligned} N_0(\gamma, f) &:= \text{the number of zeros of } f \text{ in } \Omega, \text{ and} \\ N_\infty(\gamma, f) &:= \text{the number of poles of } f \text{ in } \Omega, \end{aligned}$$

both counted with multiplicities, then we have

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}dz = N_0(\gamma, f) - N_\infty(\gamma, f).$$

For example, $f(z) = (z-i)^2/(z-4)^3$ has a zero of order 2 at i, and hence i will contribute two to the number $N_0(\gamma, f)$ if i is enclosed by γ. Similarly, 4 will contribute three to the number $N_\infty(\gamma, f)$ if 4 is enclosed by γ.

Proof. First, we note that f only has finitely many zeros in $\overline{\Omega_\gamma} = \Omega_\gamma \cup \gamma$; otherwise it would vanish identically on the component of U that contains γ by Theorem 3.1, and hence would also vanish on γ. This in particular implies that f'/f is also a meromorphic function on U.

Note also that a pole of f'/f is either a zero of f or a pole of f. If $z_0 \in \Omega_\gamma$ is a zero of f of order n_0, then we can write

$$f(z) = h(z) \cdot (z - z_0)^{n_0}$$

where h is holomorphic near z_0 with $h(z_0) \neq 0$. Thus, near z_0,

$$\frac{f'(z)}{f(z)} = \frac{h'(z) \cdot (z-z_0)^{n_0} + h(z) \cdot n_0(z-z_0)^{n_0-1}}{h(z) \cdot (z-z_0)^{n_0}} = \frac{h'(z)}{h(z)} + \frac{n_0}{z - z_0}.$$

Since h'/h is holomorphic near z_0, we then derive

$$\operatorname{Res}_{f'/f}(z_0) = n_0. \tag{3.36}$$

On the other hand, if $p \in \Omega_\gamma$ is a pole of f of order N, then we can write

$$f(z) = \frac{g(z)}{(z - z_0)^N}$$

where g is holomorphic near z_0 with $g(z_0) \neq 0$. Thus, near z_0,

$$\frac{f'(z)}{f(z)} = \frac{\frac{g'(z)}{(z-z_0)^N} - \frac{g(z)\cdot N(z-z_0)^{N-1}}{(z-z_0)^{2N}}}{g(z)/(z-z_0)^N} = \frac{g'(z)}{g(z)} - \frac{N}{z - z_0}.$$

Since g'/g is holomorphic near p, we then derive

$$\operatorname{Res}_{f'/f}(p) = -N. \tag{3.37}$$

Combining (3.36) and (3.37) with the residue theorem (Theorem 3.22), the conclusion follows. □

Note that the argument principle gives topological/analytical restriction on the distributions of zeros and poles of a meromorphic function. We mention a few consequences of the argument principles in this aspect. First, it tells us a simple fact about the limit of a sequence of non-vanishing holomorphic functions.

Corollary 3.38. Let f_n be a sequence of non-vanishing holomorphic functions on a connected open set U. If f_n converges to a function f locally uniformly on U, then f is a holomorphic function that is either non-vanishing or identically zero.

Remember, from Corollary 2.20, we know that such a limit f must be holomorphic, and from Corollary 2.23, we also have convergence of their derivatives. Now, we know that such a limit cannot vanish unless it is trivial if the sequence doesn't. A variant of it says that if all f_n's omit a value $z_0 \in \mathbb{C}$, then the locally uniform limit f also omits z_0. None of these holds in the case of real differentiable functions, as we stress many times.

Proof. Suppose f is not identically zero. We will show that f is then non-vanishing on U.

If f had a zero at $z_0 \in U$, since f is not identically zero, Proposition 3.1 would imply that 0 is an isolated zero of f. Suppose the order of z_0 is n_0, and hence we can take $r > 0$ such that $\overline{B}_r(z_0) \subseteq U$ and Then Theorem 3.35 and the convergence of f_n's imply

$$n_0 = \frac{1}{2\pi i} \int_{\partial B_r(z_0)} \frac{f'(z)}{f(z)} dz = \lim_{n\to\infty} \frac{1}{2\pi i} \int_{\partial B_r(z_0)} \frac{f_n'(z)}{f_n(z)} dz = 0$$

where the last equality is because each f_n'/f_n is holomorphic since f_n is non-vanishing. This is a contradiction since $n_0 > 0$. □

Corollary 3.38 is sometimes referred as Hurwitz's theorem, which has another part about limits of injective holomorphic functions. It can be proven similarly or be derived from its first version (Corollary 3.38), so we state it as follows and leave it to Assignment 5.

Corollary 3.39 (Hurwitz)**.** Let f_n be a sequence of injective holomorphic functions on a connected open set U. If f_n converges to a function f locally uniformly on U, then f is a holomorphic function that is either injective or constant.

The argument principle can also be used to compare the roots of two holomorphic functions that are close enough to each other. An example is the following theorem by Rouché.

Corollary 3.40 (Rouché's theorem)**.** Let γ be a simple closed path such that both γ and the bounded component Ω it bounds are contained in an open set U. Let f and g be holomorphic functions on U. If $|f| > |g|$ on γ, that is,

$$|f(z)| > |g(z)| \text{ for all } z \in \gamma, \tag{3.41}$$

then f and $f + g$ have the same number of zeros in Ω (again, counted with multiplicities).

Proof. The proof is based on a continuity argument. We consider a family

$$f_t(z) := f(z) + t \cdot g(z)$$

of holomorphic functions on U for $t \in [0, 1]$. The condition (3.41) implies that f_t does not have a zero on γ for all $t \in [0, 1]$. Then by Theorem 3.35, we know that the number of zeros of f_t in Ω is

$$n_t := \frac{1}{2\pi i} \int_\gamma \frac{f_t'(z)}{f_t(z)} dz.$$

Thus, it remains to show $n_0 = n_1$.

Since $f_t(z)$ can be viewed as a continuous function on $[0,1] \times U \ni (t,z)$, n_t is a continuous function in t. Because for each $t \in [0,1]$, n_t is an integer, the continuity forces it to be a constant. In particular, $n_0 = n_1$. $\square$

Rouché's theorem can be viewed as a stability result for numbers of zeros. The result has many applications. For example, in Assignment 5, we will see that we can give another proof of the fundamental theorem of algebra based on Rouché's theorem. Second, it can really help us to count the number of roots in a bounded region. We discuss this in the following example.

Example 3.42. Let $f(z) = z^6 - 4z^2 + 3$. We know that it has six roots in $\mathbb{C}$, and we can use Rouché's theorem to get a rough range of them. In fact, we want to take $R > 0$ such that

$$\left|z^6\right| > \left|4z^2 + 3\right|$$

on ∂B_R so that we can apply Rouché's theorem to this circle. For this, the inequality

$$\left|z^6\right| > 4\left|z^2\right| + 3$$

suffices. For example, we can take $R = 2$. Thus, Rouché's theorem implies that f and z^6 have the same number of zeros in B_2, so we know that all the roots of f lie in B_2. This is the simplest situation, and one can even deal with equations involving other functions, like exponential or trigonometric functions.

The last application that we are going to see is the open mapping theorem, which has been mentioned in Corollary 3.6 as a consequence of the inverse function theorem. Using Rouché's theorem, we can see another simple proof of the open mapping theorem.

Another proof of Corollary 3.6. Suppose $z_0 \in U$ and $y_0 = f(z_0)$. Since f is not a constant map, Proposition 3.1 imply that z_0 is an isolated zero of $f - y_0$. Thus, by taking a small enough $r > 0$, we can make $\overline{B}_r(z_0) \subseteq U$ and z_0 is the unique zero of $f - y_0$ in $\overline{B}_r(z_0)$. In particular, $f - y_0$ is non-vanishing on $\partial B_r(z_0)$ and hence

$$R := \inf_{\partial B_r(z_0)} |f - y_0| > 0.$$

Therefore, for any complex number w with $|w| < R$, Rouché's theorem (Corollary 3.40) implies that $f - y_0 + w$ also has at least a zero in $B_r(z_0)$. This means that $f(U)$ contains $B_R(y_0)$, and the conclusion follows. $\square$

A more quantitative version of the open mapping theorem, Bloch's theorem, can be found in Assignment 5. It can also be viewed as a consequence of Rouché's theorem, and we state it here.

Theorem 3.43 (Bloch's theorem)**.** Let f be a holomorphic function on U which contains $B_R(z_0)$. Then there exists a universal constant $C > 0$ such that $f\left(B_R(z_0)\right)$ contains a ball of radius $CR|f'(z_0)|$.

4. Examples and techniques of constructing meromorphic functions

We will see more examples of holomorphic functions and meromorphic functions with some of their applications in this section. It includes some explicit examples and some ways to construct holomorphic or meromorphic functions with some prescribed information.

4.1. **Logarithm and entire functions.** We will talk about an important extension of a real-valued function, the logarithm, $\log x$. Using it, we will briefly talk about how to look at the growth of a holomorphic function.

4.1.1. *Logarithm function on a simply connected domain.* When $x \in \mathbb{R}_{>0}$, we know that there is a well-defined analytic function $\log x$ such that

$$e^{\log x} = \log e^x = x. \tag{4.1}$$

It turns out that it is not straightforward to extend this to a well-defined continuous function $\log z$ for $z \in \mathbb{C} \setminus \{0\}$. An obvious reason is that for any $k \in \mathbb{Z}$, we have

$$e^{z+2k\pi i} = e^z \cdot \left(e^{2\pi i}\right)^k = e^z.$$

Thus, to make it well-defined, one has to fix a choice of the argument. For example, we can define $\log 1 = 0$ as in the real case. However, if we would like to extend this to the whole $\mathbb{C} \setminus \{0\}$, even just continuously, we will have to define

$$\log e^{it} = it$$

for $t \in [0, 2\pi)$. The problem then comes when t tends to 2π. On the one hand, continuity forces $\log e^{2\pi i} = 2\pi$. On the other hand, if the function is well-defined, we have

$$\log e^{2\pi i} = \log 1 = 0.$$

Thus, it is impossible to have a well-defined continuous log function on $\mathbb{C} \setminus \{0\}$.

We can make this more concrete if we require $\log z$ to be a holomorphic function. In fact, it has to be holomorphic if it exists as the inverse of the exponential function, based on the inverse function theorem 2.47. We can use this property to say that we cannot define log on a domain where there is a closed path that winds the origin, that is, a closed path γ such that $\mathrm{Ind}_\gamma(0) = 1$. In fact, if $\log \colon U \to \mathbb{C}$ is a holomorphic function satisfying (4.1), then differentiating (4.1) (using the complex variable z) leads to

$$1 = e^{\log z} \cdot \frac{d}{dz} \log z,$$

implying that

$$\frac{d}{dz} \log z = \left(e^{\log z}\right)^{-1} = \frac{1}{z}$$

on U. Then, if there is a closed path γ in U with $\mathrm{Ind}_\gamma(0) = 1$, then by the residue theorem (Theorem 3.22), as a meromorphic function on $\mathbb{C}$, $1/z$ satisfies

$$\int_\gamma \frac{1}{z} dz = 2\pi i,$$

while the fundamental theorem of calculus implies

$$\int_\gamma \frac{1}{z} dz = \int_\gamma \left(\frac{d}{dz} \log z\right) dz = 0$$

since γ is closed. We then get a contradiction. Formally, we would say that it is not possible to analytically continuate a logarithm function along the closed path γ. We will talk about this in more detail in Section 5.5.4.

We now formalize a related topological property, and will show that it characterizes regions on which we can define log .

Definition 4.2. A subset S of $\mathbb{C}$ is called **simply connected** if it is connected and any closed curve in S is homotopic to a constant map in S.

Recall the definition of homotopy in (2.43). We will especially talked about simply connected open subsets. Thus, an open set U is simply connected if any closed curve in U can be continuously contracted to a point, and the whole contraction process happens in U.

Example 4.3. We mention some examples of domains in $\mathbb{C}$.

(1) A convex set is simply connected.

(2) More generally (compared with (1)), a star-shaped domain is simply connected. In fact, if U is a star-shaped open set in $\mathbb{C}$, say every point p can be connected to the origin by a segment in U, which is $\overline{Op}$, then every closed curve is homotopic to the constant map $\gamma(t) = p$.

(3) A typical example which is not simply connected is a punctured disc. For example, a round circle centered at the origin is not homotopic to a constant map in $\mathbb{C} \setminus 0$.

We recall that two homotopic closed paths satisfy (2.45) as a consequence of the Cauchy theorem. It turns out that it is also true for any two homotopic paths with the same endpoints. This formulation is helpful because we have the following characterization of simply connected domains.[ex]

Proposition 4.4. A subset S in $\mathbb{C}$ is simply connected if and only if it is connected and any two curves with the same endpoints are homotopic to each other. As a consequence, combined with (2.45), it implies that for a holomorphic function f on S and two paths γ_1 and γ_2 with the same endpoints, we have

$$\int_{\gamma_1} f dz = \int_{\gamma_2} f dz.$$

As a corollary, one can generalizes Lemma 1.8 to a simply connected domain.[ex] The proof idea is to find a primitive directly, as performed in Lemma 1.8.

Corollary 4.5. Let U be a simply connected open set in $\mathbb{C}$ and let u be a harmonic function on U. Then there exists a unique harmonic conjugate of u up to a constant.

On a simply connected domain away from the origin, we can then have a well-defined logarithm. In general, we can take the logarithm of a function if its domain is simply connected and the function is non-vanishing.

Theorem 4.6. Let U be a simply connected open set in $\mathbb{C}$ and f be a non-vanishing holomorphic function on U. Then there exists a holomorphic function $\log f$ such that $e^{\log f} = f$.

Such a function $\log f$ is called a **branch** of the logarithm of f. One can also see that such a branch is unique up to an integer multiple of $2\pi i$. When $f(z) = z$ and $U = \mathbb{C} \setminus (-\infty, 0]$, we can then find a branch $g(z) = \log z$ such that $g(1) = 0$. Its value is

$$\log\left(re^{it}\right) = \log r + it$$

if $r \in \mathbb{R}_{\geq 0}$ and $t \in (-\pi, \pi)$. This is called the **principal branch** of the logarithm. One can also find out its power series expansion[ex]

$$\log z = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}(z-1)^n \tag{4.7}$$

for $z \in B_1(1)$.

Proof of Theorem 4.6. We do the case when $f(z) = z$, and leave the general case as an exercise.[ex] In this special case, the assumption implies that U does not contain the origin.

Fix a point $p \in U$. Then given any $z \in U$, we find a path $\gamma\colon [0,1] \to U$ with $\gamma(0) = p$ and $\gamma(1) = z$, and define

$$g(z) := \int_\gamma \frac{dz}{z} + C_p$$

for some $C_p \in \mathbb{C}$ to be defined. Note that this integral is valid since $1/z$ is holomorphic on U. Moreover, g is a well-defined function, since Proposition 4.4 implies that the expression does not depend on the path γ as long as it connects p to z. Thus, $g(z)$ is a continuous function whose derivative is $1/z$ (cf. the proof of Corollary 2.15), and hence g is a holomorphic function.

It remains to show that $e^{g(z)} = z$, which depends on our choice of C_p. To see this, using $g'(z) = 1/z$, we have

$$\left(ze^{-g(z)}\right)' = e^{-g(z)} + ze^{-g(z)} \cdot \left(-\frac{1}{z}\right) = e^{-g(z)}\,(1-1) = 0,$$

which implies that $ze^{-g(z)}$ is constant by the connectivity of U. Thus, we have

$$ze^{-g(z)} = pe^{-g(p)} = p \cdot e^{-C_p}. \tag{4.8}$$

We can then choose $C_p \in \mathbb{C}$ such that $e^{C_p} = p$, and then (4.8) implies $e^{g(z)} = z$. Note that the choice of C_p is unique up to an integer multiple of $2\pi i$. $\square$

Besides its many applications, the existence of a branch of logarithm on a simply connected domain allows us to define inverses of many other common functions. We record them here and omit the details.[ex]

Corollary 4.9. Let U be a simply connected open set in $\mathbb{C}$.

(1) If $f\colon U \to \mathbb{C} \setminus \{0\}$ is a holomorphic function, then there exists a holomorphic function $f^{1/n}$ such that $\left(f^{1/n}\right)^n = f$.

(2) If $g\colon U \to \mathbb{C} \setminus \{\pm 1\}$ is a holomorphic function, then there exists a holomorphic function $\cos^{-1} g$ such that $\cos\left(\cos^{-1} g\right) = g$.

As an application, we can prove the little Picard theorem. [9] It is a generalization of the fact that a non-constant entire function has a dense image (Assignment 4).

Theorem 4.10 (Little Picard)**.** If $f\colon \mathbb{C} \to \mathbb{C}$ is a non-constant entire function, then f omits at most one point in $\mathbb{C}$.

First, note that this theorem is sharp in view of the example given by the exponential function, missing exactly the origin in its image. Also, note that Theorem 4.10 is related to the Casorati–Weierstrass theorem (Proposition 3.17). In fact, as we see in Assignment 4 (cf. Example 3.13 (5) and (6)), if $f(z)$ is a non-constant entire function, then f has a pole or an essential singularity at ∞. Hence, either f is a polynomial (so its image is $\mathbb{C}$), or Proposition 3.17 implies that $f(\mathbb{C})$ is dense in $\mathbb{C}$. Theorem 4.10 then strenthens this result by providing more precise information of the image of such a function.

Proof of Theorem 4.10. We will assume the function f misses at least two points, and prove that it must be constant. After applying a linear-type transformation (of the form $az+b$), we may assume that f misses at least 0 and 1.

Since $f(\mathbb{C})$ does not include 0 and $\mathbb{C}$ is simply connected, by Theorem 4.6, we can define a holomorphic function $g := \log f$ on $\mathbb{C}$. Since $f(\mathbb{C})$ does not include 1, $g(\mathbb{C})$ then does not include any integer multiple of $2\pi i$. In particular, the image of $g/2\pi i$ does not include ± 1, so Corollary 4.9 implies that we can find a holomorphic function

$$h = \cos^{-1} \frac{1}{2\pi i} g = \cos^{-1}\left(\frac{1}{2\pi i}\log f\right)$$

on $\mathbb{C}$. Then notice that if $h(z) = \pm i\cosh^{-1} m + 2\pi n$ for some $m, n \in \mathbb{Z}$, we would have

$$f(z) = e^{2\pi i \cos h(z)} = e^{2\pi i \cos\left(\pm i \cosh^{-1} m\right)} = e^{2\pi i \cdot m} = 1,$$

where we used the formula $\cos(iz) = \cosh z$. Hence, the image of h does not contain

$$\left\{\pm i\cosh^{-1} m + 2\pi n : m, n \in \mathbb{Z}\right\}.$$

Recalling the formula$^{\text{ex}}$

$$\cosh^{-1} m = \log\left(m + \sqrt{m^2-1}\right),$$

we conclude that $h(\mathbb{C})$ does not contain an open ball of radius larger than a finite number C_h.

Now, we could apply Bloch's theorem (Theorem 3.43) to conclude that there exists $C > 0$ such that

$$CR \cdot \left|h'(z)\right| \le C_h$$

for any $z \in \mathbb{C}$ and $R > 0$, since $h\left(B_R(z)\right) \subseteq h(\mathbb{C})$ does not contain an open ball of radius at most C_h. The relation means

$$\left|h'(z)\right| \le \frac{C_h}{C} \cdot \frac{1}{R}$$

for any $R > 0$, so h' vanishes identically, and hence h is a constant. This finishes the proof. ∎

[9] The rest of the note 6 (page 57-59) is not covered in the lectures and can be viewed as optional reading.

The little Picard theorem tells us about how dense the image of an entire function has to be. There is a stronger result, the great Picard theorem, stating more precise information of the behavior of a holomorphic function near an essential singularity. See Theorem 5.24.

4.1.2. *Growth of zeros.* We study the density property of the image of an entire function in the previous section. Next, we look at the growth of an entire function. It turns out that the behavior of a holomorphic function on the boundary of a disc is related to zeros in the interior of the disc. This is encoded in the following formula by Jensen.

Theorem 4.11 (Jensen's formula)**.** Let f be a holomorphic function on an open set U and suppose $\overline{B}_R \subseteq U$ for some $R > 0$. If $f(0) \neq 0$ and $f|_{\partial B_R}$ is non-vanishing, then

$$\log|f(0)| = \sum_{z \in f^{-1}(0) \cap B_R} \log\frac{|z|}{R} + \frac{1}{2\pi}\int_0^{2\pi} \log\left|f\left(Re^{it}\right)\right| dt,$$

where the zeros are counted with multiplicities.

In fact, the formula also extends to the case of meromorphic functions. Here, since we are going to look at entire functions later, we focus on holomorphic functions. Also note that if f is non-vanishing on the disc, the formula follows directly from the mean-value property of $\log|f|$, which is the real part of a holomorphic function $\log f$. This is one way to prove the formula, which we will see in the following.

Proof. We do the case when $R = 1$, and the general case follows from scaling (and one can also get a version when the center is not the origin).

The trick we are going to use is to multiply f by a good function so that the number of its zeros decreases. That is, for $z_0 \in B_1$, consider the function

$$h_{z_0}(z) := \frac{z_0 - z}{1 - \overline{z_0} z}.$$

In Assignment 1, the following properties are proven.

(1) $h_{z_0}(0) = z_0$,

(2) h_{z_0} has a unique zero z_0 in B_1, and

(3) $|h_{z_0}| = 1$ on ∂B_1.

As a result, we can look at the function

$$F := f \cdot \prod_{z \in f^{-1}(0) \cap B_1} h_z^{-1},$$

a non-vanishing function on $\overline{B}_1$. We can then apply the mean-value theorem to the real part of $\log F$ to get

$$\log|F(0)| = \frac{1}{2\pi}\int_0^{2\pi} \log\left|F\left(e^{it}\right)\right| dt. \tag{4.12}$$

Note that for each $z \in f^{-1}(0) \cap B_1$, the properties (1), (2), and (3) imply

$$\log |h_z(0)| = \log |z| = \log |z| + \frac{1}{2\pi} \int_0^{2\pi} \log \left| h_z\left(e^{it}\right) \right| dt. \tag{4.13}$$

Combing (4.12) with (4.13) for each $z \in f^{-1}(0) \cap B_1$, the conclusion follows. □

There are many applications of Jensen's formula. For example, one can use it to prove the Liouville theorem and the fundamental theorem of algebra again. A typical usage of Jensen's formula when applied to entire functions is to relate the growth of an entire function with the density of its zeros. We state one type of this result here.

Corollary 4.14. Let f be a non-constant entire function such that for some $C, \alpha < \infty$,

$$|f(z)| \leq Ce^{C|z|^\alpha} \tag{4.15}$$

for all $z \in \mathbb{C}$. Then there exists $C_Z = C_Z(f) < \infty$ such that f contains at most $C_Z R^\alpha$ zeros in B_R.

For a function satisfying (4.15), we say f is **of order at most** α. For such a holomorphic function, if we let

$$N_0(B_R, f) := \text{the number of zeros of } f \text{ in } B_R,$$

the theorem says that the number $N_0(B_R, f)$ of zeros of f does not grow faster than R^α.

Proof. First, we assume $f(0) \neq 0$. Then applying Theorem 4.11 with radius $2R$ and putting the integral term on one side, we have

$$\log |f(0)| + \sum_{z \in f^{-1}(0) \cap B_{2R}} \log \frac{2R}{|z|} = \frac{1}{2\pi} \int_0^{2\pi} \log \left| f\left(2Re^{it}\right) \right| dt.$$

On the one hand, we can use the growth assumption (4.15) to estimate

$$\frac{1}{2\pi} \int_0^{2\pi} \log \left| f\left(2Re^{it}\right) \right| dt \leq \frac{1}{2\pi} \cdot 2\pi \cdot \log Ce^{C\cdot(2R)^\alpha} = \log C + 2^\alpha C \cdot R^\alpha.$$

On the other hand, for the summation term, we can just look at those zeros in B_R, and get

$$\sum_{z \in f^{-1}(0) \cap B_{2R}} \log \frac{2R}{|z|} \geq \sum_{z \in f^{-1}(0) \cap B_R} \log \frac{2R}{|z|} \geq \log 2 \cdot N_0(B_R, f).$$

Combining these, we get

$$\log |f(0)| + \log 2 \cdot N_0(B_R, f) \leq \log C + 2^\alpha C \cdot R^\alpha,$$

which implies

$$N_0(B_R, f) \leq \frac{1}{\log 2} \left(2^\alpha C R^\alpha - \log |f(0)|\right),$$

the desired growth.

If $f(0) = 0$, since f is non-constant, 0 must be an isolated zero of f. Thus, for small enough $r > 0$, we have $f(r) \neq 0$. Applying the argument above to the point r, we get the same form of estimates for large enough R.[ex] □