3.2. **Behavior near isolated singularities.**

3.2.1. *Laurent series.* Next, we talk about some possible singularities of a holomorphic function. For simplicity, we introduce a notation

$$\hat{B}_r(z_0) := B_r(z_0) \setminus \{z_0\}$$

for $r > 0$ and $z_0 \in \mathbb{C}$. This is not a standard notation, but we will use it from time to time.

Definition 3.7. If f is a holomorphic function on $\hat{B}_r(z_0)$ for some $r > 0$ and $z_0 \in \mathbb{C}$, then we say f has an **isolated singularity** at z_0.

This is the simplest type of singularities, since it is isolated. It already has some complicated behavior. However, we can still see some very straightforward examples.

Example 3.8. We consider the 1-ball B_1.

(1) If f is a holomorphic function on B_1, by considering $g := f|_{\hat{B}_1(0)}$, we get a holomorphic function g with an isolated singularity at the origin. We can see that this is a "fake" singularity since we can in fact extend g to the whole ball. Thus, such a singularity is not too bad.

(2) If f is a holomorphic function on B_1 with $f(0) = 0$ and $f(z) \neq 0$ for $z \in \hat{B}_1(0)$, then $g := 1/f$ defines a holomorphic function on $\hat{B}_1(0)$, and hence has an isolated singularity. We know that a non-trivial zero of a holomorphic function is isolated, so there are many such examples.

We will start from a general discussion of isolated singularities. The main tool will be Proposition 2.4 and the Cauchy theorem 2.38. After that, we will classify singularities into three categories and talk about them separately. We introduce a notation

$$A_{R,r}(z_0) := B_R(z_0) \setminus \overline{B}_r(z_0)$$

for $z_0 \in \mathbb{C}$ and $R > r \geq 0$. This is an open annulus centered at z_0. When $z_0 = 0$, we may just write $A_{R,r} := A_{R,r}(0)$. We will look at holomorphic functions on an annulus, since a holomorphic function with an isolated singularity can be viewed as a special case of these because $A_{R,0} = \hat{B}_R(0)$.

Theorem 3.9. Let f be a holomorphic function on an annulus $A_{R,r}(z_0)$. Then we can define

$$a_n := \frac{1}{2\pi i} \int_{\partial B_s(z_0)} \frac{f(z)}{(z - z_0)^{n+1}} dz$$

for any $s \in (r, R)$ and $n \in \mathbb{Z}$, so that

$$f(w) = \sum_{n=-\infty}^{\infty} a_n (w - z_0)^n \tag{3.10}$$

for all $w \in A_{R,r}$. The convergence is locally uniform in $A_{R,r}$ and the expression is unique. (In particular, the coefficients a_n's are independent of s.)

Proof. For simplicity, we assume $z_0 = 0$ and work on the annulus $A_{R,r}$. Then for $w \in A_{R,r}$, we can choose r_1 and r_2 such that $r < r_1 < |w| < r_2 < R$ and

$$f(w) = \frac{1}{2\pi i}\int_{\partial B_{r_2}} \frac{f(z)}{z-w}dz - \frac{1}{2\pi i}\int_{\partial B_{r_1}} \frac{f(z)}{z-w}dz. \tag{3.11}$$

This follows from the Cauchy theorem (Theorem 2.38) when we take $\gamma = \partial B_{r_2} - \partial B_{r_1}$, both oriented counterclockwise.

Now, we analyze the sum (3.11). We know that they are both analytic function in w by Proposition 2.4, and we can follow the proof idea of Proposition 2.4 (with $a = 0$) to get an expansion. That is, for $z \in \partial B_{r_2}$, we have

$$\frac{1}{z-w} = \frac{1}{z}\cdot\frac{1}{1-\frac{w}{z}} = \frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{w}{z}\right)^n = \sum_{n=0}^{\infty}\frac{w^n}{z^{n+1}},$$

while for $z \in \partial B_{r_1}$, we have

$$-\frac{1}{z-w} = \frac{1}{w}\cdot\frac{1}{1-\frac{z}{w}} = \frac{1}{w}\sum_{m=0}^{\infty}\left(\frac{z}{w}\right)^m = \sum_{m=0}^{\infty}\frac{z^m}{w^{m+1}}.$$

As in Proposition 2.4, these both converge locally uniformly. Thus, combining these, we derive

$$f(w) = \frac{1}{2\pi i}\sum_{n=0}^{\infty}\left(\int_{\partial B_{r_2}}\frac{f(z)}{z^{n+1}}dz\right)w^n + \frac{1}{2\pi i}\sum_{m=0}^{\infty}\left(\int_{\partial B_{r_1}} f(z)\cdot z^m dz\right)\frac{1}{w^{m+1}}.$$

By letting $m = -n-1$, we get a summation in n, and the result follows since for any $n \in \mathbb{Z}$,

$$\int_{\partial B_{r_2}}\frac{f(z)}{z^{n+1}}dz = \int_{\partial B_{r_1}}\frac{f(z)}{z^{n+1}}dz = \int_{\partial B_s}\frac{f(z)}{z^{n+1}}dz$$

for any $s \in (r, R)$ because $f(z)/z^{n+1}$ is holomorphic in $A_{R,r}$.

To get the uniqueness of the expression, suppose

$$f(w) = \sum_{n=-\infty}^{\infty} c_n (w-z_0)^n$$

is another expansion that converges absolutely and locally uniformly. Then for any $s \in (r, R)$,

$$a_j = \frac{1}{2\pi i}\int_{\partial B_s(z_0)}\frac{f(z)}{(z-z_0)^{j+1}}dz = \sum_{n=-\infty}^{\infty} c_n\cdot\frac{1}{2\pi i}\int_{\partial B_s(z_0)}\frac{(z-z_0)^n}{(z-z_0)^{j+1}}dz.$$

Note that by parametrizing $\partial B_s(z_0)$ by $z_0 + se^{it}$ for $t \in [0, 2\pi]$, we can calculate that for each n,

$$\begin{aligned}\frac{1}{2\pi i}\int_{\partial B_s(z_0)}\frac{(z-z_0)^n}{(z-z_0)^{j+1}}dz &= \frac{1}{2\pi i}\int_0^{2\pi}\left(se^{it}\right)^{n-j-1}\cdot ise^{it}dt \\ &= \frac{1}{2\pi}\int_0^{2\pi}\left(se^{it}\right)^{n-j}dt = \delta_{nj} := \begin{cases}1 & \text{if } n = j \\ 0 & \text{if } n \neq j\end{cases}.\end{aligned}$$

Putting this back to the summation, we get $a_j = \sum_n c_n \cdot \delta_{nj} = c_j$. This proves the uniqueness and the theorem follows. □

The expansion (3.10) is called the **Laurent series** of f centered at z_0. It can be thought of as a generalization of power series expansion, which is exactly when $a_n = 0$ for all $n < 0$. Note that this happens when f is a holomorphic function on the whole ball $B_R(z_0)$, and the Cauchy theorem implies $a_n = 0$ for $n < 0$. In general, if f has an isolated singularity at z_0, then based on the general behavior of the function near z_0, we can have the following classification.

Definition 3.12. Suppose f is a holomorphic function on $A_{R,0}(z_0) = \hat{B}_R(z_0)$, and let

$$f(z) = \sum_{n\in\mathbb{Z}} a_n(z - z_0)^n$$

be the Laurent series of f at z_0.

(1) We say z_0 is a **removable singularity** of f if $a_n = 0$ for $n < 0$.
(2) We say z_0 is a **pole** of f if there exists $n_0 < 0$ such that $a_{n_0} \neq 0$ and $a_n = 0$ for $n < n_0$.
(3) We say z_0 is an **essential singularity** of f if there are infinitely many $n < 0$ with $a_n \neq 0$.

Example 3.13. We mention a few examples of singularities before starting analyzing them.

(1) A rational function of the form $R(z) = P(z)/Q(z)$ with P and Q having no common factors has poles at the zeros of Q.
(2) Let $f(z) = e^{1/z}$ for $z \in \mathbb{C} \setminus \{0\}$. Then

$$e^{1/z} = \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{1}{z}\right)^n = \sum_{n=0}^{\infty} \frac{1}{n!}\frac{1}{z^n}$$

for $z \neq 0$. Thus, f has an essential singularity at 0.

Singularities are not necessarily isolated (unlike zeros). We mention two examples.

(3) The function $\frac{1}{\sin(1/z)}$ has poles at $z = 1/(2\pi n)$ for all $n \in \mathbb{N}$. Thus, 0 is an accumulated singularity of the function.
(4) A function defined as in Proposition 2.4 has a curve of singularities. That is, if γ is a path in U and $\varphi\colon U \to \mathbb{C}$ is a continuous function, then the function

$$f(w) := \int_\gamma \frac{\varphi(z)}{z - w}dz$$

is holomorphic on $U \setminus \gamma$ but has the whole path γ as its singular set.

We can also look at possible singular behavior at ∞. When a function $f(z)$ is holomorphic outside a ball, we can consider $g(w) := f(1/w)$. We say that f has a removable singularity/pole/essential singularity at ∞ if g does at 0.

(5) Given a non-constant polynomial $f(z) = \sum_{n=0}^{N} a_n z^n$, if we let $g(w) = f(1/w)$, we get

$$g(w) = \sum_{n=0}^{N} \frac{a_n}{w^n},$$

so g has a pole at 0 and hence f has a pole at ∞. We will see that this property in fact characterizes a polynomial in Assignment 4.

(6) A non-constant entire function cannot have a removable singularity at ∞; otherwise, it will be bounded, which cannot be the case by the Liouville theorem.

(7) By (2), $f(z) = e^z$ has an essential singularity at ∞.

3.2.2. *Classification of singularities.* We will now see many different properties of different types of singularities.

First, for a removable singularity of a holomorphic function f, from its definition, we know that it literally means that f can extend to a holomorphic function on the whole disc. This is also a common definition of a removable singularity. We know that such a singularity can trivially happen from Example 3.8. In that example, the function is naturally bounded near the singularity. We will see that in general, this property characterizes removable singularities.

Proposition 3.14. Let $z_0 \in U$ and f be a holomorphic function on $U \setminus \{z_0\}$. Then the following are equivalent.

(1) z_0 is a removable singularity of f.

(2) f extends to a holomorphic function on U.

(3) There exists $r > 0$ such that f is bounded on $\hat{B}_r(z_0)$.

(4) $\lim_{z \to z_0} (z - z_0) f(z) = 0$.

The implication from (3) to (1) is often referred as the simplest case of the Riemann extension theorem.

Proof. The first two conditions are equivalent, as mentioned, based on the definition. It is straightforward to see (1)$\Rightarrow$(3)$\Rightarrow$(4). Thus, we will prove (4)$\Rightarrow$(1).

One approach is to use the Laurent series directly. We instead proceed without relying on that. We again assume $z_0 = 0$ for simplicity. We consider the function

$$g(z) := \begin{cases} z^2 f(z) & \text{if } z \in \hat{B}_r(0) \\ 0 & \text{if } z = 0 \end{cases}.$$

By (4), g is a continuous function. Moreover, we can see that

$$\lim_{z \to 0} \frac{g(z) - g(0)}{z - 0} = \lim_{z \to 0} \frac{z^2 f(z)}{z} = \lim_{z \to 0} z f(z) = 0$$

by (4) again. Thus, g is a holomorphic function (since it is evidently holomorphic on $\hat{B}_r(0)$). In particular, we can write

$$g(z) = \sum_{n=2}^{\infty} a_n z^n$$

in $B_r(z_0)$. We can then define $f(0) := a_2$, which makes $f(z) = g(z)/z^2$ a holomorphic function. $\square$

We note that condition (4) is a priori weaker than (3). In fact, (4) includes the possibility that $f \sim (z - z_0)^{-\alpha}$ for some $\alpha \in (0, 1)$. It turns out that this cannot happen based on the nature of Laurent series.

Next, we study poles. If a holomorphic function f on $\hat{B}_r(z_0)$ has a pole at z_0, with

$$f(z) = \sum_{n \geq -N} a_n (z - z_0)^n$$

and $a_{-N} \neq 0$, then we say z_0 is a pole of order N. When $N = 1$, z_0 is called a **simple pole**.

Proposition 3.15. Let $z_0 \in U$ and f be a holomorphic function on $U \setminus \{z_0\}$. Then the following are equivalent.

(1) z_0 is a pole of f.

(2) There exist $N \in \mathbb{N}$ and $a_{-1}, \cdots, a_{-N}$ where $a_{-N} \neq 0$ such that

$$f(z) - \sum_{n=-N}^{-1} a_n (z - z_0)^n$$

has a removable singularity at z_0.

(3) There exists $N \in \mathbb{N}$ such that $(z - z_0)^N f(z)$ has a removable singularity and extends to a non-zero value at z_0.

(4) There exists $r > 0$ such that $1/f$ is holomorphic in $\hat{B}_r(z_0)$ and has a removable singularity and extends to zero at z_0.

(5) $\lim_{z \to z_0} f(z) = \infty$. This means $\lim_{z \to z_0} |f(z)| = \infty$.

When (2) and (3) happen, the number N is exactly the order of the pole z_0. The part of functions $\sum_{n=-N}^{-1} a_n (z - z_0)^n$ is called the **principal part** of f at z_0. As said in (2), it is a rational function $P(z)$ which is a polynomial in $1/(z - z_0)$ such that $f - P$ is holomorphic near z_0. It captures the singular behavior of the function near the pole.

Example 3.16. Let $f(z) = \frac{1}{z(z-2)}$. Then f has a simple pole at 0. Moreover, we can write

$$\frac{1}{z(z-2)} = \frac{1}{2} \left(\frac{1}{z-2} - \frac{1}{z} \right).$$

Since $1/(z - 2)$ is holomorphic near 0, we know that the principal part of f at 0 is $-\frac{1}{2z}$.

Proof of Proposition 3.15. The equivalence among (1), (2), and (3) is clearer from the Laurent series expansion, so we focus on (4) and (5). We will more or less see (3) from the proof. We again assume $z_0 = 0$ for simplicity.

For (1)⇒(5), note that if 0 is a pole of order N, say

$$f(z) = \sum_{n \geq -N} a_n z^n$$

on $\hat{B}_r(0)$ with $a_{-N} \neq 0$, then we can write $f(z) = h(z)/z^N$ if we let

$$h(z) := \sum_{n \geq 0} a_{n-N} z^n,$$

which defines a holomorphic function on B_r (which proves (3)). Since $h(0) \neq 0$, we have

$$\lim_{z \to 0} |f(z)| = \lim_{z \to 0} \frac{|h(z)|}{|z|^N} = \infty.$$

For (5)⇒(4), note that (5) implies that we can find $r > 0$ such that $|f| > 10$ on $\hat{B}_r(0)$. In particular, $f \neq 0$ on $\hat{B}_r(0)$. Thus, we can define $g := 1/f$, a holomorphic function on $\hat{B}_r(0)$. Since $|g| \leq 1/10$ on $\hat{B}_r(0)$, by Proposition 3.14 (3), g extends to a holomorphic function on B_r, with

$$g(0) = \lim_{z \to 0} \frac{1}{f(z)} = 0.$$

Hence, we get (4).

For (4)⇒(1), finally, we can write

$$g(z) = \frac{1}{f(z)} = \sum_{n \geq n_0} a_n z^n$$

in B_r with $a_{n_0} \neq 0$ for some $n_0 \geq 1$. Thus, we get a holomorphic function

$$h(z) := \frac{g(z)}{z^{n_0}} = \sum_{n \geq 0} a_{n+n_0} z^n$$

with $h(0) \neq 0$. That is, h is a non-vanishing function on B_r after shrinking r. In particular, $1/h$ is still holomorphic. We then get

$$f(z) = \frac{1}{z^{n_0}} \cdot \frac{1}{h(z)},$$

and we are done. □

Finally, we look at essential singularities.

Proposition 3.17 (Casorati–Weierstrass theorem)**.** Let $z_0 \in U$ and f be a holomorphic function on $U \setminus \{z_0\}$. Then the following are equivalent.

(1) z_0 is an essential singularity of f.

(2) For $r > 0$ such that $B_r(z_0) \subseteq U$, $\overline{f\left(\hat{B}_r(z_0)\right)} = \mathbb{C}$.

Example 3.18. Let $f(z) = e^{1/z}$. Then the image of f is $\mathbb{C} \setminus \{0\}$, which is dense in $\mathbb{C}$.

The Casorati–Weierstrass says that if z_0 is an essential singularity of f, then given any $w \in \mathbb{C}$, there exists a sequence $z_n \to z_0$ such that $f(z_n) \to w$ as $n \to \infty$. This is how the name, "essential" singularity, comes from. Note that, combined with Proposition 3.14 (2) and Proposition 3.15 (5), Proposition 3.17 also allows us to summarize the three different kinds of behavior based on the asymptotics near a singularity.

Corollary 3.19. Let $z_0 \in U$ and f be a holomorphic function on $U \setminus \{z_0\}$.

(1) z_0 is a removable singularity if and only if $\lim_{z \to z_0} f(z)$ exists and is finite.

(2) z_0 is a pole if and only if $\lim_{z \to z_0} f(z) = \infty$.

(3) z_0 is an essential singularity if and only if $\lim_{z \to z_0} f(z)$ does not exist (and can be any number sequentially).

Proof of Proposition 3.17. It is clear that (2) implies (1) based on Proposition 3.14 (2) and Proposition 3.15 (5). Thus, we will prove (1) implies (2).

We prove it by contradiction. Suppose not, that is, there exists $r > 0$ and $w \in \mathbb{C}$ such that $B_r(z_0) \subseteq U$ but $w \notin \overline{f\left(\hat{B}_r(z_0)\right)}$. Then we can find $\delta > 0$ such that

$$B_\delta(w) \subseteq \mathbb{C} \setminus \overline{f\left(\hat{B}_r(z_0)\right)}. \tag{3.20}$$

This then suggests considering the function

$$h(z) := \frac{1}{f(z) - w},$$

which is still holomorphic in $\hat{B}_r(z_0)$. Note that the property (3.20) implies that

$$|h(z)| = \frac{1}{|f(z) - w|} \leq \frac{1}{\delta}.$$

Hence, Proposition 3.14 (3) implies that h extends to a holomorphic function on the whole $B_r(z_0)$.

We can now conclude the proof by looking into two case. If $h(z_0) = 0$, then $f(z)$ has a pole at z_0, a contradiction. If $h(z_0) \neq 0$, then $f(z)$ has a removable singularity at z_0, another contradiction. Hence, the conclusion follows. □

For an essential singularity, we can't really say much, but if time permits, we will see a generalization of the Casorati–Weierstrass theorem, Picard's theorem, which gives more precise information about the image of a holomorphic function near an essential singularity. (See Theorem 4.10 and Theorem 5.24.) Among different singularities, the most interesting ones are poles. Especially, when there is a simple pole of a holomorphic function, it is important to study its Laurent coefficients. We will talk about some of them in the next section.