Theorem 2.38 (Cauchy's theorem)**.** Let f be a holomorphic function on U. If γ is a cycle in U such that

$$\operatorname{Ind}_\gamma(z) = 0 \text{ for all } z \in \mathbb{C} \setminus U, \tag{2.39}$$

then we have

(1) $\int_\gamma f dz = 0$, and

(2) $f(w) \cdot \operatorname{Ind}_\gamma(w) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-w} dz$ for $w \in U \setminus \gamma$.

Note that condition (2.39) is always true when U is a convex open set. Thus, this generalizes the special case we proved in Theorem 2.16. It is interesting to note that in the proof, we will in fact use a few consequences of the first Cauchy integral formula in Section 2.1.

To prove the theorem, we introduce a continuity lemma. It will also be used later when we prove the inverse function theorem (Theorem 2.47).

Lemma 2.40. Let f be a holomorphic function on an open set U. Then the function

$$F(z,w) := \begin{cases} \frac{f(z)-f(w)}{z-w} & \text{if } z \neq w \\ f'(z) & \text{if } z = w \end{cases} \tag{2.41}$$

is a continuous function on $U \times U$.

Proof of Lemma 2.40. We need to verify the continuity at a point $(z_0, z_0) \in U \times U$. Note that for $z \neq w$ in a ball $B_r(z_0) \subseteq U$, we can write $\sigma(t) := tz + (1-t)w$ to be the segment from w to z, and hence

$$F(z,w) = \frac{f(z)-f(w)}{z-w} = \frac{1}{z-w}\int_0^1 f'(\sigma(t))\,\sigma'(t)dt = \int_0^1 f'(\sigma(t))\,dt.$$

This allows us to finish the proof. Given $\varepsilon > 0$, we can find $r > 0$ such that $B_r(z_0) \subseteq U$ and $|f'(z) - f'(z_0)| \leq \varepsilon$ for $z \in B_r(z_0)$. Then given any $z, w \in B_r(z_0)$, choosing the path σ as above, which lies in $B_r(z_0)$, we have

$$|F(z,w) - F(z_0,z_0)| = \left|\int_0^1 f'(\sigma(t))\,dt - f'(z_0)\right| \leq \int_0^1 \left|f'(\sigma(t)) - f'(z_0)\right| dt \leq \varepsilon.$$

This proves the continuity of F. ☐

A crucial idea of proving Theorem 2.38 is about analytic continuation. We want to prove a function vanishes on U, and to do this, we extend it to the whole complex plane and use its structure to show that it is bounded. Thus, in the course of the proof, many consequences of the Cauchy theorem in the convex case will play a role.

Proof of Theorem 2.38. We will prove (2) first. Note that (2) is equivalent to

$$0 = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-w}dz - f(w)\cdot\frac{1}{2\pi i}\int_\gamma \frac{dz}{z-w} = \frac{1}{2\pi i}\int_\gamma \frac{f(z)-f(w)}{z-w}dz.$$

In view of this, we define F as in (2.41), and our goal is then to prove

$$h(w) := \int_\gamma F(z,w)dz = 0$$

on $U \setminus \gamma$. Note that now this function still makes sense for $w \in \gamma$ based on Lemma 2.40, and in fact, we will prove that it can be extended to the whole $\mathbb{C}$ as a globally zero function.

First, we show that h is continuous. Since γ (as its image) is a compact set, given any $w_n \to w_0$ in U, $F(\cdot, w_n) \to F(\cdot, w_0)$ uniformly on γ. Thus, $h(w_n) \to h(w_0)$, and hence h is continuous.

Next, we show that h is holomorphic. Given any closed triangle $R \subseteq U$, we can write

$$\int_{\partial R} h(w)dw = \int_{\partial R} \left(\int_\gamma F(z,w)dz \right) dw = \int_\gamma \left(\int_{\partial R} F(z,w)dw \right) dz = 0,$$

where the last equality follows from Goursat's theorem (Theorem 2.14), since for any fixed z, $F(z, \cdot)$ is holomorphic on $U \setminus \{z\}$ and continuous on U. Thus, Morera's theorem (Theorem 2.19) implies that h is holomorhpic.

Finally, we extend h to an entire function. We consider the set

$$V := \mathrm{Ind}_\gamma^{-1}(0) = \{z \in \mathbb{C} \setminus \gamma : \mathrm{Ind}_\gamma(z) = 0\}.$$

Recall that Ind_γ is an integer-valued continuous function (Corollary 2.6), so V is an open set. By the assumption (2.39), we have $\mathbb{C} \setminus U \subseteq V$, so $U \cup V = \mathbb{C}$. For $w \in V$, we consider

$$\widetilde{h}(w) := \int_\gamma \frac{f(z)}{z-w} dz,$$

which differs from $h(w)$ by Ind_γ. This implies that for $w \in U \cap V$, we have

$$h(w) = \int_\gamma \frac{f(z)}{z-w} dz - 2\pi i \cdot \mathrm{Ind}_\gamma(w) = \widetilde{h}(w).$$

Moreover, $\widetilde{h}$ is a holomorphic function on V by Proposition 2.4. Thus, we can define a function

$$h_0(w) := \begin{cases} h(w) & \text{if } w \in U \\ \widetilde{h}(w) & \text{if } w \in V \end{cases}$$

which is well-defined and holomorphic on $U \cup V = \mathbb{C}$. To see that it vanishes, note that since γ is compact, we can find $R < \infty$ such that $\gamma \subseteq B_R$. In particular, $\mathbb{C} \setminus B_R \subseteq V$, so we can estimate

$$|h_0(w)| \leq \sup_\gamma |f| \cdot \mathrm{Length}(\gamma) \cdot \frac{1}{|w| - R} \tag{2.42}$$

for $w \in \mathbb{C} \setminus B_R$. In particular, h_0 is globally bounded. By Liouville's theorem (Corollary 2.25), h_0 is a constant, and hence must be zero by letting $|w| \to \infty$ in (2.42). This finishes the proof of (2).

To prove (1), take any $z_0 \in U \setminus \gamma$ and consider the holomorphic function $f(z) \cdot (z - z_0)$. Applying (2) with $w = z_0$, we obtain

$$0 = f(z_0)(z_0 - z_0) \cdot \mathrm{Ind}_\gamma(z_0) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)(z-z_0)}{z - z_0} dz = \frac{1}{2\pi i} \int_\gamma f dz.$$

Hence, (1) follows. □

A consequence of Theorem 2.38 is that when we smoothly perturb a curve, the Cauchy integral does not change. To state it rigorously, we introduce a way to formulate such a perturbation. Let $\gamma_0, \gamma_1 \colon [0,1] \to \mathbb{C}$ be two oriented paths in an open set U. We say that these two paths are **homotopic** to each other in U if there exists a continuous map $H \colon [0,1] \times [0,1] \to U$ such that

$$H(0,\cdot) = \gamma_0 \text{ and } H(1,\cdot) = \gamma_1. \tag{2.43}$$

That is, there is a "continuous process" that deforms γ_0 to γ_1 in U.

The notion of homotopy plays an important role in the study of topology and geometry. Here, we mention one particular consequence. That is, if γ_0 and γ_1 are two closed paths that are homotopic to each other in an open set U, then[ex]

$$\mathrm{Ind}_{\gamma_0}(w) = \mathrm{Ind}_{\gamma_1}(w) \text{ for all } w \in \mathbb{C} \setminus U. \tag{2.44}$$

The reason why we separate this conclusion is that it implies

$$\int_{\gamma_0} f dz = \int_{\gamma_1} f dz \tag{2.45}$$

for any holomorphic function f on U. This is a direct consequence of Theorem 2.38 (1) if we let $\gamma := \gamma_1 - \gamma_0 \in \mathcal{P}(U)$.

Forget about two homotopic paths. In general, if two cycles γ_0 and γ_1 satisfy (2.44), then (2.45) is true for any holomorphic function f on U. Homotopic closed paths are just a special and important case of this general principle, which is usually regarded as part of the Cauchy theorem.

Example 2.46. We mention an example why this generalized Cauchy theorem is useful. It allows us to change the integral path freely. For example, consider γ_1 and γ_2 as in Figure 6.

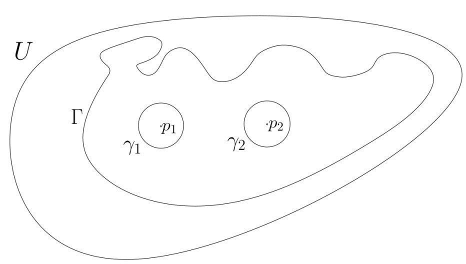

FIGURE 6. When a curve is complicated, the Cauchy theorem can be used to possibly simplify the line integral.

The open set U does not contain the two points p_1 and p_2, so a holomorphic function f on U may have a non-trivial line integral along the curve Γ in Figure 6. However, Γ and the two circles γ_1 and γ_2 satisfy (cf. Assignment 2)

$$\mathrm{Ind}_{\Gamma}(w) = \sum_{n=1}^{2} \mathrm{Ind}_{\gamma_n}(w)$$

for all $w \in \mathbb{C} \setminus U$. Hence, the Cauchy theorem implies

$$\int_{\Gamma} f dz = \sum_{n=1}^{2} \int_{\gamma_n} f dz$$

for any holomorphic function f on U. This may allow us to simplify the calculations. We will see more examples like this in the rest of this course. In particular, this idea of looking at small circles around points where a function is not defined is a key idea in the residue theorem in Section 3.3.

As mentioned, Lemma 2.40 can lead to other local information of holomorphic functions. First, it implies the inverse function theorem.

Theorem 2.47 (Inverse function theorem)**.** Let f be a holomorphic function on an open set U and $z_0 \in U$. If $f'(z_0) \neq 0$, then there exists $r > 0$ such that

(1) $f|_{B_r(z_0)}$ is injective,

(2) $f(B_r(z_0))$ is open in $\mathbb{C}$, and

(3) the inverse $g\colon f(B_r(z_0)) \to B_r(z_0)$ is holomorphic.

Note that the statement (3) is valid based on (1) and (2). In this case, $f|_{B_r(z_0)}\colon B_r(z_0) \to f(B_r(z_0))$ is called a biholomorphic function.

Proof. For (1), we apply Lemma 2.40, which implies that there exists $r > 0$ such that $B_r(z_0) \subseteq U$ and

$$\left|\frac{f(z) - f(w)}{z - w}\right| \geq \frac{1}{2}|f'(z_0)|$$

for $z \neq w$ in $B_r(z_0)$ based on the continuity of the function F defined in (2.41) at (z_0, z_0). It means

$$|z - w| \leq \frac{2}{|f'(z_0)|}|f(z) - f(w)| \tag{2.48}$$

for all $z, w \in B_r(z_0)$ (including the case when $z = w$). This implies the injectivity of f on $B_r(z_0)$.

For (2), given $w_0 \in B_r(z_0)$, we want to show that $f(w_0)$ is an interior point of $f(B_r(z_0))$. By (2.48), we can find $\rho > 0$ such that

$$\left|f(w_0 + \rho e^{it}) - f(w_0)\right| \geq \frac{|f'(z_0)|}{2}\rho =: L.$$

We claim that this implies

$$B_{L/2}\left(f(w_0)\right) \subseteq f\left(B_r(z_0)\right), \tag{2.49}$$

which implies the openness of $f(B_r(z_0))$. To see (2.49), note that for $y \in B_{L/2}(f(w_0))$, we have

$$\left|y - f(w_0 + \rho e^{it})\right| \geq \left|f(w_0 + \rho e^{it}) - f(w_0)\right| - |f(w_0) - y| > \frac{L}{2}$$

for all $t \in [0, 2\pi]$, and hence

$$\left|\frac{1}{y - f(w_0 + \rho e^{it})}\right| < \frac{2}{L} < \left|\frac{1}{y - f(w_0)}\right|. \tag{2.50}$$

This implies y is mapped by f. Otherwise, if y were not mapped by f, then $1/(y - f)$ would be a holomorphic function on $\overline{B}_\rho(w_0)$, and then (2.50) could not happen based on the maximum principle (Theorem 2.27) since f is non-constant.

For (3), fix $w_0 \in B_r(z_0)$ and let $y_0 := f(w_0)$. Since y_0 is an interior point of the image of f, given any y close to y_0, we can write

$$\frac{g(y) - g(y_0)}{y - y_0} = \frac{w - w_0}{f(w) - f(w_0)}$$

for a unique w close to w_0, with $w \to w_0$ if and only if $y \to y_0$. Hence, taking the limit, we have $g'(y_0) = 1/f'(w_0)$, and hence g is holomorphic at y_0. □

Example 2.51. We mention a direct example. That is, $f(z) = e^z$. Since $f'(z) = e^z \neq 0$ for any $z \in \mathbb{C}$, Theorem 2.47 implies that there is always a local inverse function of e^z. That is a logarithm function. An issue about it is that it is impossible to find a global inverse of e^z. We will discuss this more in Section 4.1.

Another consequence of the inverse function theorem is the open mapping theorem (Corollary 3.6). Since it pertains to the local behavior of zeros, we postpone its discussion until the next section. We summarize some key properties of holomorphic functions that we have encountered or will soon encounter later.

- Analyticity: Theorem 2.17.
- Cauchy's theorem: Theorem 2.38.
- Maximum principle: Theorem 2.27.
- Mean-value equality: Corollary 2.28.
- Morera's theorem: Theorem 2.19.
- Liouville's theorem: Corollary 2.25.
- Inverse function theorem: Theorem 2.47.
- Strong unique continuation: Corollary 3.4.
- Open mapping theorem: Corollary 3.6.

We end this section by mentioning another example of calculating definite integrals. With a more general version of the Cauchy theorem (Theorem 2.38), we can now deal with more complicated curve.

Example 2.52. We will use Theorem 2.38 to calculate

$$\int_0^\infty \frac{1 - \cos x}{x^2} dx = \frac{\pi}{2}. \tag{2.53}$$

The idea is to take the line integral of $f(z) = (1 - e^{iz})/z^2$ along the following curve γ.

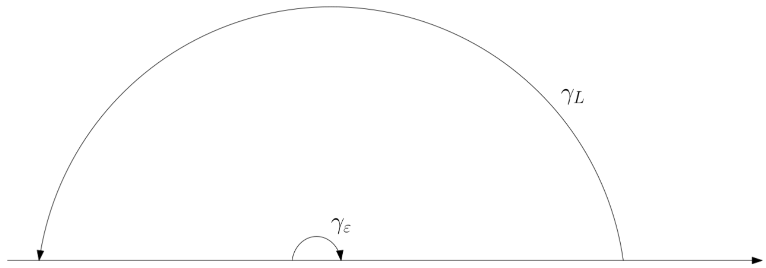

FIGURE 7. Semicircles tending to infinity and zero.

By Theorem 2.38, $\int_\gamma f dz = 0$ for any ε and L.[7] Thus, we will calculate the line integrals for γ_L and γ_ε, We can parametrize γ_L by $\gamma_L(t) = Le^{it}$, $t \in [0, \pi]$, so

$$\left|\int_{\gamma_L} f dz\right| = \left|\int_0^\pi \frac{1 - e^{iLe^{it}}}{L^2 e^{2it}} iLe^{it} dt\right| = \left|\int_0^\pi \frac{1 - e^{iL\cos t - L\sin t}}{Le^{it}} idt\right| \leq \frac{1}{L}\int_0^\pi \left(1 + e^{-L\sin t}\right) dt \leq \frac{1}{L} \cdot 2\pi \to 0 \tag{2.54}$$

as $L \to \infty$, using $\sin t \geq 0$ for $t \in [0, \pi]$. For γ_ε, we can parametrize it by $\gamma_\varepsilon(t) = \varepsilon e^{-it}$, $t \in [-\pi, 0]$. Moreover, for $z = \varepsilon e^{-it}$, we can estimate,

$$f(z) = \frac{1 - e^{iz}}{z^2} = \frac{1 - \left(1 + iz - \frac{z^2}{2} - \frac{z^3}{6} i + \cdots\right)}{z^2} = \frac{-iz + O(z^2)}{z^2} = -\frac{i}{z} + O(1)$$

as $\varepsilon \to 0$.[8] Thus, we can estimate

$$\begin{aligned} \int_{\gamma_\varepsilon} f dz &= \int_{-\pi}^0 -\frac{i}{\varepsilon e^{-it}} \cdot \left(-i\varepsilon e^{-it}\right) dt + \int_{-\pi}^0 O(1) \cdot \left(-i\varepsilon e^{-it}\right) dt \\ &= \int_{-\pi}^0 i^2 dt - \varepsilon \cdot O(1) \int_{-\pi}^0 ie^{-it} dt \to -\pi + 0 \end{aligned} \tag{2.55}$$

as $\varepsilon \to 0$. Combining (2.54) and (2.55) with Theorem 2.38, we get

$$\begin{aligned} \int_{-\infty}^\infty \frac{1 - e^{iz}}{z^2} dz &:= \lim_{L\to\infty} \lim_{\varepsilon\to 0^+} \left(\int_{-L}^{-\varepsilon} f dz + \int_\varepsilon^L f dz\right) \\ &= \lim_{L\to\infty} \lim_{\varepsilon\to 0^+} \left(-\int_{\gamma_L} f dz - \int_{\gamma_\varepsilon} f dz\right) = \pi. \end{aligned}$$

Taking the real parts of the both sides, we conclude

$$\int_{-\infty}^\infty \frac{1 - \cos x}{x^2} dx = \pi,$$

which is equivalent to (2.53).

[7]Note that without the generalization in Theorem 2.38, we cannot deal with the curve γ directly by Corollary 2.15.
[8]By definition, one can just view $O(1)$ as a bounded constant as $\varepsilon \to 0$.

3. Laurent series: local behavior near zeros and poles

We will use the tools developed in the previous sections to further study local behavior of holomorphic functions. This includes their behavior near a zero or near a singularity, a point where a priori the function is not defined.

3.1. Behavior near isolated zeros.

We start from the study of zeros of a holomorphic function. Given a function $f\colon U \to \mathbb{C}$, its zero set will be denoted by $Z(f)$. That is,

$$Z(f) := f^{-1}(0) = \{z \in U : f(z) = 0\}.$$

The first result is about the structure of $Z(f)$ when f is holomorphic.

Proposition 3.1. Let f be a holomorphic function on a connected open set U and let $Z(f)$ be its zero set. Then either $Z(f) = U$ or $Z(f)$ only contains discrete points.

Proof. This is a consequence of being an analytic function. For any $z_0 \in U$, since f is analytic, we can find $r > 0$ such that $\overline{B}_r(z_0) \subseteq U$ and for $z \in B_r(z_0)$, we have

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n.$$

There are then two situations. Either all a_n's are zeros, or we can find $n_0 \in \mathbb{Z}_{\geq 0}$ that is the smallest n such that $a_n \neq 0$. For the first case, we get $f|_{B_r(z_0)} = 0$. For the second case, we can then define a holomorphic function

$$h(z) := \begin{cases} \sum\limits_{n=0}^{\infty} a_n (z - z_0)^{n - n_0} & \text{if } z \in B_r(z_0) \\ \frac{f(z)}{(z - z_0)^{n_0}} & \text{if } z \in U \setminus \{z_0\} \end{cases}.$$

Note that the two definitions agree when $z \in B_r(z_0) \setminus \{z_0\}$. Hence, we can write

$$f(z) = h(z) \cdot (z - z_0)^{n_0},$$

where h satisfies $h(z_0) \neq 0$ (since $a_{n_0} \neq 0$), and hence we can shrink r so that $h|_{B_r(z_0)} \neq 0$. Thus, if $n_0 \geq 1$, then f has a unique zero z_0 in $B_r(z_0)$; if $n_0 = 0$, then f is non-vanishing on $B_r(z_0)$. In the first case, we say f has a zero of **order** n_0 at z_0.

Now, we can conclude the proof. We let U_1 be the set of the points in U of the first case and U_2 be the set of the points in U of the second case. That is,

$$U_1 := \{z \in U : f \text{ vanishes on } B_r(z) \text{ for some } r > 0\}, \text{ and}$$
$$U_2 := \{z \in U : f \text{ is non-vanishing on } B_r(z) \setminus \{z\} \text{ for some } r > 0\}.$$

Then by the above discussion, we know $U = U_1 \cup U_2$ and $U_1 \cap U_2 = \emptyset$. Moreover, they are both open sets by their definitions, so the connectivity implies either $U = U_1$ or $U = U_2$. In the first case, we have $f = 0$ on U. In the second case, we know that all zeros are isolated. □

We remark that in the case when f has a discrete set of zeros, we can deduce that there are at most countably many zeros.[ex]

Example 3.2. We look at the function $\sin z$ at $z = 0$. We can write

$$\sin z = z - \frac{1}{3!}z^3 + \frac{1}{5!}z^5 = z\left(1 - \frac{1}{3!}z^2 + \frac{1}{5!}z^4 - \cdots\right)$$
$$=: z \cdot h(z),$$

where h is a holomorphic function with $h(0) \neq 0$. Thus, $\sin z$ has a zero at 0 of order 1. In this case, 0 is called a simple zero of $\sin z$. One can also see this by noticing $\sin'(0) = \cos(0) \neq 0$.

Remark 3.3. Another perspective to view Proposition 3.1 is that the zero set of a holomorphic function is locally either zero-dimensional or (complex) one-dimensional. It is not easy to see why this is a distinguished property, but this amazing propery is also roughly true for several complex variables. That allows us to look at "analytic subsets," which may include singularities so are more general than complex manifolds that we are going to look at in Section 5.5. For now, a related thing is that one can use zero sets of holomorphic functions to define a topology (by viewing them as closed sets). This is called the Zariski topology.

Proposition 3.1 poses strong restriction on the structure of zero sets of holomorphic functions. Its proof then leads to important consequences, including a **strong unique continuation** result.

Corollary 3.4. Let f be a holomorphic function on a connected open set U. Then f vanishes identically on U if
(1) either there exists $z_0 \in U$ such that $f^{(n)}(z_0) = 0$ for all $n \geq 0$,
(2) or f vanishes on a subset V which admits a limit point in U.

Note that the second case includes the situation when V is an open subset of U. The reason why this result is called unique continuation is as follows. Let $\widetilde{f}$ be a holomorphic function on a connected open set U and let S be an arbitrary subset in U. Then clearly $f := \widetilde{f}|_S$ is a holomorphic function on S. In this situation, we can view $\widetilde{f}$ as an **analytic continuation** of f from S to a larger set U, a process we have seen many times. What Corollary 3.4 is telling us is as follows.
(1) If we are finding an extension F of f such that all of its derivatives are prescribed at a fixed point, then $\widetilde{f}$ is the unique possible extension.
(2) If we are finding an extension F of f from a subset V admitting a limit point, then $\widetilde{f}$ is still the unique possible extension.
In either case, if there is another extension F, then the difference $F - \widetilde{f}$ satisfies the condition in Corollary 3.4 so it must vanish identically, implying $F = \widetilde{f}$ on U. Thus, Corollary 3.4 is a very strong unique continuation property of holomorphic functions.

We can further understand the behavior of a non-constant holomorphic function near a zero. It will tell us that near a zero of finite order, the map has a "multi-sheeted" structure.

Theorem 3.5. Let f be a non-constant holomorphic function on a connected open set U. Suppose $z_0 \in U$, $y_0 = f(z_0)$, and as a zero of $f - y_0$, the order of z_0 is n_0. Then we can find $r > 0$ and a holomorphic function φ on $B_r(z_0) \subseteq U$ such that

(1) $f(z) = y_0 + \varphi(z)^{n_0}$ for $z \in B_r(z_0)$, and

(2) φ' is non-vanishing in $B_r(z_0)$ and is biholomorphic onto a disc.

Note that by Proposition 3.1, the order of z_0 as a zero of $f - f(z_0)$ is finite since f is non-constant. The theorem is a consequence of the inverse function theorem 2.47. As a corollary, we have the following open mapping theorem.[ex]

Corollary 3.6 (Open mapping theorem)**.** Let f be a non-constant holomorphic function on a connected open set U. Then $f(U)$ is still an open set.

The construction in the proof of Theorem 3.5 is related to defining **logarithm** for complex numbers. We will talk about this in a more general context later, but for a given complex number, this is not a big deal. Recall that the map $t \in [0, 2\pi] \mapsto e^{it}$ is a surjective map onto the unit circle on the complex plane. Thus, given any non-zero complex number $z \in \mathbb{C} \setminus \{0\}$, we can always find $t \in [0, 2\pi]$ such that

$$z = |z| \cdot \frac{z}{|z|} = |z| \cdot e^{it},$$

so it makes sense to define

$$\log z := \log |z| + it,$$

where $\log |z|$ is the usual logarithm for positive real numbers. The problem of globally defining such a function is finding a continuous choice of $t \in [0, 2\pi]$, but this is not an issue if we only look at a point.

Proof of Theorem 3.5. By Proposition 3.1, z_0 is an isolated zero of $f - y_0$. Thus, we can find $r > 0$ such that $B_r(z_0) \subseteq U$ and write

$$f(z) = y_0 + h(z) \cdot (z - z_0)^{n_0}$$

where n_0 is the order of z_0 as a zero of $f - y_0$ and h is a non-vanishing holomorphic function on $B_r(z_0)$. Note that it remains to take the "n_0-th root" of h. This can be done by first taking the logarithm of h.

To do this, first note that h'/h is holomorphic (since h does not vanish) and hence analytic on $B_r(z_0)$. Thus, we know that h'/h admits a primitive, that is, a holomorphic function F such that $F' = h'/h$. This implies

$$\left(he^{-F}\right)' = h'e^{-F} - h \cdot e^{-F} \cdot \frac{h'}{h} = 0.$$

Thus, we can shift F so that $h = e^F$. In fact, since we have

$$he^{-F} = h(z_0)e^{-F(z_0)}$$

for all $z \in B_r(z_0)$, by considering $F_1 := F - F(z_0) + \log h(z_0)$, it implies

$$h = e^{F + \log h(z_0) - F(z_0)} = e^{F_1}.$$

Thus, (1) follows by letting $\varphi := e^{F_1/n_0} \cdot (z - z_0)$.

For (2), note that

$$\varphi'(z_0) = e^{F_1(z_0)/n_0} \neq 0,$$

so the conclusion follows from Theorem 2.47, after possibly shrinking r, and the fact that the power map $z \mapsto z^{n_0}$ is an open map. ∎