Theorem 2.14 (Goursat'stheorem, missing a point)**.** Let $f\colon U \to \mathbb{C}$ be a continuous function and $p \in U$. Suppose f is holomorphic in $U \setminus \{p\}$. If R is a (solid) closed triangle in U, then

$$\int_{\partial R} f dz = 0,$$

where ∂R is the boundary of R and we view it as a piecewise smooth curve.

In the proof, for two points A and B on the complex plane, we let $\overline{AB}$ be the segment connecting A and B. Given three points, we let ΔABC be the triangle with A, B, and C being its vertices, and its boundary is oriented in the direction from A to B, and so on.

Proof. If p is not in R, then the theorem already follows from Theorem 2.9.

Now, suppose p is one of the vertices of R, and let the other two vertices be B and C so that ΔpBC is oriented counterclockwise. Given $\varepsilon > 0$, we then choose $X \in \overline{pB}$ and $Y \in \overline{pC}$ such that the lengths of them are less than ε. We can then estimate

$$\int_{\partial R} f dz = \int_{\partial \Delta pXY} f dz + \int_{\partial \Delta XBY} f dz + \int_{\partial \Delta BCY} f dz = \int_{\partial \Delta pXY} f dz,$$

where we apply Theorem 2.9 to the triangles ΔBCY and ΔpXY. Since f is bounded on R, say bounded by $M < \infty$, we can then estimate

$$\left| \int_{\partial R} f dz \right| = \left| \int_{\partial \Delta pXY} f dz \right| \le M \cdot 4\varepsilon.$$

This is true for all ε, so the conclusion follows.

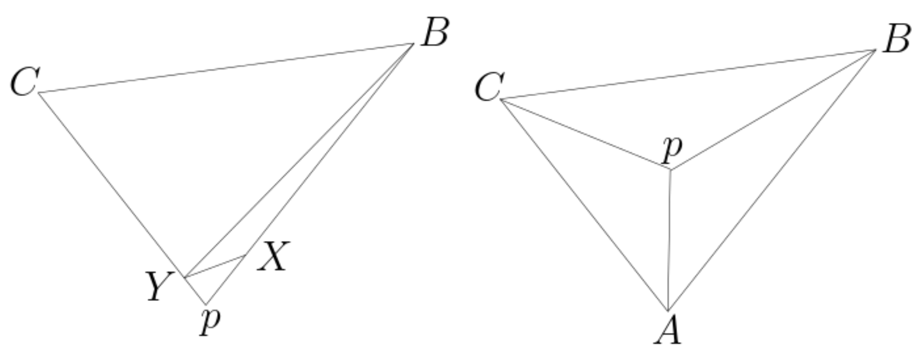

FIGURE 4. Possible choices of X and Y in the second case and the picture in the third case.

Finally, we deal with the case when $p \in R \setminus \{A, B, C\}$ where we write $R = \Delta ABC$. Then we can apply the previous case to $\Delta ABp, \Delta BCp$, and ΔCAp to get

$$\int_{\partial R} f dz = \int_{\partial \Delta ABp} f dz + \int_{\partial \Delta BCp} f dz + \int_{\partial \Delta CAp} f dz = 0 + 0 + 0 = 0.$$

This finishes the proof of the theorem. $\square$

One can imagine that Goursat's theorem basically implies the line integral of a holomorphic function along any reasonable curve should vanish, since every bounded domain can be approximated by triangles as long as its boundary is not too bad. We can make this guess rigorous by using the simple observation, Proposition 2.8. That is, we can now use Goursat's theorem to get a primitive of a holomorphic function. We do it in a convex set.

Corollary 2.15. Let $f\colon U \to \mathbb{C}$ be a continuous function and $p \in U$. Suppose f is holomorphic in $U \setminus \{p\}$. If U is a convex set, then

$$\int_\gamma f dz = 0$$

for any closed path γ in U.

Proof. We will show that f admits a primitive and apply Proposition 2.8. Given two points A and B, we let $\overline{AB}$ be the segment oriented from A to B.

Fix a point O in U. Given any $w \in U$, we define

$$F(w) := \int_{\overline{Ow}} f dz.$$

We will show that $F' = f$. To see this, fix $z_0 \in U$. Given any $\varepsilon > 0$, by the continuity of f, there exists $\delta > 0$ such that $B_\delta(z_0) \subseteq U$ and $|f(z) - f(z_0)| < \varepsilon$ for all $z \in B_\delta(z_0)$. For such a $z \in B_\delta(z_0)$, we then have

$$\begin{aligned}\left|\frac{F(z)-F(z_0)}{z-z_0} - f(z_0)\right| = \left|\frac{1}{z-z_0}\int_{\overline{zz_0}} f dz - f(z_0)\right| &= \left|\frac{1}{z-z_0}\int_{\overline{zz_0}} (f(z)-f(z_0))\, dz\right| \\ &\le \frac{1}{|z-z_0|}\cdot \varepsilon \cdot |\overline{zz_0}| = \varepsilon,\end{aligned}$$

where we apply Theorem 2.14 to the triangle formed by O, z, and z_0 and use the triangle inequality (2.2). This then shows that

$$F'(z_0) = \lim_{z\to z_0} \frac{F(z)-F(z_0)}{z-z_0} = f(z_0),$$

and the result follows from Proposition 2.8. □

We remark that the proof also works for a star-shaped set in $\mathbb{C}$. This does not make a big difference, as we are going to apply it to a local open ball to get the Cauchy integral formula.

Theorem 2.16 (Cauchy's integral formula for convex sets)**.** Let γ be a closed path in a convex open set U. If f is a holomorphic function on U, then for $w \in U \setminus \gamma$,

$$f(w)\cdot \mathrm{Ind}_\gamma(w) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-w} dz.$$

In particular, if $\overline{B}_r(a) \subseteq U$, we have

$$f(w) = \frac{1}{2\pi i}\int_{\partial B_r(a)} \frac{f(z)}{z-w} dz$$

for any $w \in B_r(a)$.

Proof. Fix $w \in U$. We consider the following function

$$g(z) := \begin{cases} \frac{f(z)-f(w)}{z-w} & \text{if } z \neq w \\ f'(w) & \text{for } z = w \end{cases}.$$

By definition, g is a continuous function and is holomorphic on $U \setminus \{w\}$. Thus, Corollary 2.15 implies $\int_\gamma g dz = 0$. Since $w \notin \gamma$, this means

$$\frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-w}dz = \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{z-w}dz = f(w)\cdot \operatorname{Ind}_\gamma(w),$$

where we use the definition of $\operatorname{Ind}_\gamma$ in Corollary 2.6. □

2.1.3. *Consequences of Cauchy's formula.* We will now see that even only stated for convex sets, Corollary 2.15 and Theorem 2.16 already have many applications, including the fact that holomorphic functions are analytic. The general case of the Cauchy integral formula will be dealt with in Theorem 2.38.

First, as mentioned many times, we can now show that holomorphic functios are analytic.

Theorem 2.17. Let f be a holomorphic function on an open set U. Then f is analytic. In fact, given any $\overline{B}_r(z_0) \subseteq U$, $f(z)$ is equal to a convergent power series

$$f(z) = \frac{1}{2\pi i}\sum_{n=0}^{\infty}\left(\int_{\partial B_r(z_0)} \frac{f(w)}{(w-z_0)^{n+1}}dw\right)(z-z_0)^n.$$

Proof. This is a direct consequence of Theorem 2.16 and Proposition 2.4. □

Before seeing more consequences, we first note that by using the expansion techniques in Proposition 2.4, we can get a formula of remainder terms in the Taylor expansion if we expand $1/(z-w)$ up to finitely many terms.[ex] We record it as a corollary of this theorem and the proof of Proposition 2.4.

Corollary 2.18. Let f be a holomorphic function on an open set U. Then given any $\overline{B}_r(z_0) \subseteq U$ and $k \in \mathbb{N}$, we can write

$$\begin{aligned} f(z) &= \sum_{n=0}^{k-1}\frac{f^{(n)}(z_0)}{n!}(z-z_0)^n + \frac{1}{2\pi i}\left(\int_{\partial B_r(z_0)} \frac{f(w)}{(w-z_0)^k(w-z)}dw\right)(z-z_0)^k \\ &= \frac{1}{2\pi i}\sum_{n=0}^{k-1}\left(\int_{\partial B_r(z_0)} \frac{f(w)}{(w-z_0)^{n+1}}dw\right)(z-z_0)^n + \frac{1}{2\pi i}\left(\int_{\partial B_r(z_0)} \frac{f(w)}{(w-z_0)^k(w-z)}dw\right)(z-z_0)^k. \end{aligned}$$

Now we know that holomorphic functions and analytic functions are the same. We derive more interesting properties of them. First, we can get a kind of converse to the Cauchy–Goursat theorem.

Theorem 2.19 (Morera's theorem)**.** Let $f\colon U \to \mathbb{C}$ be a continuous function. If for any closed triangle $R \subseteq U$, it satisfies $\int_{\partial R} f dz = 0$, then f is holomorphic.

Proof. For any $\overline{B}_r(z_0) \subseteq U$, following the proof in Corollary 2.15, we can find a converse F of f by taking $F(z) := \int_{\overline{z_0 z}} f dz$. Since $F' = f$, F is holomorphic, and hence analytic. Thus, f is holomorphic on $B_r(z_0)$. This is true for any $\overline{B}_r(z_0) \subseteq U$, so the conclusion follows. $\square$

We remark that both Theorem 2.17 and Theorem 2.19 are improvements of "regularity" of a function that is a priori only differentiable or even continuous. This is commonly seen for second-order elliptic operators in PDE theories. However, here, note that holomorphic functions satisfy a first-order equation, either the Cauchy–Riemann equation (1.7) or the $\overline{\partial}$-equation $\partial_{\overline{z}} f = 0$.

We also remark that Morera's theorem does not require convexity for U. Morera's theorem can be used to prove that a locally uniformly convergent sequence of holomorphic functions has a nice limit. This result is due to Weierstrass.

Corollary 2.20 (Weierstrass)**.** Let f_n be a sequence of holomorphic functions on an open set U. If f_n converges to a function f locally uniformly, then the limit f is also holomorphic.

Proof. Let R be a closed triangle in U. Then since each f_n is holomorphic, by the convergence on the compact set R, we have

$$\int_{\partial R} f dz = \lim_{n\to\infty} \int_{\partial R} f_n dz = \lim_{n\to\infty} 0 = 0.$$

This is true for any closed triangle in U, so Theorem 2.19 implies that f is holomorphic. $\square$

This corollary can be used to produce holomorphic functions. In fact, we can say more about the convergence in the corollary. To see this, we first see that the power series expansion gives us good estimates on high derivatives of a holomorphic function.

Proposition 2.21. Let f be a holomorphic function on an open set U. Then for any $\overline{B}_r(z_0) \subseteq U$, we have

$$|f^{(n)}(z_0)| \leq \frac{n!}{r^n} \cdot \sup_{\overline{B}_r(z_0)} |f|.$$

Proof. By the expansion in Theorem 2.17, we have

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_{\partial B_r(z_0)} \frac{f(w)}{(w - z_0)^{n+1}} dw.$$

If we parametrize $\partial B_r(z_0)$ by $\gamma(t) := z_0 + re^{it}$ for $t \in [0, 2\pi]$, we have

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_0^{2\pi} \frac{f(z_0 + re^{it})}{(re^{it})^{n+1}} \cdot ire^{it} dt = \frac{n!}{2\pi} \int_0^{2\pi} \frac{f(z_0 + re^{it})}{(re^{it})^n} dt. \tag{2.22}$$

The estimate then follows directly. $\square$

Using Proposition 2.21, we can then strengthen Corollary 2.20 to the following. As Corollary 2.20, this is also sometimes attributed to Weierstrass.

Corollary 2.23 (Weierstrass)**.** Let f_n be a sequence of holomorphic functions on an open set U. If f_n converges to a function f locally uniformly, then the limit f is holomorphic and for all $k \in \mathbb{N}$, $f_n^{(k)}$ converges to $f^{(k)}$ locally uniformly.

Proof. Let K be a compact subset in U. We can then take $r > 0$ such that $\overline{B}_r(z) \subseteq U$ for all $z \in K$, and hence $\overline{B}_r(K) := \cup_{z\in K}\overline{B}_r(z)$ is still a compact set.$^{\text{ex}}$ Then for any $z_0 \in K$, and any n, Proposition 2.21 implies

$$|f'(z_0) - f_n'(z_0)| \leq \frac{1}{r} \cdot \sup_{\overline{B}_r(K)} |f - f_n|. \tag{2.24}$$

Now we can conclude the proof. Given any ε, by the uniform convergence of f_n on $\overline{B}_r(K)$, there exists $N \in \mathbb{N}$ such that $|f - f_n| < r\varepsilon$ on $\overline{B}_r(K)$ for $n \geq N$. This and (2.24) then imply $|f' - f_n| \leq \varepsilon$ on K for $n \geq N$. This shows that f_n' converges uniformly on K. The argument for $f_n^{(k)}$ is similar. $\square$

Note that both Corollarys 2.20 and 2.23 are far from being true in the real case. The uniform convergence of the sequence itself definitely says nothing about their derivatives.

The estimate in Proposition 2.21 is very powerful. First, it gives a growth restriction for non-trivial entire holomorphic function. The proof only uses the $n = 0$ case, and will be left as an exercise in Assignment. One can also try to look at a version that depends on the high-order derivative estimates.

Corollary 2.25 (Liouville's theorem)**.** Let f be a holomorphic function on $\mathbb{C}$. If f is bounded, then f is constant.

In view of the estimate in Proposition 2.21, one can in fact prove Liouville's theorem for entire holomorphic functions with "sublinear" growth. On the other hand, based on Liouville's theorem, one can prove the fundamental theorem of algebra in a relatively simple way. We will also see this in Assignment 2, and here just list it as another corollary.

Corollary 2.26 (Fundamental theorem of algebra)**.** Every non-constant polynomial with complex coefficients has at least a complex root.

Another useful consequence of Proposition 2.21 is the **maximum principle**. This kind of result says that the maximum of a function must be achieved on its boundary. It has a stronger version, usually called the **strong maximum principle**, saying that if the maximum is achieved in the interior, then the function must be constant. We will show this stronger version for holomorphic function.

Theorem 2.27 (Strong maximum principle)**.** Let f be a holomorphic function on a **connected** open set U. If $\sup_U |f|$ is achieved at a point in U, then f is a constant function.

From its proof, we will see that one can also replace $|f|$ with Re f or Im f. Before proving the theorem, we first single out the useful formula (2.22) especially when $n = 0$.

Corollary 2.28. Let f be a holomorphic function on an open set U. Then for any $\overline{B}_r(z_0) \subseteq U$, we have

$$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{it}) dt.$$

This formula is sometimes also called Cauchy's integral formula, as a special case of its more generalized version. Note that it is also a mean-value-type equality, a property that is also true for harmonic functions.

Proof of Theorem 2.27. Suppose the supremum of $|f|$ is achieved at a point $w \in U$, say $|f(w)| = M$. We will show that

$$|f|^{-1}(M) := \{z \in U : |f(z)| = M\} = U.$$

First, if $z_0 \in |f|^{-1}(M)$, then by taking $r > 0$ such that $\overline{B}_r(z_0) \subseteq U$, Corollary 2.28 implies that for all $\varepsilon \in (0, r)$,

$$M = |f(z_0)| = \left| \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + \varepsilon e^{it}) dt \right| \leq \frac{1}{2\pi} \int_0^{2\pi} \left| f(z_0 + \varepsilon e^{it}) \right| dt.$$

In particular, this implies

$$\frac{1}{2\pi} \int_0^{2\pi} \left(\left| f\left(z_0 + \varepsilon e^{it}\right) \right| - M \right) dt \geq 0.$$

However, since $|f(z_0 + \varepsilon e^{it})| \leq M$ for all $t \in [0, 2\pi]$, the integrand is a non-positive continuous function, so it must vanish. This is true for all $\varepsilon \in (0, r)$ and $t \in [0, 2\pi]$, so we have $B_r(z_0) \subseteq |f|^{-1}(M)$. This shows that $|f|^{-1}(M)$ is an open set in U.

On the other hand, $|f|^{-1}(M)$ is automatically a closed set in U since $|f|$ is continuous. Since U is connected, we have either $|f|^{-1}(M) = \emptyset$ or $|f|^{-1}(M) = U$. We know $w \in |f|^{-1}(M)$, so $|f|^{-1}(M)$ can only be U, and the proof is complete. ☐

A particular consequence of the strong maximum principle is the weak maximum principle. It says that if f is a holomorphic function on U and K is a compact set in U, then

$$\max_K |f| = \max_{\partial K} |f|.$$

This follows almost directly from Theorem 2.27.$^{\text{ex}}$

There are a lot more applications of the maximum principle. For example, one can use it to prove the following Schwarz lemma (Assignment 2), which implies a classification result of the automorphism group of the unit disc D.

Corollary 2.29 (Schwarz lemma)**.** Let D be the unit disc and $f \colon D \to D$ be a holomorphic function. If $f(0) = 0$, then

(1) for all $z \in D$, $|f(z)| \leq |z|$;

(2) $|f'(0)| \leq 1$;

(3) if either $|f(z)| = |z|$ for some $z \in D \setminus \{0\}$ or $|f'(0)| = 1$, then $f = e^{it}$ is a rotation about 0.

As a consequence, one can derive that the **automorphism group** of D,

$$\operatorname{Aut}(D) := \{f\colon D \to D \text{ is biholomorphic}\},$$

consists of elements of the form

$$f(z) = e^{it} \cdot \frac{a - z}{1 - \overline{a}z}$$

for some $t \in \mathbb{R}$ and $a \in D$.

We mention another immediate application of Theorem 2.17 and the Cauchy integral formula (Corollary 2.28). Recall that from Lemma 1.8, we see a connection between harmonic functions and holomorphic functions. Based on this, we can obtain the same regularity property for harmonic functions from these Cauchy's theorems, and these are also left in Assignment 2.

Corollary 2.30. Let $u\colon U \to \mathbb{R}$ be a harmonic function on an open set U. Then u is (real) analytic. Also, u satisfies the mean-value equality, in the sense that given any $\overline{B}_r(z_0) \subseteq U$, we have

$$u(z_0) = \frac{1}{2\pi}\int_0^{2\pi} u\left(z_0 + re^{it}\right) dt.$$

This is another example of elliptic regularity. A priori, a harmonic function is a C^2 function that satisfies $\Delta u = 0$. This corollary tells us that they are in fact automatically analytic. More properties and applications of harmonic functions will be discussed in Section 5.3.

Finally, we mention an application of the Cauchy–Goursat theorem that is about Cauchy's original motivation. It is about using the integral formula to calculate some definite integrals. Currently, we have Corollary 2.15 that only works for convex sets, but we will see that it is enough to derive some non-trivial integrals.

Example 2.31. We will use Corollary 2.15 to calculate

$$e^{-\pi\xi^2} = \int_{-\infty}^{\infty} e^{-\pi x^2} e^{-2\pi i x\xi} dx \tag{2.32}$$

for any $\xi \in \mathbb{R}$. This means that the Fourier transform of $e^{-\pi x^2}$ is still itself. To see this, we consider a rectangular curve γ with vertices $L, L + i\xi, -L + i\xi$, and $-L$.

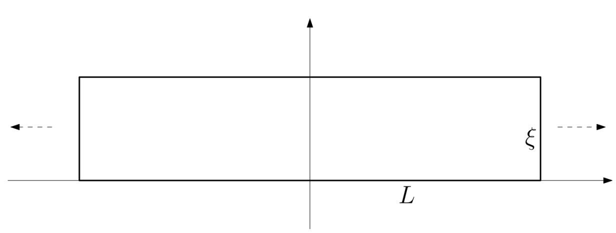

FIGURE 5. A rectangular curve whose top and bottom sides are longer and longer.

By Corollary 2.15, we know that the line integral of $f(z) = e^{-\pi z^2}$ along γ vanishes, and we can calculate the line integrals of the four sides directly. The bottom part of γ can be parametrized by

$\gamma_{\text{bot}}(t) = t$, $t \in [-L, L]$, so

$$\int_{\gamma_{\text{bot}}} f dz = \int_{-L}^{L} e^{-\pi t^2} dt \to 1 \tag{2.33}$$

as $L \to \infty$. For the right side, we can parametrize it by $\gamma_{\text{right}}(t) = L + it$ for $t \in [0, \xi]$, and we derive

$$\int_{\gamma_{\text{right}}} f dz = \int_0^{\xi} e^{-\pi(L+it)^2} i\, dt = i \int_0^{\xi} e^{-\pi\left(L^2 - t^2\right) + 2\pi L t i} dt.$$

Its magnitude can then be estimated by

$$\left| \int_{\gamma_{\text{right}}} f dz \right| = \int_0^{\xi} e^{-\pi L^2 + \pi t^2} dt \le e^{-\pi L^2} \cdot C_\xi \to 0 \tag{2.34}$$

as $L \to \infty$, where $C_\xi = \int_0^{\xi} e^{\pi t^2} dt$ is a constant depending only on ξ. One can similarly show that

$$\int_{\gamma_{\text{left}}} f dz \to 0 \text{ as } L \to \infty. \tag{2.35}$$

Finally, for the top part, we parametrize it by $\gamma_{\text{top}}(t) = -t + i\xi$ for $t \in [-L, L]$ and get

$$\begin{aligned} \int_{\gamma_{\text{top}}} f dz = \int_{-L}^{L} e^{-\pi(-t+i\xi)^2} \cdot (-1) dt &= -e^{\pi \xi^2} \int_{-L}^{L} e^{-\pi t^2 + 2\pi i \xi t} dt \\ &= -e^{\pi \xi^2} \int_{-L}^{L} e^{-\pi x^2 - 2\pi i \xi x} dx, \end{aligned} \tag{2.36}$$

where in the last step, we change the variables $x := -t$. Combining (2.33), (2.34), (2.35), and (2.36) with Corollary 2.15, we get

$$0 = \lim_{L \to \infty} \int_{\gamma} f dz = 1 - e^{\pi \xi^2} \int_{-\infty}^{\infty} e^{-\pi x^2 - 2\pi i \xi x} dx,$$

which implies (2.32).

2.2. **Global Cauchy theory.** We will develop some integral theories and prove a general form of the Cauchy integral formula in this section. The formula will be used many times in the rest of the course, and the tools developed will lead to further nice properties of holomorphic functions, as we will see at the end of this section.

To allow a general class of curves to integrate over, given an open set $U \subseteq \mathbb{C}$, we consider the **free abelian group**, $\mathcal{P}(U)$, generated by all oriented paths in U. That is, any element $\gamma \in \mathcal{P}(U)$ can be formally written as

$$\gamma = c_1 \cdot \gamma_1 + \cdots + c_N \cdot \gamma_N \tag{2.37}$$

where $c_n \in \mathbb{Z}$ and each γ_n is a path in U. An element in $\mathcal{P}(U)$ will be called a chain in U. When each path γ_n in (2.37) is a closed path, then γ will be called a **cycle** in U.

The reason that we introduce these is that it allows us to deal with line integral over a less restricted class of curves. For example, we can now talk about the integral over the union of two or more curves directly, without a limiting process. This will be helpful when we apply the Cauchy theorem in different situations. **However**, the proof of the general Cauchy theorem (Theorem 2.38) still works for a single closed path, the object we studied before, so if one is not comfortable with the notion of a general cycle, there is no problem to skip it for a while.

Given a continuous function f on U, for a chain γ of the form in (2.37), we define

$$\int_\gamma f dz := \sum_{n=1}^{N} c_n \int_{\gamma_n} f dz.$$

When γ is a cycle, we define

$$\mathrm{Ind}_\gamma(w) : = \sum_{n=1}^{N} c_n \cdot \mathrm{Ind}_{\gamma_n}(w)$$

for $w \in U \setminus \gamma$. Here, we still use the same notation γ to denote the union of all the images of γ_n's. One can then interpret Ind_γ as the "total winding number" that all γ_n's wind around w. When $\gamma = 1 \cdot \gamma_1$ is just a usual (closed) path, the line integral and the index are just the ones we used to work with.

Theorem 2.38 (Cauchy's theorem)**.** Let f be a holomorphic function on U. If γ is a cycle in U such that

$$\mathrm{Ind}_\gamma(z) = 0 \text{ for all } z \in \mathbb{C} \setminus U, \tag{2.39}$$

then we have

(1) $\int_\gamma f dz = 0$, and

(2) $f(w) \cdot \mathrm{Ind}_\gamma(w) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z-w} dz$ for $w \in U \setminus \gamma$.

Note that condition (2.39) is always true when U is a convex open set. Thus, this generalizes the special case we proved in Theorem 2.16. It is interesting to note that in the proof, we will in fact use a few consequences of the first Cauchy integral formula in Section 2.1.