1.2.3. *Power series.* We've encountered some special power series in calculus. For example, we know

$$\frac{1}{1-r} = \sum_{n\geq 0} r^n \tag{1.13}$$

when $|r| < 1$. This is also an example of writing a rational function in the form of a power series.

Remark 1.14. A direct calculation (the same as the real case) shows that (1.13) also works for complex r with $|r| < 1$. That is,

$$\frac{1}{1-z} = \sum_{n\geq 0} z^n. \tag{1.15}$$

Note, however, that the left hand side of (1.15) makes sense for any $z \neq 1$, for which it even defined a holomorphic function. Therefore, one can view $1/(1-z)$ as a "natural" extension of the power series to a much larger domain. This process is called an **analytic continuation**, and we will encounter this a few times later.

We will first see that, in general, when a power series converges, it defines a holomorphic function and its derivative can be derived easily. We recall the notation of an open ball

$$B_r(z_0) := \{z \in \mathbb{C} : |z - z_0| < r\},$$

and we sometimes write $B_r := B_r(0)$ when the center is the origin. We let $\overline{B}_r(z_0)$ be the corresponding closed ball, which is the closure of $B_r(z_0)$.

Theorem 1.16 (Abel's theorem)**.** Given a sequence $a_n \in \mathbb{C}$, consider the power series $\sum_{n=0}^{\infty} a_n z^n$. The limit

$$R := \left(\limsup_{n\to\infty} |a_n|^{1/n}\right)^{-1} \in [0, \infty] \tag{1.17}$$

is called the **radius of convergence** of the power series and satisfies the following properties.

(1) The power series converges absolutely when $z \in B_R$ and diverges when $z \in \mathbb{C} \setminus \overline{B}_R$.

(2) For $\rho < R$, the convergence is uniform on $\overline{B}_\rho$.

(3) The function

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

is a holomorphic function in B_R, and its derivative is given by

$$f'(z) = \sum_{n=0}^{\infty} n a_n z^{n-1}.$$

The formula (1.17) is Hadamard's formula. It is direct to see why the radius should be of the form, since we hope for, at least, $|a_n z^n| \leq 1$ for large enough n, compared with the geometric series

(1.13). It turns out that it is also a sufficient condition for the convergence. We also recall that given a sequence b_n, its limit superior is defined by

$$\limsup_{n\to\infty} b_n := \lim_{n\to\infty}\left(\sup_{i\geq n} b_i\right).$$

Proof of Theorem 1.16. We prove (1) and (2) first. If $z \in B_R$, then we can take $\rho \in (|z|, R)$. This means that $1/\rho > 1/R$, so there exists $N \in \mathbb{N}$ such that for $n \geq N$, $|a_n|^{1/n} < 1/\rho$, which implies[ex]

$$|a_n z^n| < \left(\frac{|z|}{\rho}\right)^n.$$

Thus, the convergence follows from the root test. For the uniform convergence, by taking $\overline{\rho} \in (\rho, R)$, we can similarly show that for $z \in \overline{B}_\rho$,

$$|a_n z^n| < \left(\frac{\rho}{\overline{\rho}}\right)^n$$

for large enough n, and the uniform convergence follows from Weierstrass' M-test.[ex]

On the other hand, if $|z| > R$, we can take $\rho \in (R, |z|)$ and since $1/\rho < 1/R$, we can find a subsequence a_{n_i} of a_n such that $|a_{n_i}|^{1/n_i} > 1/\rho$ for all i, which means

$$|a_{n_i} z^{n_i}| > \left(\frac{|z|}{\rho}\right)^{n_i}.$$

The divergence then also follows from the root test.

Finally, we prove (3). First, from the limit $\lim_{n\to\infty} n^{1/n} = 1$, it is clear that the power series $\sum_{n=0}^\infty n a_n z^{n-1}$ has the same radius of convergence. Then at a point $z_0 \in B_R$, for a nearby point $z \in B_R \setminus \{z_0\}$, we analyze the difference

$$\frac{f(z)-f(z_0)}{z-z_0} - \sum_{n=0}^\infty n a_n z_0^{n-1} = \sum_{n=0}^\infty \left(a_n \frac{z^n - z_0^n}{z-z_0} - n a_n z_0^{n-1}\right).$$

Using the relation $\frac{z^n - z_0^n}{z-z_0} = \sum_{j=0}^{n-1} z^j z_0^{n-1-j}$ twice, if we take $\rho \in (|z_0|, R)$, then for $z \in B_\rho$, we can estimate

$$\begin{aligned}\left|a_n \frac{z^n - z_0^n}{z-z_0} - n a_n z_0^{n-1}\right| &= |a_n| \left|\sum_{j=0}^{n-1}\left(z^j z_0^{n-1-j} - z_0^{n-1}\right)\right| \\ &= |a_n| \left|\sum_{j=0}^{n-1} z_0^{n-1-j}\left(z^j - z_0^j\right)\right| \\ &= |a_n| \left|\sum_{j=1}^{n-1} z_0^{n-1-j}\cdot (z-z_0)\sum_{k=0}^{j-1} z^k z_0^{j-1-k}\right| \leq |a_n|\cdot|z-z_0|\cdot\frac{1}{2}n(n-1)\rho^{n-2}.\end{aligned}$$

We only need to do this for $n \geq 2$, since the term vanishes when $n = 0$ and 1. Thus, we can estimate

$$\left|\frac{f(z)-f(z_0)}{z-z_0} - \sum_{n=0}^\infty n a_n z_0^{n-1}\right| \leq \frac{1}{2}|z-z_0|\sum_{n=2}^\infty n(n-1)|a_n|\rho^{n-2}.$$

The sum converges since it comes from the power series expansion of $f''(z)$. Thus, we see that the difference tends to zero as $z \to z_0$, and the conclusion follows. $\square$

We give two remarks here. First, we talk about power series centered at the origin. One can do the same thing for power series centered at any other point, like

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n,$$

and all the arguments are the same. Second, note that we do not talk about the case when $|z|$ is exactly the radius of convergence. That is because many things can happen in this case. One can look at the series

$$\sum_{n=0}^{\infty} z^n, \sum_{n=0}^{\infty} \frac{1}{n} z^n, \text{ and } \sum_{n=0}^{\infty} \frac{1}{n^2} z^n$$

when $|z| = 1$ and see whether they converge or not.

As a consequence of Theorem 1.16, we know that given a power series

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

with R being its radius of convergence, f is a holomorphic function on B_R, and

$$f'(z) = \sum_{n=0}^{\infty} n a_n z^{n-1}$$

is also a holomorphic function on B_R. Inductively, we can show that $f^{(n)}(z)$ is holomorphic on B_R for all $n \in \mathbb{N}$. In particular, f is a smooth function, though being analytic is a much more rigid condition than smoothness. We note that we can also integrate and get a "primitive" of f. That is,

$$F(z) = \sum_{n=0}^{\infty} \frac{a_n}{n+1} z^{n+1}$$

is holomorphic on B_R and satisfies $F' = f$.

A function that can locally be defined by a power series is called an **analytic function**. From above, we know that a complex analytic function is a holomorphic function. We will later see that any holomorphic function is also an analytic function, so basically all of them are locally represented by power series. That will then show a fundamental difference between holomorphic functions and real differentiable functions. Many interesting properties of holomorphic functions can, in fact, be derived now using power series, but we delay this until actually showing that holomorphic functions are analytic, and at that time we will have more tools to get these consequences directly. See Section 2.1.3.

At this point, using power series, we can extend many real-valued functions we know to holomorphic function.

Example 1.18. One can check that the following functions all have radii of convergence being ∞ and hence they are all holomorphic on the whole $\mathbb{C}$.

(1) $\sin z = \frac{e^{iz}-e^{-iz}}{2i} = z - \frac{1}{3!}z^3 + \frac{1}{5!}z^5 - \cdots$.
(2) $\cos z = \frac{e^{iz}+e^{-iz}}{2} = 1 - \frac{1}{2!}z^2 + \frac{1}{4!}z^4 - \cdots$.
(3) One can check directly, as in the real case, that $(e^z)' = e^z$, $(\sin z)' = \cos z$, and $(\cos z)' = \sin z$.

2. Cahchy–Goursat theory: integral representation of holomorphic functions

We will show that a holomorphic function can be locally represented by an integral given by the Cauchy integral formula. As a consequence, we will see that a holomorphic function is analytic. This will then tell us a lot of good properties of holomorphic functions.

2.1. Local results and consequences.

2.1.1. *Line integrals and indices.* We first talk about how to integrate a holomorphic function along a reasonable curve. In this course, we mostly talk about **piecewise smooth curves**, meaning that they are continuous curves and are smooth away from finitely many times.

Definition 2.1. We say $\gamma\colon [a,b] \to \mathbb{C}$ is a **path** if it is continuous and piecewise smooth, in the sense that there exist finitely many times $t_i \in [a,b]$ for $i = 0, \cdots, \ell$ such that $a = t_0 < t_1 < \cdots < t_\ell$ and $\gamma|_{[t_{i-1},t_i]}$ is a smooth curve. If the curve further satisfies $\gamma(a) = \gamma(b)$, then we say γ is a **closed path**.

For simplicity, we start from a smooth curve $\gamma\colon [a,b] \to U \subseteq \mathbb{C}$. We will sometimes abuse the notation and denote the image of γ, that is, $\gamma([a,b])$, also by γ. The image is definitely independent of the parametrization of γ, so sometimes we have to be careful. If there is a continuous function $f\colon U \to \mathbb{C}$, then the **line integral** of f along γ is defined to be

$$\int_\gamma f dz := \int_a^b f(\gamma(t)) \cdot \gamma'(t) dt.$$

Note that this definition does on depend on the parametrization of γ. That is, given any strictly increasing smooth map $\varphi\colon [c,d] \to [a,b]$ between two intervals, which leads to a reparametrization $\widetilde{\gamma} := \gamma \circ \varphi$ of γ, then one can check that

$$\int_{\gamma\circ\varphi} f dz = \int_\gamma f dz$$

based on the chain rule.[ex] Thus, this justifies the usage of γ to denote its image since the line integral only depends on the curve and its orientation. We remark that most of the curves in this course will be compact.

When γ is a piecewise smooth curve (or a path), this means that we can find a parametrization $\gamma\colon [a,b] \to U$ and finitely many times $t_i \in [a,b]$ for $i = 0, \cdots, \ell$ such that $a = t_0 < t_1 < \cdots < t_\ell$ and $\gamma|_{[t_{i-1},t_i]}$ is a smooth curve for all $i = 1, \cdots, \ell$. We can then define

$$\int_\gamma f dz := \sum_{i=1}^{\ell} \int_{\gamma|_{[t_{i-1},t_i]}} f dz,$$

which is again well-defined in the sense that it is invariant under reparametrization.

We remark that the line integral of a function along a curve can be defined for much more general functions and curves, as Riemann integrals make sense for functions with less regularity. In most situations, we deal with paths. We discuss a general notion of the length of a curve in Assignment 2, and one can similarly discuss a general notion of line integrals. We end the introduction by mentioning a common property that we will use, the triangle inequality

$$\left|\int_\gamma f\,dz\right| \le \int_\gamma |f|\,|dz|. \tag{2.2}$$

Here, the line integral of a real-valued scalar function with respect to arc length is defined by the same way. That is, if $h\colon U \to \mathbb{R}$ is continuous and $\gamma\colon [a,b] \to U$ is smooth, then

$$\int_\gamma h|dz| := \int_a^b h\left(\gamma(t)\right) \cdot \left|\gamma'(t)\right| dt,$$

which can be similarly generalized to the path case. A proof of (2.2) will be discussed in Assignment 2.

Example 2.3. We see some examples of line integrals.

(1) We do a specific example. Suppose γ is the part of the unit circle $\partial B_1(0)$ from 1 to i counterwise and let $f(z) = \mathrm{Re} z$. We can parametrize γ by $\gamma(z) = e^{it}$ for $t \in [0, \pi/2]$. Then

$$\int_\gamma f\,dz = \int_0^{\pi/2} \left(\mathrm{Re}\, e^{it}\right) \cdot ie^{it} dt = \int_0^{\pi/2} \cos t \cdot (-\sin t + i \cos t)\, dt = -\frac{1}{2} + \frac{\pi}{4} i.$$

(2) Given a path $\gamma\colon [a,b] \to U \subseteq \mathbb{C}$, it comes with an orientation. We can look at the same curve with the other orientation, denoted by

$$\begin{aligned} -\gamma\colon [-b,-a] &\to U \\ t &\mapsto \gamma(-t). \end{aligned}$$

The images of γ and $-\gamma$ are the same, and one can derive that

$$\int_{-\gamma} f\,dz = -\int_\gamma f\,dz$$

for any continuous function f on U.

(3) Given two paths $\gamma_1\colon [a,b] \to U$ and $\gamma_2\colon [b,c] \to U$, one can consider their concatenation

$$\begin{aligned} \gamma_1 + \gamma_2\colon [a,c] &\to U \\ t &\mapsto \begin{cases} \gamma_1(t) & \text{if } t \in [a,b] \\ \gamma_2(t) & \text{if } t \in [b,c] \end{cases} \end{aligned}$$

if $\gamma_1(b) = \gamma_2(b)$. The image of $\gamma_1 + \gamma_2$ is the union of those of γ_1 and γ_2, and one can show that

$$\int_{\gamma_1+\gamma_2} f\,dz = \int_{\gamma_1} f\,dz + \int_{\gamma_2} f\,dz$$

for any continuous function f on U.

The notations in (2) and (3) look natural, and we will see that they are compatible with a more general operation acting on the space of "cycles," introduced in Section 2.2.

The next example is separated, since it tells us that line integrals always provide ways to contruct more holomorphic functions.

Proposition 2.4. Let γ be a path in an open set U and $\varphi\colon U \to \mathbb{C}$ be a continuous function. Then the function

$$f(w) := \int_\gamma \frac{\varphi(z)}{z-w} dz$$

is analytic (and hence holomorphic) on $U \setminus \gamma$.

Proof. Fix an open ball $B_r(a) \Subset U \setminus \gamma$. Then for $z \in \gamma$ and $w \in B_r(a)$, we can write

$$\frac{\varphi(z)}{z-w} = \frac{\varphi(z)}{z-a} \cdot \frac{z-a}{z-w} = \frac{\varphi(z)}{z-a} \cdot \frac{1}{1-\frac{w-a}{z-a}}.$$

Note that the conditions $z \in \gamma$ and $w \in B_r(a)$ then imply

$$\left|\frac{w-a}{z-a}\right| \leq \frac{|w-a|}{r} < 1.$$

Thus, we get a power series expansion

$$\frac{\varphi(z)}{z-w} = \sum_{n=0}^\infty \frac{\varphi(z)}{(z-a)^{n+1}}(w-a)^n.$$

As we know in Abel's theorem, the convergence is uniform in any compact set in $U \setminus \gamma$, so we can take the integral and switch it with the sum, deriving

$$\int_\gamma \frac{\varphi(z)}{z-w} dz = \sum_{n=0}^\infty \left(\int_\gamma \frac{\varphi(z)}{(z-a)^{n+1}} dz\right)(w-a)^n.$$

This proves that f is locally a power series, and hence holomorphic. □

Note that the assumption on φ is very weak, only requiring its continuity. Thus, the proposition does produce many holomorphic functions. We remark that the proof extends to the case when φ is a complex-valued function on a more general space when the integration makes sense. That will need more advanced measure theory.

Since this local notion of uniform convergence will be used a lot in the rest of the course, we formally give it a definition here.

Definition 2.5. Let f_n's and f be functions on an open set U. We say f_n converges **locally uniformly** to f if given any compact set $K \subseteq U$, f_n converges uniformly to f on K.

We then look at a special example for Proposition 2.4. It is called the **index** function, and will be important later.

Corollary 2.6. Let γ be a closed path in an open set U. Then the function

$$\mathrm{Ind}_\gamma(w) := \frac{1}{2\pi i}\int_\gamma \frac{dz}{z-w}$$

is integer-valued for $w \in U \setminus \gamma$. In particular, it is constant on each component of $U \setminus \gamma$.

This index function "counts" the number of times that γ winds around w. It is thus sometimes called the winding number. A particular case is when γ is the boundary of a small ball, and we have

$$\frac{1}{2\pi i}\int_{\partial B_\varepsilon(a)}\frac{dz}{z-w}=\begin{cases}1 & \text{if } w\in B_r(a)\\ 0 & \text{if } w\in\mathbb{C}\setminus\overline{B}_r(a)\end{cases}$$

where $\partial B_\varepsilon(a)$ is oriented counterclockwise.

Proof. We let $\gamma\colon [a,b]\to U$ be a parametrization, and we have

$$\mathrm{Ind}_\gamma(w)=\frac{1}{2\pi i}\int_a^b\frac{\gamma'(t)}{\gamma(t)-w}dt$$

for $w\in U\setminus\gamma$. We then look at the function

$$h(s):=\int_a^s\frac{\gamma'(t)}{\gamma(t)-w}dt.$$

Suppose γ is smooth on $[a,b]\setminus S$ where S is a finite set. Then we know that for $s\in[a,b]\setminus S$,

$$h'(s)=\frac{\gamma'(s)}{\gamma(s)-w}.$$

This implies

$$0=\gamma'-(\gamma-w)h'=e^h\left((\gamma-w)e^{-h}\right)'$$

on $[a,b]\setminus S$. Thus, $(\gamma-w)e^{-h}$ is constant on $[a,b]\setminus S$ and hence on $[a,b]$. We then, using $h(a)=0$, get

$$(\gamma(b)-w)\,e^{-h(b)}=(\gamma(a)-w)\,e^{-h(a)}=\gamma(a)-w.$$

Then by $\gamma(a)=\gamma(b)$, we derive $e^{h(b)}=1$, so $\frac{1}{2\pi i}h(b)\in\mathbb{Z}$. The second conclusion then follows from this and Proposition 2.4 since analytic functions are continuous. □

We end this section by mentioning a common situation in which it is clear how the index function behaves. We say γ is a **simple closed path** if it is a closed path that does not have self-intersection. That is, it is of the form $\gamma\colon[a,b]\to\mathbb{C}$ with $\gamma(a)=\gamma(b)$ and $\gamma|_{[a,b)}$ being an injective map. For such a curve, it divides the plane $\mathbb{C}$ into exactly two parts. This is a special case of the Jordan curve theorem, which in fact holds in a much more general situation for continuous curves. Some related discussion will be in Assignment 2.

Theorem 2.7 (Jordan curve theorem, simple version)**.** Let γ be a simple closed path. Then $\mathbb{C}\setminus\gamma$ has exactly two components. One of them is a bounded open set on which $\mathrm{Ind}_\gamma=\pm1$ (depending on the orientation of γ); the other is an unbounded component on which $\mathrm{Ind}_\gamma=0$.

From now on, when the curve γ we are integrating along is a Jordan curve (that is, a simple closed curve), we will always implicitly assume that it is oriented so that on the bounded component it encloses, the index function Ind_γ is $+1$. For example, when we write

$$\int_{\partial B_1(0)}f\,dz,$$

unless otherwise stated, we implicitly assume the circle $\partial B_1(0)$ is oriented counterclockwise.

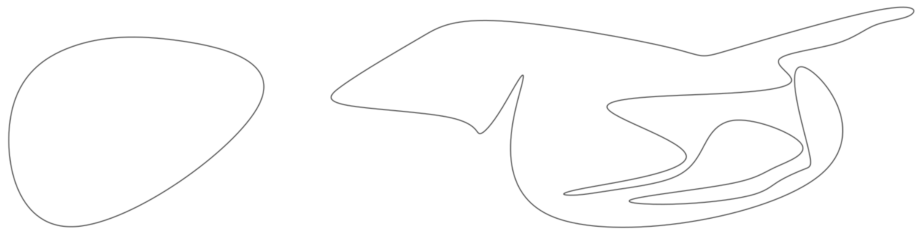

FIGURE 2. Usually it is intuitive to believe the content of the Jordan curve theorem, but even for a smooth curve, sometimes it is not completely trivial, as there are many more complicated curves than the right one that one can find on Google.

2.1.2. *Cauchy's integral formula.* The important property we are going to investigate is the dependence of the line integral on the curve. This reminds us of Green's theorem in calculus. Similar arguments tell us that if we can "integrate" the function, then the line integral should only depend on the endpoints of the curve. We know that we can integrate a convergence power series, and this is in general true if a continuous function f has a **primitive**, in the sense that $f(z) = F'(z)$ for some differentiable function F. We remark that, unlike the one-dimensional case, this is usually not a trivial condition.

Proposition 2.8. Let $f\colon U \to \mathbb{C}$ be a continuous function and γ be a closed path in U. If f has a primitive on U, then $\int_\gamma f dz = 0$.

The proof will tell us that the line integral only depends on the endpoints of the curves, and hence vanishes when the endpoints are the same.

Proof. We do the case when γ is a smooth curve and let $\gamma\colon [a, b] \to U$ be a parametrization. The piecewise smooth case is similar.

Let F be a primitive of f on U. Then by definition, we have

$$\begin{aligned}\int_\gamma f dz = \int_a^b f(\gamma(t))\,\gamma'(t)dt &= \int_a^b F'(\gamma(t))\,\gamma'(t)dt \\ &= \int_a^b (F\circ\gamma)'(t)dt = F(\gamma(b)) - F(\gamma(a)) = 0,\end{aligned}$$

where we use the chain rule and the fundamental theorem of calculus. □

This observation then directly applies to those holomorphic functions defined by power series. Our goal in this section is to show that the property is in fact correct for all holomorphic functions. To achieve, we first deal with the following special but important case when the curve is triangular.

Theorem 2.9 (Goursat's theorem)**.** Let $f\colon U \to \mathbb{C}$ be a holomorphic function. If R is a (solid) closed triangle in U, then

$$\int_{\partial R} f dz = 0,$$

where ∂R is the boundary of R and we view it as a piecewise smooth curve.

We remark that the orientation of a curve is usually important for a line integral, but in this case it doesn't matter since the integral is always zero. In the proof, all triangles are closed and solid (so not just the boundary).

Proof. We write $R = R_1^0$, and we divide it into the four triangles R_1^1, R_2^1, R_3^1, and R_4^1 of the same shapes and similar to R^0. We can inductively do this once we have R_k^i $(k = 1, \cdots, 4^i)$. For each $i \in \mathbb{N}$, we let D^i be the diameter of R_1^i (and hence R_k^i for all k). Note that $D^i = D^1/2^i \to 0$ as $i \to \infty$. We orient all R_k^i's counterclockwise.

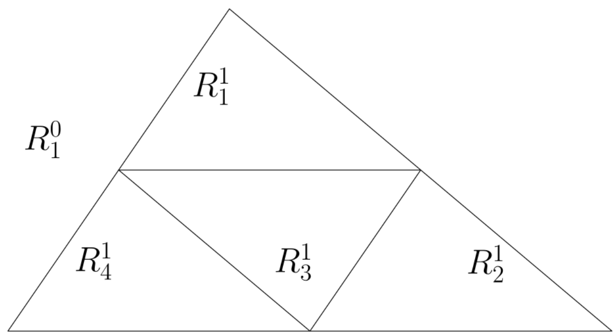

FIGURE 3. One can divide a triangle into four similar smaller triangles.

The orientation implies the equality

$$\int_{\partial R_1^0} f dz = \sum_{k=1}^{4} \int_{\partial R_k^1} f dz.$$

Thus, we know that at least one of the k's satisfies

$$\left| \int_{\partial R_1^0} f dz \right| \leq 4 \left| \int_{\partial R_k^1} f dz \right|.$$

After reordering, we may assume this $k = 1$. We can inductively, for each i, choose R_1^{i+1} such that

$$\left| \int_{\partial R_1^i} f dz \right| \leq 4 \left| \int_{\partial R_1^{i+1}} f dz \right|$$

for all $i \in \mathbb{N}$. In particular, we have

$$\left| \int_{\partial R} f dz \right| \leq 4^i \left| \int_{\partial R_1^i} f dz \right|. \tag{2.10}$$

Next, we look at the sequence $R_1^1 \supseteq R_1^2 \supseteq \cdots \supseteq R_1^i \supseteq \cdots$ of decreasing triangles. From the compactness of each R_1^i, the completeness of $\mathbb{R}$, and the fact that $D^i \to 0$ as $i \to \infty$, we know that the intersection

$$\bigcap_{i \in \mathbb{N}} R_1^i$$

consists of exactly one point. After translation, we may assume this point is the origin, that is,

$$\bigcap_{i\in\mathbb{N}} R_1^i = \{0\}\,. \tag{2.11}$$

We now look at the first order approximation of f at the origin. That is, we can write

$$f(z) = f(0) + f'(0)z + h(z)$$

where $h(z) = o(z)$ as $z \to 0$. Recall that this means $h(z)/z \to 0$ as $z \to 0$. Using the fact that $f(0) + f'(0)z$ has a primitive (since it is linear), Proposition 2.8 then implies

$$\int_{\partial R_1^i} f dz = \int_{\partial R_1^i} \left(f(0) + f'(z)z\right) dz + \int_{\partial R_1^i} h dz = \int_{\partial R_1^i} h dz. \tag{2.12}$$

We start the analysis now. Given $\varepsilon > 0$, using $h(z) = o(z)$ as $z \to 0$, we can take $\rho > 0$ such that

$$|h(z)| \le \varepsilon |z| \tag{2.13}$$

for all $z \in B_\rho$. Using (2.11), the fact that R_1^i's are all similar, and the fact that $D^i \to 0$, we can find $N \in \mathbb{N}$ such that $R_1^i \subseteq B_\rho$ for $i \ge N$. Combining this with (2.12) and (2.13), we obtain that for $i \ge N$,

$$\left|\int_{\partial R_1^i} f dz\right| \le \int_{\partial R_1^i} |h|\,|dz| \le 4D^i \cdot \varepsilon D^i$$

where we use the triangle inequality (2.2) and a rough bound of the perimeter of ∂R_1^i by $4D^i$. Putting this together with (2.10) and using $D^i = D^1/2^i$, we get

$$\left|\int_{\partial R} f dz\right| \le 4^i \left|\int_{\partial R_1^i} f dz\right| \le 4^i \cdot \left(4D^i \cdot \varepsilon D^i\right) = \varepsilon \cdot 4 \left(D^1\right)^2.$$

This is true for any $\varepsilon > 0$, so we are done. □

We remark that if f is assumed to be C^1, then one can derive Theorem 2.9 based on Green's theorem in calculus and the Cauchy–Riemann equations. Here, we do not make this assumption, although later we will see that a holomorphic function is always C^1.

Before exploring more consequences of this important theorem, we first notice that we can make it slightly more general. That is, we do not need the function to be holomorphic on the whole triangle.

Theorem 2.14 (Goursat'stheorem, missing a point)**.** Let $f\colon U \to \mathbb{C}$ be a continuous function and $p \in U$. Suppose f is holomorphic in $U \setminus \{p\}$. If R is a (solid) closed triangle in U, then

$$\int_{\partial R} f dz = 0,$$

where ∂R is the boundary of R and we view it as a piecewise smooth curve.

In the proof, for two points A and B on the complex plane, we let $\overline{AB}$ be the segment connecting A and B. Given three points, we let ΔABC be the triangle with A, B, and C being its vertices, and its boundary is oriented in the direction from A to B, and so on.