5.5. **Riemann surfaces and uniformization.** We are now going to talk about what a Riemann surface really is. Roughly speaking, such an object is a space that locally looks like an open set in the complex plane. This local structure is based on the notion of local charts, on which there is a coordinate system from the Euclidean space.

Definition 5.79. Let M be a Hausdorff[17] topological space.

(1) A **holomorphic atlas** on M is an open cover U_α ($\alpha \in A$) of M and a family of homeomorphisms $\varphi_\alpha \colon U_\alpha \to V_\alpha$ where each V_α is an open set in $\mathbb{C}$ such that for any $\alpha, \beta \in A$, the **transition map**
$$\varphi_\beta \circ \varphi_\alpha^{-1} \colon \varphi_\alpha \left(U_\alpha \cap U_\beta\right) \to \varphi_\beta \left(U_\alpha \cap U_\beta\right)$$
is a holomorphic map between subsets in $\mathbb{C}$. With U_α's and φ_α's, (M, φ_α) is called a **Riemann surface**. Each U_α is called a **holomorphic chart**.

(2) Suppose M and N are two Riemann surfaces, with holomorphic charts $\varphi_\alpha \colon U_\alpha \to V_\alpha$ ($\alpha \in A$) and $\varphi'_{\alpha'} \colon U'_{\alpha'} \to V'_{\alpha'}$ ($\alpha' \in A'$). A continuous map $f \colon M \to N$ is called a **holomorphic map** between M and N if given any $\alpha \in A$ and $\alpha' \in A'$, the map
$$\varphi'_{\alpha'} \circ f \circ \varphi_\alpha^{-1} \colon \varphi_\alpha \left(f^{-1}\left(U'_{\alpha'}\right) \cap U_\alpha\right) \to V'_{\alpha'}$$
is a holomorphic map between subsets in $\mathbb{C}$.

(3) A holomorphic map $f \colon M \to N$ between Riemann surfaces is called a **(complex) diffeomorphism** if f^{-1} exists, is continuous, and is holomorphic.

(4) If M admits two holomorphic atlases (M, φ_α) and $(M, \varphi'_{\alpha'})$, we say that these two atlases are equivalent if the identity map $(M, \varphi_\alpha) \to (M, \varphi'_{\alpha'})$ is a complex diffeomorphism An equivalent class of holomorphic atlases is called a **complex structure** on M.

Example 5.80. We have encountered many Riemann surfaces. Let's find some holomorphic structures on them.

(1) The complex plane $\mathbb{C}$ is a Riemann surface, and itself can be a holomorphic atlas with a single chart $\mathbb{C}$. Similarly, any open subset of $\mathbb{C}$ is a Riemann surface.

(2) The Riemann sphere $\hat{\mathbb{C}}$ is a Riemann surface. We take an atlas that consists of two charts $U_1 = \mathbb{C}$ and $U_2 = \hat{\mathbb{C}} \setminus \{0\}$, with
$$\varphi_1 \colon U_1 \to \mathbb{C}, z \mapsto z, \text{ and}$$
$$\varphi_2 \colon U_2 \to \mathbb{C}, z \mapsto \frac{1}{z}.$$
One can then check that the transition map is
$$\varphi_2 \circ \varphi_1^{-1} \colon \mathbb{C} \setminus \{0\} \to \mathbb{C} \setminus \{0\}, z \mapsto \frac{1}{z},$$
which is a holomorphic map. Thus, they form a holomorphic atlas and $\hat{\mathbb{C}}$ is a Riemann surface. Without specification, when we mention $\hat{\mathbb{C}}$ as a Riemann surface, we mean using the standard complex structure induced by this holomorphic atlas.

[17] Recall that this means that any two different points can be separated by disjoint open sets.

(3) A complex cylinder $\mathbb{C}/\mathbb{Z}$ is a Riemann surface (see Example 5.74(3)). It is topologically equivalent to $S^1 \times \mathbb{R}$.

(4) Another common example are complex tori. Given two linearly independent complex numbers ω_1 and ω_2, they generate a lattice

$$\Lambda := \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2.$$

Then $\mathbb{C}/\Lambda$ is a complex torus. Note that all these complex tori are homeomorphic to each other, but these homeomorphisms may not be complex diffeomorphisms.

Now after Riemann surfaces are rigorously introduced, we can talk about holomorphic maps into $\hat{\mathbb{C}}$. Suppose U is an open set in $\mathbb{C}$ and $f\colon U \to \hat{\mathbb{C}}$ is a holomorphic map between the two Riemann surfaces U and $\hat{\mathbb{C}}$ with their standard complex structures. Then by definition, this means that f is continuous and when composed with the local charts in Example 5.80(2), it is holomorphic. That is, both

$$\varphi_1 \circ f\colon \{z \in U : f(z) \neq \infty\} \to \mathbb{C}, z \mapsto f(z), \text{ and}$$
$$\varphi_2 \circ f\colon \{z \in U : f(z) \neq 0\} \to \mathbb{C}, z \mapsto \frac{1}{f(z)}$$

are holomorphic functions. By Proposition 3.15, this means that f is either a meromorphic function on U or identically ∞. One can similarly treat the case when U is an open set in $\hat{\mathbb{C}}$.$^{\text{ex}}$

5.5.1. *Model spaces and maps between them.* We now mention one of the most important results in complex analysis of one variable. It is the **uniformization theorem**, which classifies all possible simply connected Riemann surfaces up to complex diffeomorphisms.

Theorem 5.81 (Uniformization theorem)**.** Let M be a simply connected Riemann surface. Then M is complex diffeomorphic to $\mathbb{C}$, $\hat{\mathbb{C}}$, or the unit disc D.

As open sets in $\mathbb{C}$ are also naturally Riemann surfaces, one can view this as a much stronger generalization of the Riemann mapping theorem (Theorem 5.77). We will first see how we can use it to roughly get a whole classification of all Riemann surfaces. This will also be based on topological theories about covering spaces, and to implement these, we have to understand possible automorphisms of these model spaces.

Given a Riemann surface M, we define its automorphism group to be

$$\mathrm{Aut}(M) := \{f\colon M \to M \text{ is a complex diffeomorphism}\}.$$

Note that $\mathrm{Aut}(M)$ has a natural group structure by composition. We will combine our knowledge for the automorphism groups of the three model spaces with the following fact from topology about the universal cover of a topological space.

Theorem 5.82. Let M be a connected Riemann surface. Then there exists a simply connected Riemann surface $\widetilde{M}$ and a covering map $\pi\colon \widetilde{M} \to M$.

The non-trivial part is about the existence of $\widetilde{M}$ as a topological space and the covering map π. Once such a covering map is found, it can be used to pull back the complex structure from M to $\widetilde{M}$.$^{\text{ex}}$ As a corollary of Theorems 5.81 and 5.82, we obtain a classification result.

Corollary 5.83. Let M be a connected Riemann surface. Then there exist a model space $\widetilde{M} \in \left\{\mathbb{C}, \hat{\mathbb{C}}, D\right\}$ and a discrete subgroup Γ of $\operatorname{Aut}(\widetilde{M})$ such that M is complex diffeomorphic to $\widetilde{M}/\Gamma$.

Here, the discrete group Γ has to act **freely** on M so that the quotient is still a manifold. That is, any non-identity element has to have no fixed points when acting on M.[18] We will explain more in Corollary 5.86.

To say more in this direction, we quickly find out the automorphism groups of these model spaces. For automorphisms of D, recall that from Schwarz's lemma (Corollary 2.29), we know that

$$\operatorname{Aut}(D) = \left\{ f(z) = e^{it} \cdot \frac{a-z}{1-\overline{a}z} : t \in \mathbb{R} \text{ and } a \in D \right\}.$$

Next, we look at automorphisms of $\hat{\mathbb{C}}$. For the cases of both $\mathbb{C}$ and $\hat{\mathbb{C}}$, the following lemma will be useful.

Lemma 5.84. Let $f\colon \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a holomorphic map. Then f is either a rational function or identically ∞.

Proof. Based on the discussion above, $f|_{\mathbb{C}}$ is a meromorphic function or identically ∞. Suppose f is not identically ∞. Then $f^{-1}(\infty)$ is a discrete set in $\hat{\mathbb{C}}$, so it is a finite set (by the compactness of $\hat{\mathbb{C}}$.

Let $p_1, \cdots, p_k$ be the poles of f and suppose $P_1, \cdots, P_k$ are the principal parts of f at p_j's (cf. Proposition 3.15). Then we know that the function

$$h(z) := f(z) - \sum_{j=1}^{k} P_j(z)$$

is a holomorphic map between $\hat{\mathbb{C}}$ and is a holomorphic function on $\mathbb{C}$. If $h(\infty) = f(\infty) \neq \infty$, then $h|_{\mathbb{C}}$ is a bounded holomorphic function and hence a constant by Liouville's theorem (Corollary 2.25), so f is a rational function. If $h(\infty) = f(\infty) = \infty$, then h has a pole at ∞, and hence h is a polynomial so f is still a rational function. $\square$

We can use this lemma to find out automorphisms of $\hat{\mathbb{C}}$ and $\mathbb{C}$.

Corollary 5.85. An automorphism of $\hat{\mathbb{C}}$ is a Möbius transformation, and an automorphism of $\mathbb{C}$ is a non-constant linear function. That is,

$$\begin{aligned} &\operatorname{Aut}(\hat{\mathbb{C}}) = \left\{ f(z) = \frac{az+b}{cz+d} : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{C}) \right\} \text{ and} \\ &\operatorname{Aut}(\mathbb{C}) = \left\{ f(z) = az + b : a \neq 0 \right\}. \end{aligned}$$

[18] The group Γ is the fundamental group of M.

Proof. Suppose $f \in \mathrm{Aut}(\hat{\mathbb{C}})$. Then by Lemma 5.84, $f = \frac{P(z)}{Q(z)}$ where P and Q are polynomials. Assume P and Q do not have common roots. Since f is injective, P and Q must be linear, so we can write

$$f(z) = \frac{az+b}{cz+d}$$

for some $a, b, c, d \in \mathbb{C}$. As f is non-constant, we have $ad \neq bc$. Thus, after rescaling, we may assume $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{C})$.

For $\mathrm{Aut}(\mathbb{C})$, by Assignment 4, we already know that an element must be of the form $az + b$ due to its injectivity. We include an argument here for completeness. Suppose $f \in \mathrm{Aut}(\mathbb{C})$. By the open mapping theorem (Corollary 3.6), $f(B_1)$ is a non-trivial open set in $\mathbb{C}$. In particular, since f is bijective, we have $f(\mathbb{C} \setminus \overline{B}_1)$ does not contain the open set $f(B_1)$. Hence, f does not have an essential singularity at ∞ by the Casorati–Weierstrass theorem (Proposition 3.17), noting that $\mathbb{C} \setminus B_1$ is an open (punctured) neighborhood of ∞. Thus, f has either a removable singularity or a pole at ∞. For the former case, f is bounded and hence constant, which cannot happen. Thus, f has a pole at ∞, so f is a polynomial. Since f is bijective, f must be a non-constant linear function. ☐

We can now use this information about automorphism groups to get a finer classification result.

Corollary 5.86. A connected Riemann surface is complex diffeomorphic to one of the following spaces.

(1) (Elliptic type) The Riemann sphere $\hat{\mathbb{C}}$.

(2) (Parabolic type) The complex plane $\mathbb{C}$, the punctured plance C^*, and a complex torus $\mathbb{C}/\Lambda$.

(3) (Hyperbolic type) A hyperbolic surface D/G for some $G \subseteq \mathrm{Aut}(D)$.

We basically do not get anything more specific in the third case. However, we do get cleaner lists for the other two types.

Proof. Suppose $M = \hat{\mathbb{C}}/\Gamma$ for some discrete $\Gamma \leq \mathrm{Aut}(\hat{\mathbb{C}})$ that acts freely on $\hat{\mathbb{C}}$. However, in $\mathrm{Aut}(\hat{\mathbb{C}})$, all the non-identity elements have at least a fixed point (cf. Problem 10(3) in Assignment 1). Thus, the only possibility of Γ is the trivial group, so $M = \hat{\mathbb{C}}$.

Suppose $M = \mathbb{C}/\Gamma$ for some discrete $\Gamma \leq \mathrm{Aut}(\mathbb{C})$ that acts freely on $\mathbb{C}$. Since a non-identity element cannot have a fixed point, an automorphism $az + b$ must satisfy $a = 1$, and hence it is of the form $f_c(z) = z + c$. Since Γ is discrete, we can find f_{c_0} so that

$$|c_1| = \min_{f_c \in \Gamma \setminus \{0\}} |c|.$$

After a conjugation, we may assume $c_1 = 1$ (cf. Problem 10(5) in Assignment 1). We now deal with three situations.

If Γ is generated by f_1, then $\Gamma = \mathbb{Z}$ and hence $M = \mathbb{C}/\mathbb{Z}$, which is equivalent to $\mathbb{C}^*$ as observed in Example 5.74(3).

If there exists f_c that is not generated by f_1 (so that $c \in \mathbb{C} \setminus \mathbb{Z}$ with $|c| \geq 1$), then we can take such a $c = c_2$ with the smallest length (again using the discrete property) and conclude that $\Gamma = \mathbb{Z} \cdot f_1 + \mathbb{Z} \cdot f_{c_2}$. In fact, if there were another element not generated by f_1 and f_{c_2}, then Γ won't be discrete.[ex]. Thus, $M = \mathbb{C}/\left(\mathbb{Z} \cdot f_1 + \mathbb{Z} \cdot f_{c_2}\right)$ is a complex torus (cf. Example 5.80(4)). □

5.5.2. *Green's function on a Riemann surface.* We will talk about half of the proof of Theorem 5.81. The proof is based on **Green's function**. This is related to finding harmonic functions on a Riemann surface. Crucial ingredients include most of the tools we develop in Section 5.3. Many of them are local, and hence we expect that they also hold in a general Riemann surface. We first talk about what harmonicity means on M, on which sometimes we will use z to represent a holomorphic chart (when it is not too confusing, otherwise, we will still use φ).

Definition 5.87. Let M be a Riemann surface, $u\colon M \to \mathbb{R}$ be a continuous function, and $p \in M$.

(1) We say u is harmonic near p if there is an open neighborhood U of p and a holomorphic chart $\varphi\colon U \to V \subseteq \mathbb{C}$ such that $u \circ \varphi^{-1}$ is a harmonic function on V.

(2) We say $u\colon U \to [-\infty, \infty)$ is subharmonic on an open set $U \subseteq M$ if for any connected compact set $K \subseteq U$ with $\partial K \neq \emptyset$, given a harmonic function v on $\operatorname{int}K$ with $u|_{\partial K} \leq v|_{\partial K}$, it follows that $u \leq v$ on K.

Importantly, the definition of harmonicity does not depend on the choice of holomorphic charts, as this is equivalent to having a local harmonic conjugate by Lemma 1.8. Also note that here we use the maximum principle to define subharmonic functions, which makes sense as we have Theorem 5.56 in the planar case.

We need a way to produce harmonic functions. In fact, Perron's method introduced in Section 5.3.3 is general enough for a large class of families of functions. To say this, we recall the harmonic lifting introduced in Proposition 5.58. Let v be a subharmonic function on M. We say an open set $U \subseteq M$ is a **coordinate ball** if there exists a holomorphic chart $\varphi\colon U \to B$ where B is an open ball in $\mathbb{C}$. Then the **harmonic lifting** of v in U is defined as in the planar case. To be explicit, $\widetilde{v}|_{M\setminus U} = v|_{M\setminus U}$, and for $p \in U$,

$$\widetilde{v}(p) = \frac{1}{2\pi}\int_0^{2\pi} v \circ \varphi^{-1}\left(re^{it}\right) \cdot \frac{r^2 - |\varphi(p)|^2}{|re^{it} - \varphi(p)|^2}dt$$

if $B = B_r(0) \subseteq \mathbb{C}$ and $v|_{\partial U} > -\infty$. Thus, we can always find a harmonic lifting of a (non-trivial) subharmonic function in a coordinate ball.

We say a family $\mathcal{F}$ of subharmonic functions on M is a **Perron family** if it satsifies the following closure properties.

(1) If $v_1, v_2 \in \mathcal{F}$, then $\max\{v_1, v_2\} \in \mathcal{F}$.

(2) If $v \in \mathcal{F}$ and U is a coordinate ball, then the harmonic lifting of v in U is also in $\mathcal{F}$.

Then the proof of Theorem 5.61 essentially tells us the following result.

Theorem 5.88. Let $\mathcal{F}$ be a Perron family on M and consider

$$u(p) := \sup\{v(p) : v \in \mathcal{F}\}.$$