As mentioned in Remark 5.60, it is not always possible to have the nice boundary behavior as in (2). However, we can first see that (1) is in general true.

Theorem 5.61. The function $\mathcal{P}f$ constructed in (5.59) is harmonic in U. That is, (1) is correct.

The function $\mathcal{P}f$ is called the **Perron solution** on U with boundary f.[16] Note that when the Dirichlet problem (5.42) is solvable, the solution must equal $\mathcal{P}f$ by the maximum principle.

Proof. First, for each $v \in \mathcal{S}(f)$, the maximum principle (applied to $v-0$) implies $v \leq \sup_{\partial U} f \leq M$. As a result, we first know that $\mathcal{P}f \colon U \to \mathbb{R}$ is a bounded well-defined function. We write $u := \mathcal{P}f$ and will prove that it is harmonic.

Given any $B := B_r(z_0)$ with $\overline{B} \subseteq U$, by the definition of u, we can find a sequence of subharmonic functions $v_n \in \mathcal{S}(f)$ such that

$$\lim_{n\to\infty} v_n(z_0) = u(z_0). \tag{5.62}$$

We will now produce a harmonic function from these v_n's, and show that it agrees with u.

We may assume v_n is an increasing sequence (i.e., $v_1 \leq v_2 \leq \cdots$) by replacing v_n with $\max\{v_1, \cdots, v_n\}$ inductively. The new sequence is still in $\mathcal{S}(f)$ by Proposition 5.58 and still produces the same limit (5.62) at z_0. We then consider $\widetilde{v}_n$, the harmonic lifting of v_n in B. Since $v_n \leq \widetilde{v}_n$ and $\widetilde{v}_n$ is still in $\mathcal{S}(f)$, we have the (monotone) limit (5.62) is still preserved; that is,

$$\lim_{n\to\infty} \widetilde{v}_n(z_0) = u(z_0). \tag{5.63}$$

Moreover, for any $z \in B$, the maximum principle implies

$$\widetilde{v}_n(z) - \widetilde{v}_{n+1}(z) \leq \max_{\partial B}(\widetilde{v}_n - \widetilde{v}_{n+1}) = \max_{\partial B}(v_n - v_{n+1}) \leq 0 \tag{5.64}$$

by the monotonicity of v_n's. Thus, $\widetilde{v}_n|_B$ is an increasing sequence of harmonic functions that are uniformly bounded by M. Harnack's compactness theorem (Corollary 5.54) then tells us that $\widetilde{v}_n|_B$ locally uniformly converges to a harmonic function $\widetilde{v}$ on B with $\widetilde{v}(z_0) = u(z_0)$ by (5.63).

We are going to prove $\widetilde{v} = u$ on the whole ball B. Given another point $z_1 \in B$, we can do a similar process as above, with v_n's added in the construction. This allows us to compare the corresponding limit with $\widetilde{u}$. To be more specific, we can take $w_n \in \mathcal{S}(f)$ such that

$$\lim_{n\to\infty} w_n(z_1) = u(z_1) \tag{5.65}$$

and by replacing w_n with $\max\{v_j, w_j : j \leq n\}$ inductively, we get an increasing sequence w_n with the same limit (5.65) and $w_n \geq v_n$. Then again, consider $\widetilde{w}_n$, the harmonic lifting of w_n in B. The same reason implies

$$\lim_{n\to\infty} \widetilde{w}_n(z_1) = u(z_1) \tag{5.66}$$

and $\widetilde{w}_n$ is increasing. Harnack's compactness theorem then implies that $\widetilde{w}_n|_B$ locally uniformly converges to a harmonic functions $\widetilde{w}$ on B with $\widetilde{w}(z_1) = u(z_1)$ by (5.66).

Now, we compare $\widetilde{w}$ and $\widetilde{v}$. Since $w_n \geq v_n$, the same argument in (5.64) implies $\widetilde{w}_n \geq \widetilde{v}_n$ in B. In particular, $\widetilde{v}_n(z_0) \leq \widetilde{w}_n(z_0)$. Moreover, we have $\widetilde{w}_n(z_0) \leq u(z_0) = \widetilde{v}(z_0)$ by (5.63). Combining these

[16]This is a bit misleading since we do not know any actual boundary information about $\mathcal{P}f$.

two, we get $\widetilde{w}(z_0) = \widetilde{v}(z_0)$. On the other hand, $\widetilde{w}_n \geq \widetilde{v}_n$ implies $\widetilde{w} \geq \widetilde{v}$ in B. Thus, the maximum principle implies $\widetilde{w} = \widetilde{v}$. This implies

$$\widetilde{v}(z_1) = \widetilde{w}(z_1) = u(z_1)$$

by (5.66). This is true for any $z_0 \in B$, so we get $u = \widetilde{v}$ on the whole ball B. In particular, u is harmonic near z_0, and the proof is complete. □

Note that we do not use much explicit information of f in the construction of $\mathcal{P}f$ except the bound $|f| \leq M$. Thus, Theorem 5.61 is true for very arbitrary f, and we may need more to achieve (2). To understand what really happens on the boundary, we first try to find some necessary conditions for (2) to be true. To this end, we say $p_0 \in \partial U$ is a **regular boundary point** if given any bounded function $f\colon \partial U \to \mathbb{R}$ that is continuous near p_0, we have

$$\lim_{z \to p_0} (\mathcal{P}f)(z) = f(p_0).$$

That is, (2) is true at p_0 when f is continuous. We will see that this boundary regularity is related to some "barrier property" near the boundary point. We start with the following observation.

Suppose $p_0 \in \partial U$ is a regular boundary point. We look at the continuous function $f(p) := |p - p_0|$ for $p \in \partial U$ and get its Perron solution $u := \mathcal{P}f$. The boundary regularity implies

$$\lim_{z \to p_0} u(z) = f(p_0) = 0.$$

Note that by Example 5.57(2), the function $|z - p_0|$ is subharmonic and hence in $\mathcal{S}(f)$. This implies $u(z) \geq |z - p_0|$ for all $z \in U$. Hence, the function u satisfies the following properties.

(1) u is harmonic on U.

(2) $u \geq |(\cdot) - p_0|$ on U.

(3) $\lim_{z \to p_0} u(z) = 0$.

This means that u has a limit at p_0 that is strictly less than its limit sup as z tends to other boundary points. A relaxation of this necessary condition is recorded in the following definition.

Definition 5.67. A **harmonic barrier** at $p_0 \in \partial U$ is a continuous function w on $\overline{U}$ such that

(1) w is harmonic on U,

(2) $w > 0$ on $\partial U \setminus \{p_0\}$, and

(3) $w(p_0) = 0$.

We remark thata we will only need local barrier functions for the purpose of solving the Dirichlet problem. The definition above can also be localized.

Example 5.68. In a few important situations, such a harmonic barrier function exists.

(1) When $U = B_1$, one can check that $w(z) := 1 - \operatorname{Re}(\overline{p_0} z)$ is a harmonic barrier for $p_0 \in \partial B_1$.

(2) Suppose that at $p_0 \in \partial U$, there is a line segment $I \subseteq \mathbb{C} \setminus U$ with $p_0 \in \partial I$. We claim that there is a subharmonic barrier at p_0. To see this, after a rotation and a translation, we may assum $p_0 = 0$ and $I = [-1, 0]$ is contained in the real axis. Then one can check that $w(z) := \mathrm{Re}\sqrt{\frac{z}{z+1}}$ is a harmonic barrier.

(3) Based on (2), if ∂U is a union of finitely many paths (see Definition 2.1), then every point on ∂U admits a harmonic barrier.

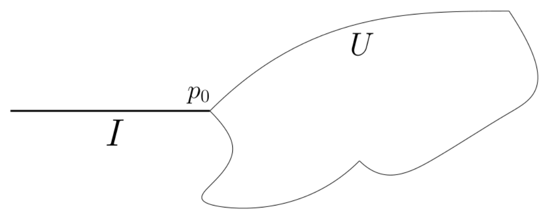

FIGURE 13. When the boundary of an open set is a path, in particular, every boundary point admits a harmonic barrier.

The existence of subharmonic barriers, as we know, is a necessary consequence of boundary regularity. It turns out that it characterizes boundary regularity and hence gives us a general condition to answer (2) and solve the Dirichlet problem.

Theorem 5.69. *Let U be a bounded, connected, open set and let $p_0 \in \partial U$. If there is a harmonic barrier at p_0, then p_0 is a regular boundary point on ∂U.*

The idea is that $\mathcal{P}f$ can be almost bounded by f near its continuous point. This boundedness can be achieved with the help of a barrier function.

Proof. Let f be a bounded function on ∂U that is continuous near p_0 and let $u := \mathcal{P}f$. Let $w\colon U \to \mathbb{R}$ be a harmonic barrier at p_0. Given any $\varepsilon > 0$, we will prove

$$\limsup_{z\to p_0} u(z) \le f(p_0) + \varepsilon \tag{5.70}$$

and

$$\liminf_{z\to p_0} u(z) \ge f(p_0) - \varepsilon. \tag{5.71}$$

Clearly, they imply $u(z) \to f(p_0)$ as $z \to p_0$, which is what we want.

To see (5.70), we first take $\delta > 0$ such that

$$|f(p) - f(p_0)| < \varepsilon \text{ for } p \in \partial U \cap B_\delta(p_0).$$

We take

$$w_0 := \min_{\partial U \setminus B_\delta(p_0)} w,$$

which is positive (by (2)). Then we consider a continuous function

$$w_1(z) := f(p_0) + \frac{M - f(p_0)}{w_0} w(z) + \varepsilon,$$

which is still harmonic in U. Note that for $p \in \partial U \cap B_\delta(p_0)$,

$$w_1(p) \geq f(p_0) + \varepsilon \geq f(p),$$

while for $p \in \partial U \setminus B_\delta(p_0)$,

$$w_1(p) \geq f(p_0) + (M - f(p_0)) + \varepsilon \geq f(p).$$

Hence, given $v \in \mathcal{S}(f)$, the maximum principle (Theorem 5.56) applied to $v - w_1$ implies $v \leq w_1$ also in U. Taking the supremum over $v \in \mathcal{S}(f)$, we get $u(z) \leq w_1(z)$. Thus,

$$\limsup_{z \to p_0} u(z) \leq \limsup_{z \to p_0} w_1(z) = w_1(p_0) = f(p_0) + \varepsilon.$$

This proves (5.70).

For (5.71), the argument is similar by considering

$$w_2(z) := f(p_0) - \frac{M + f(p_0)}{w_0} w(z) - \varepsilon$$

and proving $w_2 \leq f$ on ∂U.$^{\text{ex}}$ Hence, $w_2 \in \mathcal{S}(f)$, so $u(z) \geq w_2$, implying

$$\liminf_{z \to p_0} u(z) \geq \limsup_{z \to p_0} w_2(z) = w_2(p_0) = f(p_0) - \varepsilon.$$

This proves (5.71) and the theorem. □

Combining Theorem 5.61, Example 5.68(3), and Theorem 5.69, we can derive that the Dirichlet problem is solvable for a continuous function on ∂U, which is a union of finitely many paths. We record this as a corollary.

Corollary 5.72. When U is a bounded connected open set with ∂U being a union of finitely many paths, given a continuous function f on ∂U, there exists a unique solution to the Dirichlet problem (5.42).

Later, we will see how this is related to finding conformal maps between complicated domains. See Theorem 5.75 in Section 5.4.

5.4. **Riemann mapping theorem.** We defined a local notion of conformality in Section 5.3. Now, we look at a global version of it that gives us a way to classify spaces.

Definition 5.73. A **conformal mapping** $f\colon U \to V$ is a bijective C^1 map that is conformal at any $z \in U$. In this case, we say U is **conformally equivalent** to V.

Thus, the set of conformal mappings is a more restrictive class, instead of those maps that are conformal at any point. One can find a map that is conformal at any point but fails to be injective globally. Note that based on the inverse function theorem (Theorem 2.47), the inverse of a conformal mapping is also holomorphic, so a conformal mapping is actually the same as a biholomorphic map.

Example 5.74. We mention examples of conformal mappings that can be written down explicitly.

(1) Any automorphism $f \in \mathrm{Aut}(D)$ is a conformal equivalence between D and itself.

(2) Let $H := \{z \in \mathbb{C} : \mathrm{Im} z > 0\}$ be the upper half plane. Then the map

$$\begin{aligned} \varphi \colon H &\to D \\ z &\mapsto \frac{z-i}{z+i} \end{aligned}$$

is a conformal equivalence between H and D. Its inverse is $w \mapsto \frac{1+w}{1-w} i$.

(3) The "cylinder" $\mathbb{C}/\mathbb{Z}$, which is defined by

$$\{a + bi : a \in \mathbb{R} \text{ and } b \in \mathbb{R}/\mathbb{Z}\},$$

is conformally equivalent to the punctured plane $\mathbb{C}^* := \mathbb{C} \setminus \{0\}$ by the map $e^{2\pi i z}$.

(4) The exponential map e^z defines a conformal equivalence between the (horizontal) strip $\{z \in \mathbb{C} : \mathrm{Im} z \in (0, \pi/2)\}$ and the sector $\{z \in \mathbb{C} : \mathrm{Re} z, \mathrm{Im} z > 0\}$.

We also provide a case when a conformal equivalence does not exist.

(5) The unit disc D is not conformally equivalent to $\mathbb{C}$. Otherwise, there would exist a non-constant holomorphic $f \colon \mathbb{C} \to D$. This cannot happen because of the Liouville theorem (Corollary 2.25).

We will see that these examples are not special, based on the Riemann mapping theorem we will prove in this section. At the beginning of Section 5.3, we mention that this was one of Riemann's original motivations to study the Dirichlet problem. When he did this, he was not able to get a general result due to the lack of some topological facts. Let's now try to implement his ideas at least in a very nice and still fairly general situation.

Theorem 5.75 (Riemann mapping theorem, simple version)**.** Let U be a bounded, connected, open set in $\mathbb{C}$ such that ∂U is a union of finitely many paths. If U is simply connected, then U is conformally equivalent to the (open) unit disc D.

Proof. After a translation, we may assume $0 \in U$. Then the function $f(z) = \log |z|$ is a continuous function on ∂U. Thus, by Corollary 5.72, we can find a harmonic function $u \colon U \to \mathbb{R}$ with $u|_{\partial U} = f$. Next, since U is simply connected, by Corollary 4.5, we can find a harmonic conjugate $v \colon U \to \mathbb{R}$ of u on U. We then consider the function

$$\varphi(z) := z \cdot e^{-(u(z)+iv(z))},$$

a holomorphic function on U. Given $z_0 \in \partial U$, we have

$$\lim_{z \to z_0} |\varphi(z)| = |z_0| \cdot e^{-u(z_0)} = |z_0| \cdot e^{-\log |z_0|} = 1. \tag{5.76}$$

Thus, the maximum principle implies $|\varphi(z)| < 1$ for $z \in U$. That is, φ is a holomorphic map into the unit disc.

We now verify that φ is a conformal mapping. First, we will see φ is bijective from U to D. Notice that $\varphi(z) = 0$ if and only if $z = 0$. Moreover, given $y_0 \in D$, using (5.76), we can take $\varepsilon > 0$ so small that

$$|\varphi| > |y_0| \text{ on } \partial \varphi^{-1}(B_{1-\varepsilon}).$$

Thus, Rouché's theorem (Corollary 3.40) implies φ and $\varphi - y_0$ have the same numbers of zeros in $\varphi^{-1}(B_{1-\varepsilon})$ for any ε small enough, and hence in U. Therefore, there exists a unique $z_0 \in U$ such that $\varphi(z_0) = y_0$.

Next, note that the injectivity implies that φ' cannot vanish. This follows from the local description of holomorphic functions, see Theorem 3.5. In conclusion, φ is a conformal mapping. $\square$

We see that Theorem 5.75 deals with a large class of domains, but the method can only apply for bounded domains with nice enough boundaries. In fact, for a simply connected domain, one can show that the boundary can't be too bad, but this has not been known in Riemann's time.

The first proof of the most general form of the Riemann mapping theorem was done by Koebe, or Osgood, who proved another theorem from which the Riemann mapping theorem follows. We now state the theorem.

Theorem 5.77 (Riemann mapping theorem)**.** Let $U \subsetneq \mathbb{C}$ be a simply connected open set. Then U is conformally equivalent to the (open) unit disc D. Furthermore, given $z_0 \in U$, if we require a conformal mapping $h\colon U \to D$ to satisfy $h(z_0) = 0$ and $h'(z_0) > 0$, then such an h is unique.

We remark that all the assumptions are necessary. If U is conformally equivalent to the unit disc, then the conformal equivalence is, in particular, a homeomorphism, so U must be simply connected. Moreover, we also see that the whole complex plane $\mathbb{C}$ is not conformally equivalent to the unit disc in Example 5.74(5). Thus, Theorem 5.77 actually characterizes those domains that are conformally equivalent to D.

To motivate the proof, we reinterpret the Schwarz lemma we prove for the unit disc in Corollary 2.29. It can be viewed as a consequence of Schwarz's lemma.

Lemma 5.78. Let U be a connected open set in $\mathbb{C}$ and $z_0 \in U$. Let

$$\mathcal{F} := \{f \in \mathcal{H}(U) : |f| < 1 \text{ and } f(z_0) = 0\}.$$

If $\mathcal{F}$ contains a conformal mapping φ_0 from U to the unit disc D, then for any $f \in \mathcal{F}$,

$$|f(z)| \leq |\varphi_0(z)| \text{ for all } z \in U \text{ and } |f'(z_0)| \leq |\varphi_0'(z_0)|.$$

Moreover, if either $|f(z)| = |\varphi_0(z)|$ for some $z \in U$ or $|f'(z_0)| = |\varphi_0'(z_0)|$, then $f = e^{it}\varphi_0$ for some $t \in \mathbb{R}$.

Proof. Given $f \in \mathcal{F}$, consider the function $\widetilde{f} := f \circ \varphi_0^{-1}\colon D \to D$. Then $\widetilde{f}$ is holomorphic and satisfies $\widetilde{f}(0) = f(z_0) = 0$. Thus, Schwarz's lemma implies $|\widetilde{f}(z)| \leq |z|$ and $|\widetilde{f}'(0)| \leq 1$, with characterizations of the equality cases. After plugging in $\widetilde{f} = f \circ \varphi_0^{-1}$, this is exactly what we want. $\square$

Based on Lemma 5.78, we now fix $z_0 \in U$ and look at the class $\mathcal{F}$ defined above. If the Riemann mapping theorem is correct, then a conformal mapping $\varphi_0\colon U \to D$ will maximize the value $|f'(z_0)|$ among all f's in $\mathcal{F}$. Thus, we will try to find such an element in $\mathcal{F}$. Practically, we will restrict $\mathcal{F}$ to an even smaller class by assuming each f is injective, and that is the class that we will work on.

Proof of Theorem 5.77. Fix $z_0 \in U$. The uniqueness follows from the classification of automorphisms of D. See Corollary 2.29.

For existence, as we discussed above, we consider the family

$$\mathcal{F}_0 := \{f \in \mathcal{H}(U) : |f| < 1, f(z_0) = 0, \text{ and } f \text{ is injective}\}.$$

First, we show that $\mathcal{F}_0$ is a non-trivial family (so that later we can do an approximation argument). Since $U \subsetneq \mathbb{C}$, after a translation, we may assume $0 \notin U$. To get a map into D, we want to consider a function of the form

$$\mathrm{Inv}_{r,z_1}(z) := \frac{r}{2(z - z_1)}$$

for some $r > 0$ and $z_1 \in \mathbb{C}$, since if U is disjoint from the disc $B_r(z_1)$, then Inv_r is an injection into the unit disc D. To achieve this, we have to "squeeze" U first. (Otherwise, U may not avoid any open ball.) To this end, we note that since U is simply connected, by Corollary 4.9, we can find a holomorphic branch $s(z) = \sqrt{z}$ of the square root function on U. s is injective since s^2 is injective. Moreover, if $s(U)$ contains a ball $B_r(z_1)$ then $s(U)$ will not intersect $B_r(-z_1)$; otherwise s^2 would not be injective. We can find such $r > 0$ and $z_1 \in \mathbb{C}$ by the open mapping theorem. Thus, we are done, and

$$\mathrm{Inv}_{r,z_1} \circ s \in \mathcal{F}_0.$$

Next, by using the map $\mathrm{Inv}_{r,z_1} \circ s$, we may assume U is an open subset of D and $z_0 = 0$, and hence we are going to maximize $|f'(0)|$ for $f \in \mathcal{F}_0$. We let

$$M := \sup_{f \in \mathcal{F}_0} |f'(0)|.$$

Since $\mathrm{id}_U \in \mathcal{F}_0$, we know $M \geq 1$. Moreover, the derivative estimate, Proposition 2.21, implies that

$$|f'(0)| \leq \frac{1}{1/2} \cdot \sup_{\overline{B}_{1/2}} |f| \leq 2$$

for all $f \in \mathcal{F}_0$, so $M \leq 2$. Thus, we can now take a sequence $f_n \in \mathcal{F}_0$ such that

$$\lim_{n \to \infty} |f_n'(0)| = M.$$

Since $\mathcal{F}_0$ is a normal family by Montel's theorem (Theorem 5.13), after passing to a subsequence, we may assume f_n locally uniformly converges to a holomorphic function f on U, which satisfies $|f| \leq 1$ and $f(0) = 0$. We are now going to show that f is the desired conformal mapping by proving that f is both injective and surjective.

To see that f is injective, we note that since f_n' also converges locally uniformly (Weierstrass' theorem, Corollary 2.23), $|f'(0)| = M \geq 1$. In particular, f is not a constant map, so Hurwitz's theorem (Theorem 3.39) implies that f must be injective.

To see that f is surjective, we will use a contradiction argument and use the maximality of M. Suppose $a \in D \setminus f(U)$. A very rough idea is that the square root function will increase the derivative, to get which one needs to use the information of a missing point. We take an automorphism $A_a \in \mathrm{Aut}(D)$ that sends a to 0. Then $A_a \circ f$ is non-vanishing on a simply connected open set, so by Corollary 4.9 again, we can take a holomorphic branch $h \colon U \to D$ with

$$h^2 = A_a \circ f,$$

which means

$$h = s \circ A_a \circ f$$

where $s(z) = \sqrt{z}$ is a branch of the square root function on $A_a \circ f(U)$ (so it only makes sense after composing with $A_a \circ f$). To compare their derivatives, we now take another automorphism $A_h \in \mathrm{Aut}(D)$ that sends $h(0)$ to 0, and consider

$$\widetilde{f} := A_h \circ h = A_h \circ s \circ A_a \circ f.$$

Then $\widetilde{f}$ is also an injective map from D to D with $\widetilde{f}(0) = 0$, and if we let $t(z) := z^2$ be the squaring function, we get

$$f = A_a^{-1} \circ t \circ A_h^{-1} \circ \widetilde{f}.$$

Now, $A_a^{-1} \circ t \circ A_h^{-1}$ is a map from D to D that sends 0 to 0 and it is not a rotation (because of t), so Schwarz's lemma implies

$$\left| \left(A_a^{-1} \circ t \circ A_h^{-1} \right)' (0) \right| < 1.$$

This implies

$$M = |f'(0)| = \left| \left(A_a^{-1} \circ t \circ A_h^{-1} \right)' \left(\widetilde{f}(0) \right) \right| \cdot \left| \widetilde{f}'(0) \right| < \left| \widetilde{f}'(0) \right|.$$

Since $\widetilde{f} \in \mathcal{F}_0$, this is a contradiction. Thus, f is surjective, and the proof is done. $\square$

Note that, as in the existence part of the Dirichlet problem, in Theorem 5.77, no assertion about the boundary behavior of the constructed conformal map is mentioned. In some good situations, one can extend the map to the boundary; see, for example, [Ahl78, Chapter 6.2]. In general, Carathéodory's theorem asserts that this extension is always possible provided the boundary of U is a Jordan curve; see this page.

On the other hand, from Assignment 9, we know that there does not exist a simple version of the Riemann mapping theorem for multiply connected regions. It is, however, still possible to find a list of "standard models." One can see [Ahl78, Chapter 6.5].

5.5. **Riemann surfaces and uniformization.** We are now going to talk about what a Riemann surface really is. Roughly speaking, such an object is a space that locally looks like an open set in the complex plane. This local structure is based on the notion of local charts, on which there is a coordinate system from the Euclidean space.

Definition 5.79. Let M be a Hausdorff[17] topological space.

(1) A **holomorphic atlas** on M is an open cover U_α ($\alpha \in A$) of M and a family of homeomorphisms $\varphi_\alpha\colon U_\alpha \to V_\alpha$ where each V_α is an open set in $\mathbb{C}$ such that for any $\alpha, \beta \in A$, the **transition map**

$$\varphi_\beta \circ \varphi_\alpha^{-1}\colon \varphi_\alpha\left(U_\alpha \cap U_\beta\right) \to \varphi_\beta\left(U_\alpha \cap U_\beta\right)$$

is a holomorphic map between subsets in $\mathbb{C}$. With U_α's and φ_α's, (M, φ_α) is called a **Riemann surface**. Each U_α is called a **holomorphic chart**.

(2) Suppose M and N are two Riemann surfaces, with holomorphic charts $\varphi_\alpha\colon U_\alpha \to V_\alpha$ ($\alpha \in A$) and $\varphi'_{\alpha'}\colon U'_{\alpha'} \to V'_{\alpha'}$ ($\alpha' \in A'$). A continuous map $f\colon M \to N$ is called a **holomorphic map** between M and N if given any $\alpha \in A$ and $\alpha' \in A'$, the map

$$\varphi'_{\alpha'} \circ f \circ \varphi_\alpha\colon \varphi_\alpha\left(f^{-1}\left(U'_{\alpha'}\right) \cap U_\alpha\right) \to V'_{\alpha'}$$

is a holomorphic map between subsets in $\mathbb{C}$.

(3) A holomorphic map $f\colon M \to N$ between Riemann surfaces is called a **(complex) diffeomorphism** if f^{-1} exists, is continuous, and is holomorphic.

(4) If M admits two holomorphic atlases (M, φ_α) and $(M, \varphi'_{\alpha'})$, we say that these two atlases are equivalent if the identity map $(M, \varphi_\alpha) \to (M, \varphi'_{\alpha'})$ is a complex diffeomorphism An equivalent class of holomorphic atlases is called a **complex structure** on M.

Example 5.80. We have encountered many Riemann surfaces. Let's find some holomorphic structures on them.

(1) The complex plane $\mathbb{C}$ is a Riemann surface, and itself can be a holomorphic atlas with a single chart $\mathbb{C}$. Similarly, any open subset of $\mathbb{C}$ is a Riemann surface.

(2) The Riemann sphere $\hat{\mathbb{C}}$ is a Riemann surface. We take an atlas that consists of two charts $U_1 = \mathbb{C}$ and $U_2 = \hat{\mathbb{C}} \setminus \{0\}$, with

$$\varphi_1\colon U_1 \to \mathbb{C}, z \mapsto z, \text{ and}$$
$$\varphi_2\colon U_2 \to \mathbb{C}, z \mapsto \frac{1}{z}.$$

One can then check that the transition map is

$$\varphi_2 \circ \varphi_1^{-1}\colon \mathbb{C} \setminus \{0\} \to \mathbb{C} \setminus \{0\}, z \mapsto \frac{1}{z},$$

which is a holomorphic map. Thus, they form a holomorphic atlas and $\hat{\mathbb{C}}$ is a Riemann surface. Without specification, when we mention $\hat{\mathbb{C}}$ as a Riemann surface, we mean using the standard complex structure induced by this holomorphic atlas.

[17]Recall that this means that any two different points can be separated by disjoint open sets.