5.3. **Conformal maps and harmonic functions.** We will now turn to the topics about conformal maps and harmonic functions.

5.3.1. *Conformal maps and harmonic functions.* A map $f\colon U \to \mathbb{C}$ is called **conformal** if it "preserves the angles." To be precise, let $f\colon U \to \mathbb{C}$ be a C^1 map on $U \subseteq \mathbb{C}$. As before, we can write

$$f(z) = f(x,y) = u(x,y) + iv(x,y)$$

by viewing $z = x + iy \in U$. Then the tangents in the x and y directions will be sent to

$$(1,0) \mapsto \left(\frac{\partial u}{\partial x}, \frac{\partial v}{\partial x}\right), \text{ and}$$
$$(0,1) \mapsto \left(\frac{\partial u}{\partial y}, \frac{\partial v}{\partial y}\right).$$

Thus, if we let

$$J := \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}$$

be the Jacobian matrix of f, then the conformality condition means

$$\frac{\langle v, w\rangle}{|v| \cdot |w|} = \frac{\langle Jv, Jw\rangle}{|Jv| \cdot |Jw|} \tag{5.40}$$

given any $v, w \in \mathbb{R}^2 \setminus \{0\}$. By this, we can see that holomorphic functions are included when their derivatives do not vanish.

Proposition 5.41. If $f\colon U \to \mathbb{C}$ is holomorphic at $z_0 \in U$ with $f'(z_0) \neq 0$, then f is conformal at z_0.

Proof. By the Cauchy–Riemann equation (1.7), the Jacobian matrix of f at z_0 is

$$J = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix} = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ -\frac{\partial u}{\partial y} & \frac{\partial u}{\partial x} \end{pmatrix},$$

which is a rotation matrix when $f'(z_0) \neq 0$. Thus, (5.40) holds. ☐

One can similarly show that the converse statement is also true. That is, a conformal map is in fact a holomorphic function with a non-vanishing derivative.[ex]

When Riemann originally studied conformal maps, he was trying to find conformal maps using harmonic functions. For example, if one is able to solve the system

$$\begin{cases} \Delta u = 0 & \text{in } U \\ u = f & \text{on } \partial U \end{cases} \tag{5.42}$$

for a domain U with a suitably given f on ∂U, then it is possible to relate it back to finding conformal maps into a disc. We will go back to this in Section 5.4.

Besides the usage related to conformal map, the boundary value problem (5.42) is also important in many other geometric questions, and even physics. The problem of solving the boundary value problem (5.42) is called the **Dirichlet problem**. Our goal in the following two sections is to see whether we can solve it for a general class of domains and boundary data. On the other hand, we will also see that (5.42) is not always solvable, depending on the given U and f. See Remark 5.60 and Assignment 8.

Before studying harmonic functions in a systematic way, we can already use the most fundamental property of them to solve the Dirichlet problem for a disc. Recall that harmonic functions satisfy the mean-value equality based on Corollary 2.30. One can also prove this fact by starting from the definition ($\Delta u = 0$) directly.$^{\text{ex}}$ In fact, we can get a slightly stronger version than Corollary 2.30 by approximating the boundary values. We list it as a lemma here.

Lemma 5.43. Let $u\colon \overline{B}_1 \to \mathbb{R}$ be a continuous function that is harmonic in B_1. Then

$$u(0) = \frac{1}{2\pi}\int_0^{2\pi} u\left(e^{it}\right) dt.$$

The Dirichlet problem for B_1 is then related to generalizing the mean-value equality above to an arbitrary point in B_1. To achieve, we recall again that in Assignment 1, we study some good properties of the function

$$h_{z_0}(z) := \frac{z_0 - z}{1 - \overline{z_0} z},$$

which, especially sends 0 to z_0. Thus, given a continuous $u\colon \overline{B}_1 \to \mathbb{R}$ that is harmonic inside B_1, for any $z_0 \in B_1$, using h_{z_0}, we can consider

$$\widetilde{u}(w) := u\left(h_{z_0}(w)\right) = u\left(\frac{z_0 - w}{1 - \overline{z}_0 w}\right),$$

which is still harmonic in B_1.$^{\text{ex}}$ We can then calculate

$$u(z_0) = \widetilde{u}(0) = \frac{1}{2\pi}\int_0^{2\pi} \widetilde{u}\left(e^{it}\right) dt = \frac{1}{2\pi}\int_0^{2\pi} u\left(\frac{z_0 - e^{it}}{1 - \overline{z}_0 e^{it}}\right) dt. \tag{5.44}$$

Note that $\frac{z_0 - e^{it}}{1-\overline{z}_0 e^{it}}$ is still on the unit circle by the property of h_{z_0}, and by writting $\frac{z_0 - e^{it}}{1-\overline{z}_0 e^{it}} =: e^{is}$, we can get

$$e^{it} = h_{z_0}\left(e^{is}\right) = \frac{z_0 - e^{is}}{1 - \overline{z}_0 e^{is}}.$$

Thus, differentiating leads to

$$e^{it} dt = \left(\frac{-e^{is}}{1 - \overline{z}_0 e^{is}} + \frac{\left(z_0 - e^{is}\right)\cdot \overline{z}_0 e^{is}}{\left(1 - \overline{z}_0 e^{is}\right)^2}\right) ds.$$

Dividing both sides by $e^{it} = \frac{z_0 - e^{is}}{1-\overline{z}_0 e^{is}}$, we get

$$dt = \left(\frac{e^{is}}{e^{is} - z_0} + \frac{\overline{z}_0 e^{is}}{1 - \overline{z}_0 e^{is}}\right) ds = \frac{1 - |z_0|^2}{|e^{is} - z_0|^2} ds.$$

Putting this back to (5.44), we derive

$$u(z_0) = \frac{1}{2\pi}\int_0^{2\pi} u\left(e^{is}\right) \frac{1 - |z_0|^2}{|e^{is} - z_0|^2} ds. \tag{5.45}$$

That is, the value of u in B_1 is completely determined by its value on ∂B_1. The equality (5.45) is called the **Poisson formula**.

On the other hand, formula (5.45) actually can be used to produce harmonic functions with a prescribed continuous boundary datum. Thus, it provides a solution to the Dirichlet problem for B_1.

Proposition 5.46. Let $v\colon \partial B_1 \to \mathbb{R}$ be a continuous function on the unit circle. Then if we define

$$u(z) := \begin{cases} \frac{1}{2\pi}\int_0^{2\pi} v\left(e^{is}\right)\frac{1-|z|^2}{|e^{is}-z|^2}ds & \text{if } z \in B_1 \\ v(z) & \text{if } z \in \partial B_1 \end{cases},$$

then u solves (5.42) for $U = B_1$ and $f = v$.

Proof. It is relatively direct to see the harmonicity in the interior. In fact, we can write$^{\text{ex}}$

$$\frac{1-|z|^2}{|e^{is}-z|^2} = \mathrm{Re}\left(\frac{e^{is}+z}{e^{is}-z}\right),$$

so, on B_1, u is the real part of the holomorphic function

$$\frac{1}{2\pi i}\int_{\partial B_1} v(w)\cdot\frac{w+z}{w-z}\cdot\frac{1}{w}dw. \tag{5.47}$$

This part in fact does not require the continuity of v.

For the boundary continuity, we do need the continuity of v. For simplicity, we write

$$P_{v_0}(z) := \frac{1}{2\pi}\int_0^{2\pi} v_0\left(e^{is}\right)\frac{1-|z|^2}{|e^{is}-z|^2}ds$$

be the Poisson integral constructed from v_0, which makes sense for any integrable function v_0 and $z \in B_1$. Thus, viewing P as a map from the space of integrable functions on ∂B_1 to the space of harmonic functions on B_1, we can derive some basic properties of P, including the linearity and the preservation of non-negativity. Based on this, to prove the boundary continuity, that is,

$$\lim_{z\to z_0} P_v(z) = v(z_0)$$

for any $z_0 \in \partial B_1$, we may assume $v(z_0) = 0$ by possibly adding a constant to v.

Now we start to estimate and fix an arbitrarily small number $\varepsilon > 0$. By the continuity of v, we can take $\delta > 0$ such that $|v| < \varepsilon/2$ on $\partial B_1 \cap B_\delta(z_0)$. For the part in the arc $C_1 := \partial B_1 \cap B_\delta(z_0)$, we will make use of the smallness of v, and on the rest $C_2 := \partial B_1 \setminus B_\delta(z_0)$, we will estimate directly. We consider

$$v_i := v \cdot \mathrm{id}_{C_i}$$

where id_{C_i} is the characteristic function of C_i. The pointwise bound $|v_1| < \varepsilon/2$ then implies

$$|P_{v_1}| < \frac{\varepsilon}{2} \text{ on } B_1. \tag{5.48}$$

On the other hand, since v_2 vanishes on C_1, if we write $z_0 = e^{is_0}$ and

$$C_1 = \left\{e^{is} : s \in (s_0 - d, s_0 + d)\right\}$$

for some $d > 0$, we have, when z is close to z_0,

$$\begin{aligned} |P_{v_2}(z)| = \left| \frac{1}{2\pi} \int_0^{2\pi} v_2\left(e^{is}\right) \frac{1-|z|^2}{|e^{is}-z|^2} ds \right| &= \left| \frac{1}{2\pi} \int_{s_0+d}^{s_0-d+2\pi} v_2\left(e^{is}\right) \frac{1-|z|^2}{|e^{is}-z|^2} ds \right| \\ &\le \frac{2\pi - 2d}{2\pi} \cdot \sup_{\partial B_1} |v| \cdot \frac{\left|1-|z|^2\right|}{2C_d^2} \end{aligned}$$

where $C_d^2 = 2 - 2\cos(2d) = |e^{id} - 1|$ is the minimum of $|e^{is} - z_0|$ on C_2. Hence, this implies that

$$|P_{v_2}(z)| < \frac{\varepsilon}{2} \tag{5.49}$$

when $z \to z_0$. Combining (5.48) and (5.49), we get the continuity of u at z_0. $\square$

The question we want to answer by the end of next section is to what extend we can solve (5.42). We will use Perron's method of approximation by subharmonic functions to achieve in some general situations. We will first study properties of harmonic functions in this section, and then introduce subharmonic functions and Perron's method in Section 5.3.3.

First, we can use the mean-value property (the equality in Lemma 5.43) to derive the maximum principle. Note that this is also what we've done for holomorphic functions. What we can do more here is, we can instead start from the mean-value equality. For a continuous function $u\colon U \to \mathbb{R}$ on an open set $U \subseteq \mathbb{C}$, we say that u satisfies the mean-value property if for any $\overline{B}_r(z_0) \subseteq U$, it satisfies

$$u(z_0) = \frac{1}{2\pi} \int_0^{2\pi} u\left(z_0 + re^{it}\right) dt.$$

We know that harmonic functions satisfy the mean-value property by Lemma 5.43. As in Theorem 2.27, it implies the strong maximum principle, and similarly the strong minimum principle.

Lemma 5.50. Suppose U is a connected open set in $\mathbb{C}$ and $u\colon U \to \mathbb{R}$ satisfies the mean-value property. If $\sup_U u$ or $\inf_U u$ is achieved at a point in U, then u is a constant function.

The proof is the same as that of Theorem 2.27 and we do not repeat it here. As a result, we can see that the mean-value property in fact characterizes harmonicity. Note that the mean-value property does not assume C^2 regularity.

Corollary 5.51. Suppose U is an open set in $\mathbb{C}$ and $u\colon U \to \mathbb{R}$ satisfies the mean-value property. Then u is harmonic.

Proof. We fix a ball $\overline{B}_r(z_0) \subseteq U$. Since u is continuous, we can use it to produce a harmonic function by the Dirichlet problem for B_1. That is, Proposition 5.46 implies that there is a continuous function $\widetilde{u}\colon \overline{B}_r(z_0) \to \mathbb{R}$ which is harmonic in $B_r(z_0)$ and equals v on $\partial B_r(z_0)$. Thus, $u - v$ is identically zero on $\partial B_r(z_0)$ and the maximum principle (Lemma 5.50) implies $u - v$ is identically zero in $B_r(z_0)$. This implies that $u = v$, and hence u is harmonic near z_0, and this argument applies to any z_0. $\square$

Thus, functions satisfying the mean-value property are automatically harmonic. Note that the regularity then suddenly becomes much stronger (from C^1 to analytic).

Finally, we prove an important estimate for harmonic functions. It is called the Harnack inequality. It allows us to compare harmonic functions at different points.

Theorem 5.52 (Harnack inequality)**.** Let $u\colon U \to \mathbb{R}$ be a positive harmonic function. Given a compact set $K \subseteq U$, there exists $C_K < \infty$ such that

$$\sup_K u \le C_K \cdot \inf_K u.$$

The crucial point is that the constant C_K does not depend on the function itself. Thus, the estimate can be applied to a large class of harmonic functions, and thus can be useful when we want to get a uniform bound from a family or a sequence of harmonic functions.

Proof. We first deal with the case when K is a ball. Recall that we have the Poisson formula (5.45), so for any $z \in B_1$, we have

$$u(z) = \frac{1}{2\pi}\int_0^{2\pi} u\left(e^{it}\right)\frac{1-|z|^2}{|e^{it}-z|^2}dt.$$

To estimate the integrand, for $r \in (0,1)$ and $z \in B_r$, the triangle inequality implies

$$1 - r \le |e^{it} - z| \le 1 + r.$$

Thus,

$$\frac{1-r}{1+r} \le \frac{1-r^2}{(1+r)^2} \le \frac{1-|z|^2}{|e^{it}-z|^2} \le \frac{1-r^2}{(1-r)^2} \le \frac{1+r}{1-r}.$$

This and the usual mean-value property then imply

$$\frac{1-r}{1+r}u(0) \le u(z) \le \frac{1+r}{1-r}u(0).$$

This is true for all $z \in B_r$, and hence we get

$$\sup_{B_r} u \le \frac{1+r}{1-r}u(0) = \left(\frac{1+r}{1-r}\right)^2 \cdot \frac{1-r}{1+r}u(0) \le \left(\frac{1+r}{1-r}\right)^2 \inf_{B_r} u \tag{5.53}$$

for any $r \in (0,1)$.

We sketch the idea for the case of general compact sets. When we are given a compact $K \subseteq U$, we can assume K is connected by looking at each of its connected components. Then we can find z_M and z_m such that

$$u(z_M) = \sup_K u \text{ and } u(z_m) = \inf_K u.$$

Take a curve γ joining z_M and z_m, and cover the curve by finitely many balls. This curve can be bounded by at most N balls with radius $r < d_{\mathrm{Euc}}(K, \partial U)$, where N only depends on K and U. The inequality (5.53) can be applied to each of the balls, and combining them leads to the desired estimate on K. ∎

Harnack's estimate, as expected, has many applications. We end the section by mentioning a direct application about monotone sequences of harmonic functions.

Corollary 5.54. Let U be a connected open set in $\mathbb{C}$ and suppose u_n is a sequence of harmonic functions on U. Suppose u_n is an increasing sequence, in the sense that for $u_1 \le u_2 \le \cdots$. Then u_n either locally uniformly converges to a harmonic function on U, or locally uniformly converges to ∞.

Proof. We may assume $u_n > 0$; otherwise, we can replace u_n with $u_n - u_1 + 1$. Fix $z_0 \in U$ and take $\delta > 0$ so small that $\overline{B}_\delta(z_0) \subseteq U$. We look at the limit $L := \lim_{n\to\infty} u_n(z_0)$, which exists by the monotonicity.

If $L < \infty$, then for n large, we have $u_n(z_0) \le 2L$. We can apply Harnack's estimate and get a constant $C < \infty$ (independent of n) such that for all n large as above,

$$\sup_{\overline{B}_\delta(z_0)} u_n \le C \cdot \inf_{\overline{B}_\delta(z_0)} u_n \le 2CL.$$

This finite bound and the monotonicity then imply u_n converges to a limit u. The monotone convergence theorem implies that u still satisfies the mean-value property, so u is harmonic in $B_\delta(z_0)$.

If $L = \infty$, then given any $M < \infty$, we have $u_n \ge M$ for n large. We can apply Harnack's estimate and get a constant $C' > 0$ (independent of n) such that for all n large as above,

$$\inf_{\overline{B}_\delta(z_0)} u_n \ge C' \sup_{\overline{B}_\delta(z_0)} u_n \ge C'M.$$

This implies $u_n \to \infty$ uniformly on $\overline{B}_\delta(z_0)$.

By the two cases above, we know that the two sets

$$\left\{z \in U : \lim_{n\to\infty} u_n(z) < \infty\right\} \text{ and } \left\{z \in U : \lim_{n\to\infty} u_n(z) = \infty\right\}$$

are both open. The connectivity of U then forces one of them is empty, and hence exactly one of them holds for all $z \in U$. ☐

We remark that by assuming the functions in Corollary 5.54 are positive, one can drop the monotonicity assumption by using the Harnack estimate directly.

5.3.2. *Subharmonic functions.* The goal of this section is to study the Dirichlet problem (5.42) for a general domain.

The method we will study was first employed by Perron, and it is based on the approximation idea using "subharmonic functions," which are subsolutions to the Laplace equation. In the one-dimensionl case, harmonic functions are just straight lines, and then subsolutions to the Laplace equation are exactly convex functions (functions with $u'' \ge 0$). Perron's method in the (simple) one-dimensional case can be illustrated in Figure 12, which motivates the following definition.

Definition 5.55. Let U be an open set and $v\colon U \to \mathbb{R}$ be a continuous function. We say v is **subharmonic** if for any $\overline{B}_r(z_0) \subseteq U$, it satisfies the mean-value inequality

$$v(z_0) \le \frac{1}{2\pi} \int_0^{2\pi} v\left(z_0 + re^{it}\right) dt.$$

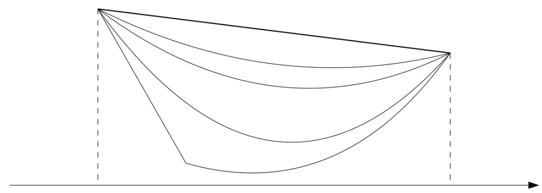

FIGURE 12. When the boundary points are fixed, the straight line, the graph of a harmonic function, maximizes the value at each point among all convex (subharmonic) graphs.

Clearly, different from the mean-value equality, subharmonic functions are much less rigid, and we do not expect the regularity can be upgraded as in the harmonic case. We will still see in Assignment 8 that subharmonic functions can be related to the Laplace equation, as mentioned above, when they are assumed to be C^2. In general, we can use the maximum principle to characterize subharmonicity.

Theorem 5.56. Let v be a continuous function on a connected open set $U \subseteq \mathbb{C}$. Then v is subharmonic if and only if for any connected subdomain $U' \subseteq U$ and any harmonic function u on U', $v - u$ satisfies the strong maximum principle on U'; that is, if $v - u$ achieves its maximum at an interior point $p \in U'$, then it must be constant on U'.

Proof. The argumet for the "only if" direction is the same as the proof of Theorem 2.27, and we go through it again as we did not repeat it for Lemma 5.50. Suppose the maximum M of $v - u$ is achieved in U'. We then consider

$$(v-u)^{-1}(M) = \left\{ z \in U' : v(z) - u(z) = M \right\}.$$

Since v satisfies the mean-value inequality and u satisfies the mean-value equality, for any $z_0 \in (v-u)^{-1}(M)$, we can take $r > 0$ such that $\overline{B}_r(z_0) \subseteq U'$ and hence for any $\varepsilon \in (0, r)$,

$$M = v(z_0) - u(z_0) \leq \frac{1}{2\pi} \int_0^{2\pi} (v-u)\left(z_0 + \varepsilon e^{it}\right).$$

The maximality of M then implies $v - u$ is constantly M in the ball. This implies $(v-u)^{-1}(M)$ is open, and hence $(v-u)^{-1}(M) = U'$.

For the "if" part, we will use the Dirichlet problem for balls. Fix a ball $\overline{B}_r(z_0) \subseteq U$. Since v is continuous, by Proposition 5.46, there is a continuous function $u \colon \overline{B}_r(z_0) \to \mathbb{R}$ which is harmonic in $B_r(z_0)$. The assumption implies that the maximum of $v - u$ is achieved on the boundary $\partial B_r(z_0)$. However, we know that $v - u = 0$ on the boundary, so we have $v - u \leq 0$ on the whole ball $B_r(z_0)$. Using the mean-value equality for u, we get

$$v(z_0) \leq u(z_0) = \frac{1}{2\pi} \int_0^{2\pi} u\left(z_0 + re^{it}\right) dt = \frac{1}{2\pi} \int_0^{2\pi} v\left(z_0 + re^{it}\right) dt,$$

and the proof is done. $\square$

Example 5.57. Although subharmonic functions do not have much rigidity, this means that it may be easier to construct them. We mention two main sources here.

(1) Any harmonic function is obviously a subharmonic function.

(2) If $f(z)$ is holomorphic and non-vanishing, then $|f(z)|$ is subharmonic and $\log|f(z)|$ is harmonic. Thus, we have a large class of examples from holomorphic functions. See Assignment 8.

Next, we list some basic constructions for subharmonic functions.

Proposition 5.58. Let v and v_1 be subharmonic functions on an open set $U \subseteq \mathbb{C}$.

(1) kv $(k \geq 0)$ is subharmonic.

(2) $\max\{v, v_1\}$ is subharmonic.

(3) Given $\overline{B}_r(z_0) \subseteq U$, replacing the values in $B_r(z_0)$ with the Poisson integral leads to another subharmonic function. That is, the function

$$v_2(z) := \begin{cases} v(z) & \text{if } z \in U \setminus B_r(z_0) \\ \frac{1}{2\pi}\int_0^{2\pi} v\left(z_0 + re^{it}\right) \frac{r^2 - |z-z_0|^2}{|re^{it} - (z-z_0)|^2} dt & \text{if } z \in B_r(z_0) \end{cases}$$

is subharmonic on U. This is called the **harmonic lifting** of v in $B_r(z_0)$.

Proof. It is not hard to see from the definition of the mean-value inequality that kv is subharmonic for $k \geq 0$.

For $s := \max\{v, v_1\}$, we look at a connected subdomain $U' \subseteq U$ and a harmonic function u on U. Suppose $s - u$ achieves its maximum at $z_0 \in U'$, say $s(z_0) = v(z_0)$. Then for any $z \in U'$,

$$v(z) - u(z) \leq s(z) - u(z) \leq s(z_0) - u(z_0) = v(z_0) - u(z_0).$$

Thus, $v - u$ also achieves its maximum at $z_0 \in U'$, so $v - u$ is constant on U'. This implies that the inequalities above are all equalities, and hence $s - u = v - u$ is also constant.

For v_2, first we know that $v \leq v_2$. We again look at a connected subdomain $U' \subseteq U$ and a harmonic function u on U'. Suppose $v_2 - u$ achieves its maximum at $z_0 \in U'$. We consider the two situations.

If $z_0 \in U \setminus B_r(z_0)$, then $v_2(z_0) = v(z_0)$, and hence for any $z \in U'$,

$$v(z) - u(z) \leq v_2(z) - u(z) \leq v_2(z_0) - u(z_0) = v(z_0) - u(z_0).$$

Thus, $v - u$ also achieves its maximum at $z_0 \in U'$, so $v - u$ is constant on U', and the inequalities above are equalities.

If $z_0 \in B_r(z_0)$, the harmonicity of v_2 implies $v_2 - u$ is constant in $U' \cap \overline{B}_r(z_0)$. We can then run the same argument in the preceding paragraph to a point on $U' \cap \partial B_r(z_0)$, on which $v_2 = v$. $\square$

5.3.3. *Perron's method to solve the Dirichlet problem.* Now, we start to set up for Perron's method of solving the Dirichlet problem. Let U be a bounded and connected open set in $\mathbb{C}$ and f be a function on ∂U. We first do not assume f is continuous, but we assume it is bounded, say $|f| \leq M < \infty$.

As we mentioned, we want to use subharmonic function to approximate the solution to the Dirichlet problem (if exists). Thus, we consider

$$\mathcal{S}(f) := \left\{ v \in \mathcal{C}\left(\overline{U}\right) : v \text{ is subharmonic in } U \text{ with } v|_{\partial U} \leq f \right\}.$$

Note that by the bound $|f| \leq M$, any constant functions less than $-M$ belong to $\mathcal{S}(f)$.

Perron's strategy consists of the following two steps.

(1) Since subharmonic functions are bounded by harmonic functions (with the same boundary data), it is reasonable to approximate the solution from below. Thus, we define a function $\mathcal{P}f$ on U by letting

$$(\mathcal{P}f)(z) := \sup_{v \in \mathcal{S}(f)} v(z) \tag{5.59}$$

for each $z \in U$. Is $\mathcal{P}f \colon U \to \mathbb{R} \cup \{\infty\}$ harmonic?

(2) Does $\mathcal{P}f$ satisfy the boundary condition? That is, is it true that

$$\lim_{z \to p} (\mathcal{P}f)(z) = f(p)$$

for $p \in \partial U$?

Remark 5.60. . We can't hope to get both of (1) and (2) for all U's. For example, if one lets $U = B_1 \setminus \{0\}$ and sets $f = 1$ on ∂B_1 and $f(0) = 0$, then one cannot solve the Dirichlet problem for such data. We will work on this example in Assignment 8.

As mentioned in Remark 5.60, it is not always possible to have the nice boundary behavior as in (2). However, we can first see that (1) is in general true.

Theorem 5.61. The function $\mathcal{P}f$ constructed in (5.59) is harmonic in U. That is, (1) is correct.

The function $\mathcal{P}f$ is called the **Perron solution** on U with boundary f.[16] Note that when the Dirichlet problem (5.42) is solvable, the solution must equal $\mathcal{P}f$ by the maximum principle.

Proof. First, for each $v \in \mathcal{S}(f)$, the maximum principle (applied to $v - 0$) implies $v \leq \sup_{\partial U} f \leq M$. As a result, we first know that $\mathcal{P}f \colon U \to \mathbb{R}$ is a bounded well-defined function. We write $u := \mathcal{P}f$ and will prove that it is harmonic.

Given any $B := B_r(z_0)$ with $\overline{B} \subseteq U$, by the definition of u, we can find a sequence of subharmonic functions $v_n \in \mathcal{S}(f)$ such that

$$\lim_{n \to \infty} v_n(z_0) = u(z_0). \tag{5.62}$$

We will now produce a harmonic function from these v_n's, and show that it agrees with u.

We may assume v_n is an increasing sequence (i.e., $v_1 \leq v_2 \leq \cdots$) by replacing v_n with $\max\{v_1, \cdots, v_n\}$ inductively. The new sequence is still in $\mathcal{S}(f)$ by Proposition 5.58 and still produces the same limit (5.62) at z_0. We then consider $\widetilde{v}_n$, the harmonic lifting of v_n in B. Since $v_n \leq \widetilde{v}_n$ and $\widetilde{v}_n$ is still in $\mathcal{S}(f)$, we have the (monotone) limit (5.62) is still preserved; that is,

$$\lim_{n \to \infty} \widetilde{v}_n(z_0) = u(z_0). \tag{5.63}$$

Moreover, for any $z \in B$, the maximum principle implies

$$\widetilde{v}_{n+1}(z) - \widetilde{v}_n(z) \leq \max_{\partial B} (\widetilde{v}_{n+1} - \widetilde{v}_n) = \max_{\partial B} (v_{n+1} - v_n) \leq 0 \tag{5.64}$$

[16] This is a bit misleading since we do not know any actual boundary information about $\mathcal{P}f$.