Lemma 5.16 then tells us that this definition agrees with Definition 5.1(2) if we replace $(\mathbb{C}, d_{\mathrm{Euc}})$ with $(\hat{\mathbb{C}}, d_{\hat{\mathbb{C}}})$. Moreover, combined with Corollary 5.15, we know that in the new definition of normality for meromorphic families, we can still use equicontinuity to characterize it.

Corollary 5.18. *Let U be a connected open set in $\mathbb{C}$ and $\mathcal{F} \subseteq \mathcal{C}(U; \hat{\mathbb{C}})$ be a family of meromorphic functions on U. Then $\mathcal{F}$ is a normal family (in the sense of Definition 5.17) if and only if $\mathcal{F}$ is equicontinuous on any compact set $K \subseteq U$.*

In the rest of the section, we will see other ways to characterize normality for meromorphic families. In the following theorem, given a function $f \in U$, we define a new function

$$\hat{f}(z) := \frac{2|f'(z)|}{1 + |f(z)|^2}.$$

We use the notation $\hat{f}$ since, as we will see soon, this is related to arc length on the Riemann sphere $\hat{\mathbb{C}}$.

Theorem 5.19 (Marty)**.** *Let $\mathcal{F} \subseteq \mathcal{C}(U; \hat{\mathbb{C}})$ be a family of meromorphic functions on a connected open set U. Then $\mathcal{F}$ is normal if and only if the family*

$$\hat{\mathcal{F}} := \left\{ \hat{f} : f \in \mathcal{F} \right\}$$

is locally bounded (in the sense of Definition 5.12).

Proof. First, we proof the "if" part. The idea is to try to use the arc length on the Riemann sphere to bound the spherical distance. To this end, we look at a smooth curve $\gamma \colon [a, b] \to \hat{\mathbb{C}}$. Recall that by (5.14), its trajectory before the stereographic is

$$\widetilde{\gamma}(t) := \left(\frac{2\gamma(t)}{1 + |\gamma(t)|^2}, 1 - \frac{2}{1 + |\gamma(t)|^2} \right).$$

To calculate its arc length, we need to know the magnitude of $\widetilde{\gamma}'(t)$, which can be given by a straightforward calculation$^{\mathrm{ex}}$

$$|\widetilde{\gamma}'(t)| = \frac{2|\gamma'(t)|}{1 + |\gamma(t)|^2}.$$

Now, to show the equicontinuity in a ball $\overline{B} \subseteq U$, given $z, w \in \overline{B}$, we take the straight line $\ell \colon [0, |z - w|] \to U$ connecting z and w in $\overline{B}$.[14] Then for any $f \in \mathcal{F}$, the curve we will consider in $\hat{\mathbb{C}}$ is $f(\ell(t))$. Since the chord is not longer than the arc, we have

$$d_{\hat{\mathbb{C}}}\left(f(z), f(w)\right) \leq \mathrm{Length}\left(f \circ \ell\right) = \int_0^{|z-w|} \frac{2|f'\left(\ell(t)\right)|}{1 + |f\left(\ell(t)\right)|^2} dt$$

using $|\ell'(t)| = 1$. By the assumption, we can find a bound $M < \infty$ such that for all $f \in \mathcal{F}$,

$$\sup_{\overline{B}} |\hat{f}| \leq M.$$

Thus, we can further estimate

$$d_{\hat{\mathbb{C}}}\left(f(z), f(w)\right) \leq M \cdot |z - w|.$$

---

[14] We can take ℓ in such a way that it has unit speed, e.g., $\ell(t) = \frac{t}{|z-w|} w + \left(1 - \frac{t}{|z-w|}\right) z$.

This proves the equicontinuity.

For the "only if" part, we again fix a ball $\overline{B} \subseteq U$. For any small ε, say $\varepsilon < 10^{-10}$, the equicontinuity gives us $\delta > 0$ such that

$$d_{\hat{\mathbb{C}}}\left(f(z), f(w)\right) < \varepsilon$$

for any $f \in \mathcal{F}$ and $z, w \in \overline{B}$ with $|z - w| \leq 2\delta$. Let z_0 be the center of the ball $\overline{B}$ and assume $\overline{B}_{2\delta}(z_0) \subseteq \overline{B} \subseteq U$ after shrinking δ if needed. Then we look at $\hat{f}$ in $B_\delta(z_0)$.

Suppose $|f(z_0)| \leq 1$. Then we have $|f(z)| \leq 1 + \varepsilon < 2$ for $z \in \overline{B}_{2\delta}(z_0)$. For $z \in B_\delta(z_0)$, we can estimate

$$|f'(z)| = \left| \frac{1}{2\pi i} \int_{\partial B_{2\delta}(z_0)} \frac{f(w)}{(w-z)^2} dw \right| \leq 2\delta \cdot \frac{2}{\delta^2} = \frac{4}{\delta}.$$

Thus,

$$\hat{f}(z) = \frac{2|f'(z)|}{1 + |f(z)|^2} \leq \frac{8/\delta}{1} = \frac{8}{\delta}.$$

Suppose $|f(z_0)| \geq 1$ on the other hand, including the case $f(z_0) = \infty$. Then $|f(z)| \geq 1 - \varepsilon > 1/2$ for $z \in \overline{B}_{2\delta}(z_0)$. Notice that if we consider $g := 1/f$ in $\overline{B}_{2\delta}(z_0)$, we get

$$\hat{g} = \frac{2|f'/f^2|}{1 + 1/|f|^2} = \frac{2|f'|}{|f|^2 + 1} = \hat{f}.$$

Moreover, we have $|g| < 2$ in $\overline{B}_{2\delta}(z_0)$, so the previous case implies $\hat{g}(z) \leq 8/\delta$ for $z \in B_\delta(z_0)$.

Combining the two cases, we get $\hat{f}(z) \leq 8/\delta$ for $f \in \mathcal{F}$ and $z \in B_\delta(p)$. This proves the local boundedness. □

Marty's theorem will later be used to get another result of Montel. To achieve that, we want to understand what happens when we are given a "non-normal" sequence. That is, a sequence f_n without a convergent subsequence. Note that by Marty's theorem, this means that the corresponding sequence $\hat{f}_n$ is not locally bounded, so we can always find a "blow-up sequence" in a compact set. By normalizing the sequence using this blow-up phenomenon, we can produce a rescaled limit, and it captures the "singular structure" of the original sequence. This will be implemented in the following result, and we remark that this technique has been extremely successful in the study of singularities in many areas of geometry and analysis.

Proposition 5.20 (Zalcman's lemma)**.** Let U be a connected open set in $\mathbb{C}$ and let f_n be a sequence of meromorphic functions on U. If the family $\{f_n : n \in \mathbb{N}\}$ is not normal, then there exist a sequence z_n in U converging to $z_0 \in U$ and a sequence of positive numbers $M_n \to \infty$ such that the functions

$$g_n(w) := f_n\left(z_n + \frac{w}{M_n}\right)$$

converge to a non-constant meromorphic function on $\mathbb{C}$ with $\hat{g} \leq 1 = \hat{g}(0)$.

Proof. Since the sequence is not normal, by Marty's theorem (Theorem 5.19), there is a compact set $K \subseteq U$ and a sequence $w_n \in K$ such that

$$\lim_{n\to\infty} \hat{f}_n(w_n) = \infty.$$

After rescaling and translating U, we may assume $\overline{B}_1(0) \subseteq U$ and $w_n \in \overline{B}_1(0)$ with $w_n \to 0$. Instead of using w_n and $\hat{f}_n(w_n)$ to rescale, we take another sequence of points z_n such that[15]

$$\max_{z\in\overline{B}_1}(1-|z|)\hat{f}_n(z) = (1-|z_n|)\hat{f}_n(z_n).$$

Note that $z_n \notin \partial B_1$ when n is large as $\hat{f}_n$ diverges. Since $\hat{f}_n(w_n) \to \infty$ and $w_n \to 0$, we know the maximum sequence above diverges. In particular, we also have

$$\lim_{n\to\infty} \hat{f}_n(z_n) = \infty,$$

and we let $M_n := \hat{f}_n(z_n)$ be the blow-up scale.

Now, we use z_n and M_n to define g_n as in the theorem. Then we have that for each n,

$$\hat{g}_n(0) = \frac{\hat{f}_n(z_n)}{M_n} = 1, \tag{5.21}$$

and for $|w| < (1-|z_n|)M_n$, we can estimate

$$\begin{aligned}\hat{g}_n(w) = \frac{\hat{f}_n\left(z_n + \frac{w}{M_n}\right)}{M_n} &\le \frac{1}{M_n}\cdot\frac{1-|z_n|}{1-\left|z_n+\frac{w}{M_n}\right|}M_n \\ &\le \frac{1-|z_n|}{1-|z_n|-\left|\frac{w}{M_n}\right|} = \frac{1}{1-\frac{|w|}{(1-|z_n|)M_n}}\end{aligned} \tag{5.22}$$

where the first inequality uses that $(1-|z_n|)M_n$ is the maximum of $(1-|z|)\hat{f}_n(z)$ in $\overline{B}_1$. Hence, for a given $R < \infty$, by taking N so large that $\frac{1}{2}(1-|z_N|)M_N > R$, we have $\hat{g}_n$ is uniformly bounded on B_R for $n \ge N$. Hence, Marty's theorem implies the conclusion. The bounds on $\hat{g}$ follow from taking the limits of (5.21) and (5.22). ☐

We can now state Montel's theorem for meromorphic families.

Theorem 5.23 (Montel)**.** Let $\mathcal{F}$ be a family of meromorphic functions on a connected open set $U \subseteq \mathbb{C}$. If $\mathcal{F}$ omits three values in $\hat{\mathbb{C}}$, then $\mathcal{F}$ is normal.

Proof. By applying a rational linear transformation (see Assignment 2), we may assume $\mathcal{F}$ omits the three points $0, 1$, and ∞ in $\hat{\mathbb{C}}$. That is, $\mathcal{F}$ is a family of non-vanishing holomorphic functions that omit the value 1.

We prove the theorem by contradiction. Suppose $\mathcal{F}$ is not normal. Then by Marty's theorem, we can find a small ball $\overline{B} \subseteq U$, a sequence $f_n \in \mathcal{F}$, and points $w_n \in \overline{B}$ with $w_n \to 0 \in \overline{B}$ such that

$$\lim_{n\to\infty} \hat{f}_n(w_n) = \infty.$$

---

[15] Directly using w_n to get $g_n := f_n(w_n + w/M_n)$ does not allow one to control $\hat{g}_n$

We now consider some new families of functions constructed from f_n. Recall that by Corollary 4.9, we can take the roots of these functions on $\overline{B}_1$. Thus, for each $k \in \mathbb{N} \cup \{0\}$, we consider the sequence

$$\mathcal{F}_k := \left\{ f_n^{1/2^k} : n \in \mathbb{N} \right\}.$$

None of these families is normal by our choice of f_n. Hence, by Zalcman's lemma (Proposition 5.20), for each $k \in \mathbb{N}$, we can get a rescaled limit g_k, a non-constant meromorphic function on $\mathbb{C}$ with

$$\hat{g}_k \leq 1 = \hat{g}_k(0)$$

Marty's theorem (Theorem 5.19) then implies that $\{g_k : k \in \mathbb{N}\}$ is a normal family, so a subsequence of it locally uniformly converges to a limit g with $\hat{g}(0) = 1$. In particular, g is a non-constant meromorphic function on $\mathbb{C}$.

Now we look at what values these functions omit. Since each element in $\mathcal{F}_k$ omits all 2^k-th roots of unity, so does their rescaled limit g_k by Hurwitz's theorem (Corollary 3.38). Similar, we know that g omits all 2^k-th roots of unity for all k. By the open mapping theorem (Corollary 3.6 and Assignment 6), we know that $\hat{\mathbb{C}} \setminus g(\mathbb{C})$ is closed. Since the set of all 2^k-th roots of unity for all k is dense in the unit circle, we can conclude that $g(\mathbb{C})$ is disjoint from the unit circle.

If $|g| < 1$, then the Liouville theorem (Corollary 2.25) implies that h is a constant. This contradicts $\hat{g}(0) = 1$.

If $|g| > 1$, by looking at $h := 1/g$, we know that $\hat{h}(0) = \hat{g}(0) = 1$, and applying Liouville to h leads to a contradiction again. ☐

As we've mentioned, Montel's theorems (Theorem 5.13 and Theorem 5.23) have many applications. Now, we can first use Theorem 5.23 to improve the little Picard theorem we prove in Theorem 4.10.

Theorem 5.24 (Great Picard)**.** If $f \colon B_1 \setminus \{0\} \to \mathbb{C}$ is a non-constant holomorphic function with an essential singularity at 0, then f omits at most one point in $\mathbb{C}$.

Another way to phrase the theorem is that if a meromorphic function omits three values on $\hat{B}_1(0)$, then it extends meromorphically to 0. We know that the little Picard theorem (Theorem 4.10) is related to Proposition 3.17. Theorem 5.24 is a stronger result than both of them.

Proof. Assume f omits at least two values in $\mathbb{C}$. We will show that f must be a constant function on $\hat{B}_1 = B_1 \setminus \{0\}$.

Applying a rational linear transformation, we may assume f omits $0, 1$, and ∞ (in $\hat{\mathbb{C}}$). Thus, f is a holomorphic function. We consider a family by rescaling f; that is, consider $f_\varepsilon(z) := f(\varepsilon z)$ and

$$\mathcal{F} := \{f_\varepsilon(z) : \varepsilon \in (0,1)\}.$$

Each f_ε is defined on a punctured ball larger than $\hat{B}_1$, but we only look at its restriction on $\hat{B}_1$. By Montel's theorem (Theorem 5.23), $\mathcal{F}$ is normal, so there exists a sequence $\varepsilon_n \to 0$ such that f_{ε_n} locally uniformly converges to a limit g. By Lemma 5.16, g is either holomorphic or identically infinity on $\hat{B}_1$.

If g is holomorphic on $\hat{B}_1$, then we can take

$$M := \sup_{\partial B_{1/2}} |g| < \infty.$$

This implies that for large n,

$$\sup_{\partial B_{1/2}} |f_{\varepsilon_n}| \leq 2M,$$

which means that for $z \in \partial B_{1/2}$,

$$|f(\varepsilon_n z)| \leq 2M.$$

Therefore, we can bound

$$\sup_{\partial B_{1/2} \cup \partial B_{\varepsilon_n/2}} |f| \leq 2M.$$

Since $\partial B_{1/2} \cup \partial B_{\varepsilon_n/2} = \partial \left(B_{1/2} \setminus B_{\varepsilon_n/2} \right)$, the maximum principle (Theorem 2.27) implies

$$\sup_{B_{1/2} \setminus B_{\varepsilon_n/2}} |f| \leq 2M.$$

Since this true for all n large and $\varepsilon_n \to 0$, we get

$$\sup_{\hat{B}_{1/2}} |f| \leq 2M.$$

By the Riemann extension theorem (Proposition 3.14), 0 is a removable singularity of f, a contradiction.

If $g \equiv \infty$ on $\hat{B}_1$, we can run the arguments above identically for $1/f$, $1/f_{\varepsilon_n}$, and $1/g$ to show that 0 is a pole of f. This contradicts the assumption again, and the proof is complete. $\square$