1. Introduction

Complex analysis is the study of functions of a complex variable, a subject that reveals a deep and beautiful interplay between algebra, analysis, and geometry. At first glance, one might think of complex numbers as merely an extension of the real number system, but the introduction of a complex variable fundamentally changes the nature of calculus. Functions that are differentiable in the complex sense—holomorphic functions—exhibit remarkable properties that set them apart from their real-variable counterparts.

Unlike real differentiable functions, holomorphic functions are automatically infinitely differentiable and analytic, meaning they can be expressed as convergent power series wherever they are defined. This rigidity leads to powerful results such as Cauchy's integral formula, which establishes profound connections between differentiation, integration, and series expansions. These properties make complex analysis not just a refinement of real analysis but an entirely new and far-reaching mathematical framework.

Beyond its theoretical elegance, complex analysis has deep applications across mathematics and physics. It plays a crucial role in number theory, algebraic geometry, dynamical systems, and representation theory, and it provides essential tools for solving problems in mathematical physics, engineering, and even fluid dynamics. Many results that seem difficult or inaccessible in real analysis become transparent through the lens of complex analysis.

In this course, we will develop the fundamental tools of complex analysis and explore their consequences. We will see why complex analysis is not only one of the most elegant subjects in mathematics but also one of the most powerful. To illustrate this, consider one of the most important functions in math, the **Riemann zeta function** $\zeta(s)$. It is defined to be

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s}$$

for $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$. At first, this sum looks like just a function of real numbers when s is real. But it turns out that by analytically continuing it, we get a function defined on almost the entire complex plane. This reveals deep insights into prime numbers. One way this connection emerges is through the the relation

$$\zeta(s) = \prod_{p \text{ is a prime}} \frac{1}{1 - p^{-s}}$$

for $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$. This is how prime numbers are encoded inside $\zeta(s)$. Using the tools of complex analysis, we can study the analytic behavior of this function and eventually arrive at profound results—such as the Prime Number Theorem, which tells us that the number of primes less than x, denoted $\pi(x)$, grows approximately like $x/\log x$.

Another beautiful application of complex analysis is its usage to classify "surfaces," that is, two-dimensional manifolds. Different from the general case, it is possible to equip a "complex structure" to any surfaces, so tools related to single-variable complex analysis can help in the classification program. We will return to both of these ideas (prime number theorem and classification of surfaces) after developing the necessary background.

In the notes, the notation $^{\text{ex}}$ means that the conclusion follows from a relatively straightforward argument, and will be left as a practice exercise. Feel free to contact me/come to my office hour if you have any question about them (or about any other things).

1.1. **Preliminaries.**

1.1.1. *Complex numbers.* We recall the notations of the common sets of numbers:

- $\mathbb{N}$: the set of positive integers.
- $\mathbb{Z}$: the set of integers.
- $\mathbb{Q}$: the set of rational numbers.
- $\mathbb{R}$: the set of real numbers.
- $\mathbb{C}$: the set of complex numbers.

Each of the constructions of a new number system from the previous one involves a new extension of some important properties.

- $\mathbb{N}$: the set $\{1, 2, 3, \cdots\}$ that is closed under addition and multiplication.
- $\mathbb{Z}$: the smallest additive "group" that contains $\mathbb{N}$.
- $\mathbb{Q}$: the smallest "field" that contains $\mathbb{Z}$.
- $\mathbb{R}$: the smallest complete metric space that contains $\mathbb{Q}$.
- $\mathbb{C}$: the smallest algebraically closed field that contains $\mathbb{R}$.

Another aspect is about solving polynomial equations. Each extension allows one to solve more polynomial equations in the extended system.

- $\mathbb{N}$ contains solutions to $x - 1 = 0$.
- $\mathbb{Z}$ contains solutions to $x + 1 = 0$.
- $\mathbb{Q}$ contains solutions to $2x - 1 = 0$.
- $\mathbb{R}$ contains solutions to $2x^2 - 1 = 0$.
- $\mathbb{C}$ contains solutions to $2x^2 + 1 = 0$.

We won't talk about the detailed constructions of any of them. Here, we briefly mention a few equivalent ways to construct $\mathbb{C}$ from $\mathbb{R}$.

(1) Field extension: $\mathbb{C} := \mathbb{R}(i) := \{a + bi : a, b \in \mathbb{R}\}$ where i satisfies $i^2 = -1$.

(2) Quotient field: $\mathbb{C} := \mathbb{R}[x]/\left\langle x^2 + 1\right\rangle$, where we let i be the image of the image x under the quotient map $\mathbb{R}[x] \to \mathbb{R}[x]/\left\langle x^2 + 1\right\rangle$.

(3) Algebraic closure: $\mathbb{C} :=$ the algebraic closure of $\mathbb{R}$.

Further discussions about these constructions will be too algebraic. For our purpose, we recall some basic operations that we will need in the course.

(1) Given a complex number $z = a + bi$ with $a, b \in \mathbb{R}$, $\mathrm{Re}z := a$ is the real part of z, and $\mathrm{Im}z := b$ is the imaginary part of z.

(2) The basic operations are based on the commutative, associative, and distributive laws, with the rule $i^2 = -1$. For example,

$$(a + bi) \cdot (c + di) = (ac - bd) + (ad + bc)i.$$

(3) The complex conjugation is a map $\overline{(\cdot)}\colon \mathbb{C} \to \mathbb{C}$ defined by $a + bi \mapsto a - bi$ for $a, b \in \mathbb{R}$. It is then direct to check

$$\mathrm{Re}z = \frac{z + \overline{z}}{2} \text{ and } \mathrm{Im}z = \frac{z - \overline{z}}{2i}$$

for any $z \in \mathbb{C}$.

(4) The length or the absolute value of a complex number z is defined by $|z| := \sqrt{z\overline{z}}$. To be explicit, if $z = a + bi$ where $a, b \in \mathbb{R}$, then $|z| = \sqrt{a^2 + b^2}$. It is its distance to the origin on the complex plane.

(5) With the distance given by $d(z, w) := |z - w|$, the complex plane $\mathbb{C}$ is a "metric space." In particular, it satisfies the triangle inequality:

$$|z + w| \leq |z| + |w|$$

for any $z, w \in \mathbb{C}$.

We recall that a metric space is a set X with a distance function $d\colon X \times X \to \mathbb{R}_{\geq 0}$[1] satisfying the following three properties for any $x, y, z \in X$.

(1) $d(x, y) = 0$ if and only if $x = y$.

(2) $d(x, y) = d(y, z)$.

(3) $d(x, z) \leq d(x, y) + d(y, z)$.

Based on the metric space structure, many analytic tools can be applied and will be crucial in the course.

1.1.2. *Review on analysis and topology.* Most of the content in this section applies to an arbitrary metric space, but we focus on the case of $\mathbb{C}$, where the distance is given by the absolute value of the difference. Much of this material should be familiar, and we encourage students to recall the concepts by working through a few **examples**, though few will be provided here since this section serves mainly as a recap.

Recall that the metric space structure gives a natural topological structure on $\mathbb{C}$. To talk about the topology in detail, we first mention that we use $B_r(z_0)$ to be the open ball centered at z_0 with radius r. That is,

$$B_r(z_0) := \{z \in \mathbb{C} : |z - z_0| < r\}.$$

We let $\overline{B}_r(z_0)$ be the corresponding closed ball, including those points on the boundary of $B_r(z_0)$. A subset U in $\mathbb{C}$ is then called **open** if for any point $z_0 \in U$, there exists $r > 0$ such that $B_r(z_0) \subseteq U$.

[1] We occasionally use subscriptions to denote different subset of the number sets. For example, $\mathbb{R}_{\geq 0} := \{r \in \mathbb{R} : r \geq 0\}$, and $\mathbb{Z}_{<0} := \{n \in \mathbb{Z} : n < 0\}$.

In general, such a point is called an **interior point** of U (even when U is not an open set, which means that not every point is an interior point). A subset K is called **closed** if its complement $\mathbb{C} \setminus K$ is an open set in $\mathbb{C}$. We leave all the other topological notions to the first time when we use them, including compactness, connectivity, and more.

Since we want to use calculus to study complex numbers and functions of them, we first recall the notions of taking limits. We will use limits of both sequences and functions.

Given a sequence a_n ($n \in \mathbb{N}$) of complex numbers, we say it converges to a complex number L, denoted by

$$\lim_{n\to\infty} a_n = L,$$

if given any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $|a_n - L| < \varepsilon$ for $n \geq N$.

Given a function $f\colon U \to \mathbb{C}$ defined on an open set U and a point $z_0 \in U$, we say f converges to a complex number L as z tend to z_0, denoted by

$$\lim_{z\to z_0} f(z) = L,$$

if given any ε, there exists $\delta > 0$ such that $|f(z) - L| < \varepsilon$ for $z \in B_\delta(z_0) \setminus \{z_0\}$. Note that the limit of f as $z \to z_0$ does not depend on $f(z_0)$. If the function $f\colon U \to \mathbb{C}$ satisfies

$$\lim_{z\to z_0} f(z) = f(z_0),$$

then we say f is **continuous** at the point z_0. We say f is a continuous function on U if f is continuous at every point $z \in U$.

Based on these two definitions, it is natural to think about the convergence of a given sequence of functions. Thus, we assume f_n is a sequence of complex-valued functions defined on U.

(1) We say f_n converges **pointwisely** to a function $f\colon U \to \mathbb{C}$ if given any $z \in U$ and $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $|f_n(z) - f(z)| < \varepsilon$ for $n \geq N$.

(2) We say f_n converges **uniformly** to a function $f\colon U \to \mathbb{C}$ if given any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for any $z \in U$, $|f_n(z) - f(z)| < \varepsilon$ for $n \geq N$.

The notion of uniform convergence arose after a mistake had been found in Cauchy's original proof of the statement that a pointwise limit of a sequence of continuous functions is still continuous. The way how Cauchy corrected his proof is the introduction of the notion of uniform convergence (if written in modern languages). We record this and another consequence of uniform convergence here, and will use them without quoting in the rest of the lecture.

Proposition 1.1. Let $f_n\colon U \to \mathbb{C}$ be a sequence of functions. Suppose f_n uniformly converges to a function $f\colon U \to \mathbb{C}$.

(1) If each f_n is continuous, then f is also continuous.

(2) If each f_n is (Riemann) integrable[2], then f is also (Riemann) integrable, and

$$\lim_{n\to\infty} \int_U f_n \, dz = \int_U f \, dz. \tag{1.2}$$

[2]Here, we mean that for each f_n, $\mathrm{Re} f_n$ and $\mathrm{Im} f_n$ are integrable on U, as a subset of $\mathbb{R}^2$. The integral in (1.2) is also interpreted in this sense.

A more useful localized notion of uniform convergence will be used later many times. We will mention it after it first appears in Section 2.1.

1.2. **Holomorphic functions: basics and examples.**

1.2.1. *Cauchy–Riemann equations.* We want to do calculus on $\mathbb{C}$, so we care about functions that are differentiable in the complex variable z.

Definition 1.3. Let U be an open set[3] in $\mathbb{C}$ and let $f\colon U \to \mathbb{C}$ be a function. We say f is **holomorphic** at a point $z_0 \in U$ if the limit

$$\lim_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0} = \lim_{h\to 0} \frac{f(z_0+h)-f(z_0)}{h} \tag{1.4}$$

exists. When it does, the limit is denoted by $f'(z_0)$.

We remark that the limit is taken by viewing $\mathbb{C}$ as a metric space using the distance function given by the absolute value $|z| := \sqrt{z\bar{z}}$. Recall that using this, we say the limit (1.4) exists and is $L \in \mathbb{C}$ if given any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\left| \frac{f(z)-f(z_0)}{z-z_0} - L \right| \leq \varepsilon$$

for $z \in B_\delta(z_0) \setminus \{z_0\}$. This limit is the **complex derivative** of f at z_0, and as mentioned, denoted by $f'(z_0)$. We say f is holomorphic on U if f is holomorphic at any point in U. In this case, we get a function $f'\colon U \to \mathbb{C}$. Most of the properties we learn for real derivatives are true, and we mention a few of them.

(1) If f is holomorphic on U, then f is, in particular, continuous.

(2) If f and g are holomorphic, then $f \pm g$ and $f \cdot g$ are holomorphic, with
$$(f \pm g)' = f' \pm g' \text{ and}$$
$$(f \cdot g)' = f' \cdot g + f \cdot g'.$$

(3) If g is holomorphic and $g \neq 0$ at a point, then at the point, $1/g$ is holomorphic with
$$\left(\frac{1}{g}\right)' = -\frac{g'}{g^2}.$$

(4) If f and g are holomorphic, then at a point z where $f \circ g$ is well-defined, $f \circ g$ is holomorphic and
$$(f \circ g)'(z) = f'\left(g(z)\right) \cdot g'(z).$$

However, holomorphic functions also behave completely differently from real differentiable functions. They are much more "rigid." To illustrate this, suppose a holomorphic function f is defined in an open set U. For $z \in U$, we write $z = x + iy$ and

$$f(z) = f(x,y) = u(x,y) + iv(x,y) \tag{1.5}$$

where u and v are real-valued functions. We will see that f being holomorphic is much stronger then u and v being differentiable. We state this observation as a lemma.

[3] In the rest of the notes, U will be an open set in $\mathbb{C}$ unless otherwise stated.

Lemma 1.6. Suppose $f\colon U \to \mathbb{C}$ is holomorphic and is written in the form (1.5). Then

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}. \tag{1.7}$$

These equations are called the **Cauchy–Riemann equations**.

Proof. The lemma follows from the existence of the limit (1.4). In fact, at a point $z_0 = x_0 + iy_0 \in U$, we can approximate z_0 from the real direction and get

$$\begin{aligned} f'(z_0) = \lim_{r\to 0} \frac{f(z_0+r) - f(z_0)}{r} &= \lim_{r\to 0} \left(\frac{u(x_0+r, y_0) - u(x_0, y_0)}{r} + i\frac{v(x_0+r, y_0) - v(x_0, y_0)}{r} \right) \\ &= \frac{\partial u}{\partial x}(x_0, y_0) + i\frac{\partial v}{\partial x}(x_0, y_0). \end{aligned}$$

We can also approximate z_0 from the imaginary direction and get

$$\begin{aligned} f'(z_0) = \lim_{r\to 0} \frac{f(z_0+ri) - f(z_0)}{ri} &= \lim_{r\to 0} \left(\frac{u(x_0, y_0+r) - u(x_0, y_0)}{ri} + i\frac{v(x_0, y_0+r) - v(x_0, y_0)}{ri} \right) \\ &= -i\frac{\partial u}{\partial y}(x_0, y_0) + \frac{\partial v}{\partial y}(x_0, y_0) \end{aligned}$$

where we use $1/i = -i$. Comparing these two equations, we get the conclusion. □

Based on Lemma 1.6, we know that holomorphic functions form a much more restricted class of differentiable functions. Let's see some consequences of the Cauchy–Riemann equation.

(1) If u and v are C^2 and satisfy (1.7), then in particular, we have

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y} = \frac{\partial^2 v}{\partial y \partial x} = -\frac{\partial^2 u}{\partial y^2},$$

so u is a **harmonic function**, in the sense that

$$\Delta u := \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

One can similarly see that v is also harmonic. Because of this property, v is sometimes called a **harmonic conjugate** of u. Harmonic functions play an important role in the study of partial differential equations, and later we will see that many good properties of them also apply to holomorphic functions. Here, we just mention the converse about the existence of harmonic conjugates as an evidence for the connection between holomorphic and harmonic functions. We remark that, however, harmonic conjugates do not always exist on a general domain (cf. Corollary 4.5).

Lemma 1.8. Let u be a harmonic function on the unit disc B_1. Then there exists a unique harmonic conjugate of u up to a constant. That is, there exists a harmonic function v such that u and v satisfy (1.7), and if $\widetilde{v}$ is another harmonic function with the same property, then $v = \widetilde{v} + c$ where c is a constant.

Proof. If such a v exists, then for any $(x_0, y_0) \in B_1$, the fundamental theorem of calculus and (1.7) imply

$$\begin{aligned} v(x_0, y_0) &= v(0) + \int_0^{x_0} \frac{\partial v}{\partial x}(x, 0)dx + \int_0^{y_0} \frac{\partial v}{\partial y}(x_0, y)dy \\ &= v(0) - \int_0^{x_0} \frac{\partial u}{\partial y}(x, 0)dx + \int_0^{y_0} \frac{\partial u}{\partial x}(x_0, y)dy. \end{aligned}$$

The same equation holds for $\widetilde{v}$, so taking their difference, we get

$$v(x_0, y_0) - \widetilde{v}(x_0, y_0) = v(0) - \widetilde{v}(0),$$

so the uniqueness follows.

For existence, inspired by the equation above, we define

$$v(x_0, y_0) := -\int_0^{x_0} \frac{\partial u}{\partial y}(x, 0)dx + \int_0^{y_0} \frac{\partial u}{\partial x}(x_0, y)dy. \tag{1.9}$$

for any $(x_0, y_0) \in B_1$. By the fundamental theorem of calculus, differentiating (1.9) in y gives

$$\frac{\partial v}{\partial y}(x_0, y_0) = \frac{\partial u}{\partial x}(x_0, y_0),$$

noting that the first term in (1.9) does not depend on y_0. For the other equation in (1.7), differentiating (1.9) in x (noting that u being C^2 allows us to do so) leads to

$$\begin{aligned} \frac{\partial v}{\partial x}(x_0, y_0) &= -\frac{\partial u}{\partial y}(x_0, 0) + \int_0^{y_0} \frac{\partial^2 u}{\partial x^2}(x_0, y_0)dy \\ &= -\frac{\partial u}{\partial y}(x_0, 0) - \int_0^{y_0} \frac{\partial^2 u}{\partial y^2}(x_0, y_0)dy \\ &= -\frac{\partial u}{\partial y}(x_0, y_0) \end{aligned}$$

where we use the harmonicity of u and the fundamental theorem of calculus again (and again). Thus, (1.7) follows. The equations and the C^2-regularity of u imply that v is also C^2, and hence harmonic. □

We go back to the discussion of the Cauchy–Riemann equation.

(2) If u and v are C^1 and satisfy (1.7), then $f := u + iv$ is holomorphic.[4] This is the converse of Lemma 1.6, and this can be seen by checking an equivalent condition of the existence of the limit (1.4). That is, f is holomorphic at z_0 if and only if there exists $L \in \mathbb{C}$ such that

$$f(z_0 + h) - f(z_0) = Lh + o(|h|)$$

as $h \to 0$. This implies that L is the first-order approximation of f at z_0 and means that $f'(z_0) = L$. Here, we recall the little o-notation, and we say that two functions $g_1(z)$ and $g_2(z)$ satisfies $g_1 = o(g_2)$ as $z \to a$ for some $a \in [-\infty, \infty]$ if $|g_1(z)|/|g_2(z)| \to 0$ as $z \to a$. By writing out the first-order approximations of u and v, one can see that their derivatives together determine the derivative of of f.[ex]

[4] Note that the a priori regularity here is important. Otherwise, there are counterexamples. See Assignment 1.

(3) Cauchy–Riemann equations also allow us to introduce the first-order operator

$$\partial_z := \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) \text{ and}$$
$$\partial_{\overline{z}} := \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

With these notations, one can calculate that

$$\partial_{\overline{z}} f = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)(u+iv) = \frac{1}{2}\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) + \frac{1}{2}i\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right).$$

Thus, a holomorphic function satisfies (1.7) if and only if $\partial_{\overline{z}} f = 0$. Intuitively, this means that if we write

$$f(z,\overline{z}) = f(x,y) = f\left(\frac{z+\overline{z}}{2}, \frac{z-\overline{z}}{2i}\right),$$

then it is "independent" of $\overline{z}$.

Example 1.10. We end this section by mentioning a few examples.

(1) A **polynomial** $f(z) = a_0 + a_1 z + \cdots + a_n z^n$ is a holomorphic function. One can check that, as in the real case, if $f(z) = z^n$, then $f'(z) = nz^{n-1}$. Polynomials are **entire** functions, functions that are holomorphic on the whole $\mathbb{C}$.

(2) The conjugate function $f(z) = \overline{z}$ is not holomorphic. This can be seen by either the definition or by the Cauchy–Riemann equation. In the languages introduced in (3), this is because $\partial_{\overline{z}} f = 1 \neq 0$. A problem in Assignment 1 will ask you to check whether a function in z and $\overline{z}$ is holomorphic based on the definition (of complex differentiability).

(3) A **rational function** $f(z) = P(z)/Q(z)$ is holomorphic at which Q does not vanish, where P and Q are polynomials. The zeros of Q are called the **poles** of f. The study of the behavior of f near its poles is crucial. Common examples are when P and Q are polynomials of degree one, and we will see some discussion in Assignment 1. Some properties of polynomial and rational functions are discussed in Section 1.2.2.

(4) The exponential function

$$e^z := \sum_{n\geq 0} \frac{1}{n!} z^n = 1 + z + \frac{z^2}{2} + \frac{z^3}{6} + \cdots$$

is a holomorphic function. It is direct to verify it is a well-defined function for all $z \in \mathbb{C}$. By the binomial formula, as in the real case, one can verify that

$$e^z \cdot e^w = e^{z+w}$$

for any $z, w \in \mathbb{C}$. We will see that, in general, a **power series**

$$f(z) = \sum_{n\geq 0} a_n z^n$$

is holomorphic at which the series converges. This will be in Section 1.2.3. This will also be a main topic in Section 2.1, as we will see all holomorphic functions are locally power series.

(5) We mention a few examples of harmonic conjugates.

$$u(x,y) = x,\ v(x,y) = y+1;$$
$$u(x,y) = y,\ v(x,y) = -x;$$
$$u(x,y) = x^2 - y^2,\ v(x,y) = 2xy;$$
$$u(x,y) = e^x \cos y,\ v(x,y) = e^x \sin y.$$

One can try to find out holomorphic functions they represent.

1.2.2. *Polynomial and rational functions.* We will have a brief discussion about polynomial and rational functions, as they are the first non-trivial class of holomorphic functios we can work on. They are also, respectively, finite cases for power series and Laurent series, which are important expansions for holomorphic functions and meromorphic functions, and some notions we introduce here will be generalized later as we proceed.

If $P(z)$ is a polynomial of degree n, then the fundamental theorem of algebra[5] implies that P has exactly n roots, counted with multiplicities. If a zero z_1 appears exactly n_1 times, that is, we can write $P(z) = P_1(z)\cdot(z-z_1)^{n_1}$ for another polynomial P_1 with $P_1(z_1) \neq 0$, then we say z_1 is a zero of P of **order** n_1. Here, we mention an interesting results about zeros of P and its derivatives.

Lemma 1.11 (Gauss–Lucas theorem)**.** Let P be a non-constant polynomial. Then any zero of P' lies in the **convex hall** of the set of all zeros of P.

The convex hall of a subset $X \subseteq \mathbb{C}$ is the smallest convex set that contains X. In practice, one can take the intersection of all half-planes that contain X to find the convex hall of X.

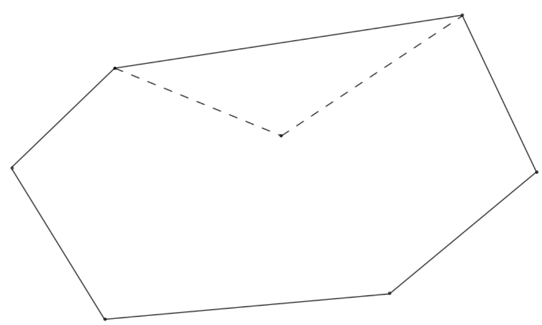

FIGURE 1. When there is a finite set, its convex hall is a convex polygon that contains all the segments between any two points of the set.

Proof. It is sufficient to show that if all the zeros of P lie in a half-plane, then all the zeros of P' also do. Thus, we assume all the zeros $z_1, \cdots, z_n$ of P lie in the half-plane defined by $\mathrm{Im} z \geq 0$. Take a point z_0 that is not in the half-plane defined by $\mathrm{Im} z \geq 0$, that is,

$$\mathrm{Im} z_0 < 0. \tag{1.12}$$

[5]The theorem can be proven based on calculus, so we take it for granted here. Later, we will see that it can be proven based on the tools we develop for holomorphic functions, cf. Section 2.1.2.

In particular, we know $\operatorname{Im}(z_0 - z_k) < 0$ for all $k = 1, \cdots, n$ and $P(z_0) \neq 0$, so

$$\frac{P'(z_0)}{P(z_0)} = \frac{1}{z_0 - z_1} + \cdots + \frac{1}{z_0 - z_n}$$

has positive imaginary part. Thus, $P'(z_0) \neq 0$. □

Next, we talk about rational functions. For a rational function $R(z) = P(z)/Q(z)$, where we assume P and Q do not have common zeros, we know that $R(z)$ is holomorphic away from its poles. One can also show that at a pole p_1 of R, we have $\lim_{z \to p_1} R(z) = \infty$, which we will rigorously define next. It is thus sometimes convenient to introduce the space with a "formal infinity." That is, we consider

$$\hat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}.$$

We only formally add it to the complex plane so that it matches all the calculations we have here.[6] For ∞-valued point, we define $R(z_0) = \infty$ if

$$\lim_{z \to z_0} \frac{1}{R(z)} = 0.$$

For ∞-input, we define

$$R(\infty) := \lim_{w \to 0} R\left(\frac{1}{w}\right).$$

When $z_0 \in \mathbb{C}$, $R(z_0) = \infty$ if and only if $Q(z_0) = 0$. If z_0 is a zero of Q of order n_0, then we say z_0 is a pole of R of **order** n_0.

We can write down $R(\infty)$ very explicitly. Suppose $\deg P = n$ and $\deg Q = m$, and

$$R(z) = \frac{P(z)}{Q(z)} = \frac{a_n z^n + \cdots + a_1 z + a_0}{b_m z^m + \cdots + b_1 z + b_0}.$$

Then we know that

$$R\left(\frac{1}{w}\right) = \frac{a_n w^{-n} + \cdots + a_1 w^{-1} + a_0}{b_m w^{-m} + \cdots + b_1 w^{-1} + b_0} = w^{m-n} \cdot \frac{a_n + \cdots + a_1 w^{n-1} + a_0 w^n}{b_m + \cdots + b_1 w^{m-1} + b_0 w^m},$$

so we can conclude that

$$R(\infty) = \lim_{w \to 0} R\left(\frac{1}{w}\right) = \begin{cases} 0 & \text{if } m > n \text{ (so } R \text{ has a zero of order } m-n) \\ \frac{a_n}{b_m} & \text{if } m = n \\ \infty & \text{if } m < n \text{ (so } R \text{ has a pole of order } n-m) \end{cases}.$$

The reason why this information is useful is as follows. We let $d := \max\{m, n\}$, called the **degree** of R. Then for any $y_0 \in \hat{\mathbb{C}}$, we have

$$\#\left\{z \in \hat{\mathbb{C}} : R(z) = y_0\right\} = d.$$

For example, when $y_0 = \infty$, we know that $R(z) = \infty$ if either $z \in \mathbb{C}$ and $Q(z) = 0$, and we get m solutions, or $z = \infty$, which happens only when $m < n$, and we get $n - m$ solutions. Thus, there are $\max\{m, n\} = d$ solutions. The case for a general y_0 is similar.

[6] It does have some geometric meaning, and as a Riemann surface, $\hat{\mathbb{C}}$ is an important model space, cf. Section 5.5.