Weil II, perverse sheaves

Kevin Chang

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1 Derived categories of ℓ-adic sheaves

Before we can say anything about ℓ-adic sheaves or complexes of ℓ-adic sheaves, we have to specify the category we're working in. The abelian category of $\overline{\mathbb{Q}}_\ell$-sheaves isn't too hard to define. Let X be a variety over a field k. Given a finite extension $E/\mathbb{Q}_\ell$, an $\mathcal{O}_E$-sheaf $\mathcal{F}$ is an inverse system of étale sheaves $\mathcal{F}_r$ on X, where each $\mathcal{F}_r$ is a Λ_r-sheaf (here, $\Lambda_r = \mathcal{O}_E/(\pi^r)$). We get the category of E-sheaves from the category of $\mathcal{O}_E$-sheaves by inverting ℓ, and we get the category of $\overline{\mathbb{Q}}_\ell$-sheaves by taking the direct limit of the categories of E-sheaves over all E.

Since we have a lot of derived functors, we need a triangulated category of complexes to work with. A naïve guess would be to work with the derived category of the abelian category of $\overline{\mathbb{Q}}_\ell$-sheaves. However, this doesn't work because this abelian category doesn't have enough injectives or even acyclic objects, meaning that we can't define the derived functors we want like f_*.

The correct category $D^b_c(X, \overline{\mathbb{Q}}_\ell)$ is defined as follows. We fix a finite extension $E/\mathbb{Q}_\ell$. We start with the categories $D^b_c(X, \Lambda_r)$ consisting of complexes of étale Λ_r-sheaves on X with bounded constructible cohomology. We restrict to the subcategories $D^b_{ctf}(X, \Lambda_r)$ consisting of complexes with finite Tor-dimension. Then we can define the derived category of $\mathcal{O}_E$-sheaves $D^b_c(X, \mathcal{O}_E) := \varprojlim_r D^b_{ctf}(X, \Lambda_r)$, where the $\varprojlim_r$ is a 2-limit, i.e. an element $K \in D^b_c(X, \mathcal{O}E)$ consists of a sequence $K_r \in D^b_{ctf}(X, \Lambda_r)$ and isomorphisms $K_{r+1} \otimes^L_{\Lambda_{r+1}} \Lambda_r \xrightarrow{\sim} K_r$. Finally, we take $D^b_c(X, \overline{\mathbb{Z}}_\ell) := \varinjlim_E D^b_c(X, \mathcal{O}_E)$ and $D^b_c(X, \overline{\mathbb{Q}}_\ell) := D^b_c(X, \overline{\mathbb{Z}}_\ell)[\ell^{-1}]$.

The derived categories $D^b_c(X, \overline{\mathbb{Q}}_\ell)$ carry the usual six functors $f_*, f^*, f_!, f^!, \otimes^L, R\mathcal{H}om$ and a Verdier duality operation $\mathbb{D} := R\mathcal{H}om(-, a^!\overline{\mathbb{Q}}_\ell)$, where a is the map $X \to \operatorname{Spec} k$.

2 Weil II

We will now specify to varieties over a finite field $k := \mathbb{F}_q$. Given a variety X_0/k, we will use X to denote the base change of X_0 to $\overline{k}$. Given a $\overline{\mathbb{Q}}_\ell$-sheaf $\mathcal{F}_0$ on X_0, we have the all-important **Grothendieck trace formula**:

$$\sum_{x \in X_0(k)} \operatorname{tr} F_x | \mathcal{F} = \sum_{i=0}^{2 \dim X} (-1)^i \operatorname{tr} F | H^i_c(X, \mathcal{F}).$$

If we consider all the base changes of X_0 to finite extensions of k, the trace formula lets us define the L-function of $\mathcal{F}_0$:

$$L(X_0, \mathcal{F}_0, t) := \prod_{x \in |X_0|} \det(1 - t^{d(x)} F_x, \mathcal{F}) = \prod_{i=0}^{2 \dim X} \det(1 - tF, H^i_c(X, \mathcal{F}))^{(-1)^{i+1}}.$$

The trace formula illustrates an important local-to-global principle: the (local) pointwise Frobenius actions give us information about the (global) Frobenius action on the compactly supported cohomology. Deligne's work on the Riemann hypothesis over finite fields, which is vastly generalized in Weil II, is another example of this principle: controlling the pointwise Frobenius eigenvalues gives us control of the Frobenius eigenvalues on cohomology.

To state the main theorem of Weil II, we'll need to review some terminology. A $\overline{\mathbb{Q}}_\ell$-sheaf $\mathcal{F}_0$ on X_0 is **(pointwise) pure** of weight w if for all closed points $x \in |X_0|$ and all isomorphisms $\overline{\mathbb{Q}}_\ell \cong \mathbb{C}$, the eigenvalues of F_x on $\mathcal{F}_{\overline{x}}$ get taken to complex numbers of absolute value $q^{w/2}$. A $\overline{\mathbb{Q}}_\ell$-sheaf is **mixed** of weight $\leqslant w$ if it admits a finite filtration by pure sheaves of weight $\leqslant w$. Some examples of pure sheaves (of weight 0) are lisse sheaves whose eigenvalues are roots of unity, e.g. the constant sheaf, Kummer sheaves, Artin-Schreier sheaves. Mixedness arises when we take cohomology of pure sheaves, thanks to the following theorem of Deligne.

Theorem 1 (Main theorem of Weil II)**.** Let $f_0 : X_0 \to Y_0$ be a map of varieties over k. If $\mathcal{F}_0$ is a pure sheaf on X_0 of weight w, then for each i, $R^i f_{0!}\mathcal{F}$ is mixed of weight $\leqslant w + i$. The weights that appear differ from w by an integer.

When $Y_0 = \operatorname{Spec} k$ and $\mathcal{F}_0 = \overline{\mathbb{Q}}_\ell$, this says that the Frobenius eigenvalues of $H^i_c(X, \overline{\mathbb{Q}}_\ell)$ have absolute value at most $q^{i/2}$. If X_0 is smooth and proper, then Poincaré duality says that the Frobenius eigenvalues of $H^i(X, \overline{\mathbb{Q}}_\ell)$ and $H^{2n-i}(X, \overline{\mathbb{Q}}_\ell)$ pair to q^n ($n = \dim X$), so that the Frobenius eigenvalues on $H^i(X, \overline{\mathbb{Q}}_\ell)$ have absolute value exactly $q^{i/2}$ (this is the Riemann hypothesis).

We can state this theorem in terms of complexes. A complex $K \in D^b_c(X_0, \overline{\mathbb{Q}}_\ell)$ is **mixed** of weight $\leqslant w$ if the cohomology sheaves $H^i K$ are mixed of weight $\leqslant w + i$. We also say that K is mixed of weight $\geqslant w$ if $\mathbb{D}K$ is mixed of weight $\leqslant -w$ and **pure** of weight w if K is mixed of weight $\leqslant w$ and of weight $\geqslant w$. Be careful: the cohomology sheaves of a pure complex might not be pointwise pure!

We let $D^b_m(X_0, \overline{\mathbb{Q}}_\ell) \subset D^b_c(X_0, \overline{\mathbb{Q}}_\ell)$ denote the full subcategory of mixed complexes, and we let $D^b_{\leqslant w}(X_0, \overline{\mathbb{Q}}_\ell)$ (resp. $D^b_{\geqslant w}(X_0, \overline{\mathbb{Q}}_\ell)$) denote the full subcategory of mixed complexes of weight $\leqslant w$ (resp. of weight $\geqslant w$). These subcategories are stable with respect to our usual operations:

- $f_{0!}, f_0^* : D^b_{\leqslant w} \to D^b_{\leqslant w}$: $f_{0!}$ is the main theorem of Weil II; f_0^* is because pullbacks preserve weights.
- $\mathbb{D} : D^b_{\leqslant w} \leftrightarrow D^b_{\geqslant w}$: By definition.
- $f_{0*}, f_0^! : D^b_{\geqslant w} \to D^b_{\geqslant w}$: Use the fact that $f_{0*} = \mathbb{D} f_{0!} \mathbb{D}$ and $f_0^! = \mathbb{D} f_{0!} \mathbb{D}$.

For an analytic number theorist, weights are useful for bounding things that can be expressed in terms of traces of Frobenii (e.g. exponential sums). However, even if you're allergic to bounds, you can get us some really nice stuff from purity.

Theorem 2. If $\mathcal{F}_0$ is a pointwise pure lisse sheaf on a normal variety X_0, then the base change $\mathcal{F}$ is semisimple, i.e. $\mathcal{F}_0$ is geometrically semisimple.

Corollary 1 (Decomposition theorem for smooth proper maps)**.** Let S be a normal variety over an algebraically closed field of characteristic $p \neq \ell$, and let $f : X \to S$ be a smooth proper map. Then the lisse sheaves $R^i f_* \mathbb{Q}_\ell$ are semisimple.

By applying this corollary to a Lefschetz pencil for a smooth projective variety X (and knowing a bit about Lefschetz pencils), we get hard Lefschetz.

Corollary 2 (Hard Lefschetz for étale cohomology)**.** If X is a smooth projective variety of dimension n over an algebraically closed field and $\eta \in H^2(X, \mathbb{Q}_\ell)$ is the hyperplane class, then $- \smile \eta^i : H^{n-i}(X, \mathbb{Q}_\ell) \to H^{n+i}(X, \mathbb{Q}_\ell)$ is an isomorphism.

3 Perverse sheaves

For a variety X/k, **perverse $\overline{\mathbb{Q}}_\ell$-sheaves** are objects $K \in D^b_c(X, \overline{\mathbb{Q}}_\ell)$ satisfying the following conditions for all i:

- (Support) $\dim \operatorname{supp} H^{-i}K \leqslant i$
- (Co-support) $\dim \operatorname{supp} H^{-i}\mathbb{D}K \leqslant i$

The support and co-support conditions actually correspond to being bounded above by 0 and below by 0, respectively, in the **perverse t-structure**, so that perverse sheaves form the heart of the perverse t-structure. In particular, perverse sheaves form an abelian category. We'll denote the truncation functors by ${}^p\tau_{\leqslant i}, {}^p\tau_{\geqslant i}$, and we'll denote the perverse cohomology functor by ${}^pH^i := {}^p\tau_{\leqslant i}\, {}^p\tau_{\geqslant i}$.

If $i : Y \hookrightarrow X$ is a smooth closed subvariety of dimension d and $\mathcal{L}$ is a lisse sheaf on Y, then $i_*\mathcal{L}[d]$ is a perverse sheaf. In particular, if X is smooth of dimension n, then any shifted lisse sheaf $\mathcal{L}[n]$ is perverse.

To get perverse sheaves on non-smooth varieties X, we use an operation called the **intermediate extension**. For an open embedding $j : U \hookrightarrow X$ and a perverse sheaf K on U, the intermediate extension $j_{!*}K$ is defined as $\operatorname{im}({}^pH^0 j_! K \to {}^pH^0 K)$, which is the same as $\operatorname{im}({}^p\tau_{\geqslant 0} j_! K \to {}^p\tau_{\leqslant 0} j_* K)$ because $j_!$ (resp. j_*) is perverse t-right exact (resp. t-left exact) (to see this, note that $j_!$ doesn't increase supports). For a locally closed embeddding $h : V \hookrightarrow X$ factoring as $h = i \circ j$, we define $h_{!*} := i_* j_{!*}$. If V is smooth of dimension d and $\mathcal{L}$ is a lisse sheaf on V, then the perverse sheaf we get from intermediate extension is called an **intersection complex**: $IC(V, \mathcal{L}) := h_{!*}\mathcal{L}[d]$. In particular, for a smooth open subvariety $j : U \hookrightarrow X$, we have the intersection complex of X: $IC_X := j_{!*}\overline{\mathbb{Q}}_\ell[n]$; its hypercohomology is called the **intersection cohomology** of X.

It can be shown that the abelian category of perverse sheaves is noetherian and artinian and that the intersection complexes with $\mathcal{L}$ simple are the simple objects. Thus, all perverse sheaves are built out of lisse sheaves on smooth subvarieties. Perverse sheaves are useful because they enjoy a lot of the same properties as lisse sheaves on smooth varieties while being applicable when there is no smoothness.

4 Weil II and perverse sheaves

We return to the situation over a finite field k. Since perverse sheaves are objects of $D^b_c(X_0, \overline{\mathbb{Q}}_\ell)$, it makes sense to talk about mixed and pure perverse sheaves. The next couple results are direct generalizations of what we saw earlier for lisse sheaves on smooth varieties.

Theorem 3. A pure perverse sheaf is geometrically semisimple.

If $f_0 : X_0 \to S_0$ is a proper map, then $f_{0!} = f_{0*}$, so that f_{0*} takes pure complexes to pure complexes. This gets us the following corollary.

Corollary 3 (Decomposition theorem over a finite field)**.** Let $f_0 : X_0 \to Y_0$ be a proper map, and let K_0 be a pure perverse sheaf on X_0. Then f_*K is a direct sum of shifted simple perverse sheaves on Y_0.