The Swan conductor and the Grothendieck-Ogg-Shafarevich formula

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1 Review of ramification theory

Let $\mathcal{O}_K$ be a henselian DVR of characteristic $p > 0$ with fraction field K and residue field k. Then there is a tower $K^{\mathrm{sep}} \supset K^{\mathrm{tr}} \supset K^{\mathrm{ur}} \supset K$, where K^{ur} is the maximal unramified extension of K and K^{tr} is the maximal tamely ramified extension (i.e. the degree of any finite subextension of $K^{\mathrm{tr}}/K^{\mathrm{ur}}$ is coprime to p). We have the following description of Galois groups:

$$\begin{aligned}\mathrm{Gal}(K^{\mathrm{ur}}/K) &\cong \mathrm{Gal}(k^{\mathrm{sep}}/k)\\ \mathrm{Gal}(K^{\mathrm{tr}}/K^{\mathrm{ur}}) &\cong \prod_{\ell \neq p} \mathbb{Z}_\ell(1).\end{aligned}$$

We denote the inertia group $I := \mathrm{Gal}(K^{\mathrm{sep}}/K^{\mathrm{ur}})$ and the wild inertia group $P := \mathrm{Gal}(K^{\mathrm{sep}}/K^{\mathrm{tr}})$.

Since we only care about ramification, we'll assume that k is separably closed, so that $\mathcal{O}_K$ is strictly henselian. Let $v : K \to \mathbb{Z}$ be the valuation on K. Let L/K be a finite Galois extension with Galois group G. We can define a decreasing filtration $G = G_0 \supset G_1 \supset G_2 \supset \cdots$ as follows: $G_i = \{\sigma \in G | v(\sigma(\pi_L) - \pi_L) \geqslant i + 1\}$. Any $\sigma \in G_0$ sends π_L to a unit multiple $u\pi_L$; the map $\sigma \mapsto u \pmod{\pi_L}$ defines an injective homomorphism $G_0/G_1 \hookrightarrow k^\times$. Similarly, any $\sigma \in G_i$ sends π_L to $\pi_L + a\pi_L^{i+1}$; the map $\sigma \mapsto a \pmod{\pi_L}$ defines an injective homomorphism $G_i/G_{i+1} \hookrightarrow k$. Hence, G_1 is a maximal p-subgroup of G a.k.a. the wild inertia subgroup.

2 The Swan conductor

Let X be a smooth projective curve with function field K over an algebraically closed field k, and let $U \subset X$ be a nonempty open subset. Let $\mathbb{F}$ be a finite field of characteristic $\ell \neq p$. Given an $\mathbb{F}$-local system $\mathcal{F}$ on U, we get an $\mathrm{Gal}(K^{\mathrm{sep}}/K)$-representation M by restricting to the generic point, which factors through some finite $\mathrm{Gal}(L/K)$. For every closed point $x \in U$, the inertia group I_x acts trivially on M. Now fix a closed point $x \in X - U$, and let $G = I_x$.

Theorem 1. The rational number

$$\sum_{i \geqslant 1} \frac{1}{[G_0 : G_i]} \dim_{\mathbb{F}}(M/M^{G_i})$$

is independent of the choice of L and is an integer.

Remark 1. We can also define the Swan conductor using the upper numbering G^v, defined as follows. First we say that for $u \in \mathbb{R}_{\geqslant 0}$, $G_u := G_{\lceil u \rceil}$. We then define the unique continuous piecewise linear function $\phi : \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$ such that $\phi(0) = 0$ and $\phi'(u) = (G_0 : G_u)^{-1}$. The upper numbering is defined by $G^v := G_{\phi^{-1}(v)}$. The set of **jumps** of G is defined as the image $\phi(\mathbb{N}) \subset \mathbb{R}_{\geqslant 0}$, and we can define $\mathrm{Swan}_x(\mathcal{F}) := \sum_v v \dim_{\mathbb{F}}(M^{G^v}/M^{G^{>v}})$, where v ranges over the jumps.

The convenient thing about the upper numbering is that for a normal subgroup $N \subset G$, $(G/N)^v = \mathrm{im}(G^v \to G/N)$, so that we can define the upper numbering on the entire Galois group $\mathrm{Gal}(K_x^{\mathrm{sep}}/K_x)$. Also, there's the Hasse-Arf theorem, which states that for G abelian, the jumps are integers.

This integer is called the **Swan conductor** of $\mathcal{F}$ at x and is denoted $\mathrm{Swan}_x(\mathcal{F})$. Note that

- $\mathrm{Swan}_x(\mathcal{F}) = 0$ iff the wild inertia subgroup acts trivially, i.e. $\mathcal{F}$ is tamely ramified at x.
- For $\mathcal{F}$ tamely ramified at x and $\mathcal{G}$ another local system on some open subset of X, $\mathrm{Swan}_x(\mathcal{F} \otimes \mathcal{G}) = \mathrm{Swan}_x(\mathcal{G}) \cdot \mathrm{rk}\,\mathcal{F}$

Given a $\overline{\mathbb{Q}}_\ell$-local system $\mathcal{F}$ on U with fiber V, we get a corresponding representation $I_x \to \mathrm{GL}(V)$ defined over some finite extension $\mathcal{O}_E/\mathbb{Z}_\ell$ (here, $V = F_0 \otimes_{\mathcal{O}_E} \overline{\mathbb{Q}}_\ell$ for some free $\mathcal{O}_E$-module F_0). We have an action of I_x on the $\mathbb{F}$-vector space $M_0 := F_0/\pi_E F_0$. We define $\mathrm{Swan}_x(\mathcal{F}) := \mathrm{Swan}_x(M_0)$. It's not too hard to show that this is independent of choice of F_0.

Example 1. (1) For a finite field $\mathbb{F}_q$, the **Kummer cover** is the finite étale cover $\mathbb{G}_{m0} \to \mathbb{G}_{m0}$ given by taking $(q-1)$th powers. It is Galois with Galois group $\mathbb{F}_q^\times$. For a multiplicative character $\chi : \mathbb{F}_q^\times \to \mathbb{C}^\times$, we can define the **Kummer sheaf** $\mathcal{L}(\chi)$ on $\mathbb{G}_{m0}$. Since the Kummer cover is tamely ramified above 0 and ∞ (it has degree $q-1$), $\mathrm{Swan}_0(\mathcal{L}(\chi)) = \mathrm{Swan}_\infty(\mathcal{L}(\chi)) = 0$.

(2) The **Artin-Schreier cover** is the finite étale cover $\mathbb{A}^1_0 \to \mathbb{A}^1_0$ given by $x \mapsto x^q - x$ with Galois group $\mathbb{F}_q$. Given an additive character $\psi : \mathbb{F}_q \to \mathbb{C}^\times$, we can define the **Artin-Schreier sheaf** $\mathcal{L}(\psi)$. For ψ nontrivial, we can show that $\mathrm{Swan}_\infty(\mathcal{L}(\psi)) = 1$ as follows. We identify $\mathbb{F}_q \cong \mathrm{Gal}(\overline{\mathbb{F}}_q[[u^{-1}]]/\overline{\mathbb{F}}_q[[t^{-1}]])$, where $u^q - u = t$. Now u^{-1} is a uniformizer of the big field, and for $a \in \mathbb{F}_q$, the corresponding automorphism sends $u^{-1} \mapsto (u+a)^{-1} = u^{-1}(1 + au^{-1})^{-1}$, so that $(u+a)^{-1} - u^{-1} = -au^{-2} + O(u^{-3})$. Thus, $G_1 = G$, and $G_2 = 0$. Since ψ is nontrivial, there are no G_1-fixed points, and everything is G_2-fixed. Thus, the terms in the above sum are 0 except for $i = 1$, so we get $\mathrm{Swan}_\infty(\mathcal{L}(\psi)) = 1$.

3 The Grothendieck-Ogg-Shafarevich formula

Let $\mathcal{F}$ be a $\overline{\mathbb{Q}}_\ell$-local system on U. The Grothendieck-Ogg-Shafarevich formula is an index formula for the Euler characteristic $\chi_c(U, \mathcal{F})$ (it can be shown that $\chi_c(U, \mathcal{F}) = \chi(U, \mathcal{F})$).

Theorem 2 (Grothendieck-Ogg-Shafarevich)**.**

$$\chi_c(U, \mathcal{F}) = \chi_c(U)\,\mathrm{rk}\,\mathcal{F} - \sum_{x \in X - U} \mathrm{Swan}_x(\mathcal{F})$$

4 Applications to exponential sums

As Deligne noted very early on, étale cohomology is extremely useful for bounding exponential sums, since exponential sums can often naturally be expressed as sums of traces of Frobenius.

4.1 Gauss sums

We'll start with a classical example. Fix a multiplicative character $\chi : \mathbb{F}_q^\times \to \mathbb{C}^\times$ and a nontrivial additive character $\psi : \mathbb{F}_q \to \mathbb{C}^\times$. The **Gauss sum** $G(\chi, \psi)$ is defined as the sum $\sum_{a \in \mathbb{F}_q^\times} \chi(a)\psi(a)$. This can also be thought of as the sum over $\mathbb{G}_m(\mathbb{F}_q)$ of the traces of Frobenius of the rank 1 local system $\mathcal{F} := \mathcal{L}(\chi^{-1}) \otimes \mathcal{L}(\psi^{-1})$, which is $\sum_{i=0}^2 (-1)^i \operatorname{tr} \operatorname{Fr} H_c^i(\mathbb{G}_m, \mathcal{F})$ by the Grothendieck trace formula. By the properties in Section 2, we have $\mathrm{Swan}_0(\mathcal{F}) = 0$ and $\mathrm{Swan}_\infty(\mathcal{F}) = 1$. Since $\chi_c(\mathbb{G}_m) = 0$, the GOS formula tells us that $\chi_c(\mathbb{G}_m, \mathcal{F}) = -1$. Since $\mathbb{G}_m$ is non-proper, $H_c^0(\mathbb{G}_m, \mathcal{F}) \cong 0$. Since $\mathcal{F}^\vee \cong \mathcal{L}(\chi) \otimes \mathcal{L}(\psi)$ is nontrivial, $H_c^2(\mathbb{G}_m, \mathcal{F}) \cong H^0(\mathbb{G}_m, \mathcal{F}^\vee)^\vee \cong 0$. Thus, $H_c^1(\mathbb{G}_m, \mathcal{F})$ is 1-dimensional. Similarly, $H^1(\mathbb{G}_m, \mathcal{F})$ is 1-dimensional.

Consider $j : \mathbb{G}_m \hookrightarrow \mathbb{P}^1$. We have a distinguished triangle $j_!\mathcal{F} \to Rj_*\mathcal{F} \to K$ of complexes on $\mathbb{P}^1$, where K is supported at $\{0, \infty\}$. Since the map $j_!\mathcal{F} \to j_*\mathcal{F}$ is an isomorphism ($\mathcal{F}$ is a nontrivial rank 1 local system), $\mathcal{H}^0 K \cong 0$, so that the map $H_c^1(\mathbb{G}_m, \mathcal{F}) \to H^1(\mathbb{G}_m, \mathcal{F})$ is an injection and thus an isomorphism. By Weil II, since $\mathcal{F}$ is pure of weight 0, we know that $H_c^1(\mathbb{G}_m, \mathcal{F})$ is mixed of weight $\leqslant 1$ and that $H^1(\mathbb{G}_m, \mathcal{F})$ is mixed of weight $\geqslant 1$ (by Poincaré duality). Our isomorphism tells us that both groups are pure of weight 1, so that the Gauss sum $G(\chi, \psi) = \sum_{i=0}^2 (-1)^i \operatorname{tr} \operatorname{Fr} H_c^i(\mathbb{G}_m, \mathcal{F}) = -\operatorname{tr} \operatorname{Fr} H_c^1(\mathbb{G}_m, \mathcal{F})$ has absolute value $q^{1/2}$, as is classically known about Gauss sums.

4.2 Kloosterman sums

Again fix a nontrivial additive character $\psi : \mathbb{F}_q \to \mathbb{C}^\times$. For $n \geqslant 1$ and $a \in \mathbb{F}_q$, the **Kloosterman sums** are defined

$$K_{n,a} := \sum_{x_1 \cdots x_n = a} \psi(x_1 + \cdots + x_n).$$

We can easily compute $K_{n,0} = (-1)^{n-1}$, but the other Kloosterman sums are more mysterious. The trivial bound is $|K_{n,a}| \leqslant (q-1)^{n-1}$, but Deligne improved this to $|K_{n,a}| \leqslant nq^{\frac{n-1}{2}}$ using étale cohomology.

We consider the Kloosterman manifolds $V_a^{n-1} = \{x_1 \cdots x_n = a\} \subset \mathbb{A}^n$. There is a map $\sigma : \mathbb{A}^n \to \mathbb{A}^1$ sending $(x_1, \ldots, x_n) \mapsto x_1 + \cdots + x_n$ and a map $\pi : \mathbb{A}^n \to \mathbb{A}^1$ sending $(x_1, \ldots, x_n) \mapsto x_1 \cdots x_n$. $K_{n,a}$ is the sum over $V_a^{n-1}(\mathbb{F}_q)$ of the traces of Frobenius of the pullback $\sigma^*\mathcal{L}(\psi^{-1})$. Since it doesn't matter whether we use ψ or ψ^{-1}, we'll set $\mathcal{F} = \mathcal{F}(\psi\sigma) := \sigma^*\mathcal{L}(\psi)$ for convenience. Deligne's estimate on Kloosterman sums follows from (a) and (c) of the following geometric result, along with Weil II.

Theorem 3. $H_c^*(V_a^{n-1}, \mathcal{F})$ satisfies the following:

(a) $H_c^i \cong 0$ for $i \neq n-1$.

(b) $H_c^* \xrightarrow{\sim} H^*$.

(c) For $a \neq 0$, $\dim H_c^{n-1} = n$.

(d) For $a = 0$, $\dim H_c^{n-1} = 1$.

This theorem is proven by a simultaneous induction on n with the following theorem on the pushforward $R^{n-1}\pi_!\mathcal{F}$.

Theorem 4. (a) $R^{n-1}\pi_!\mathcal{F}|_{\mathbb{G}_m}$ is lisse of rank n.

(b) The extension by 0 to $\mathbb{P}^1$ is the same as the direct image sheaf.

(c) $\mathrm{Swan}_0(R^{n-1}\pi_!\mathcal{F}) = 0$; the monodromy around 0 is unipotent with a single Jordan block.

(d) $\mathrm{Swan}_\infty(R^{n-1}\pi_!\mathcal{F}) = 1$; the wild inertia has no fixed points.

(e) $R^i\pi_!\mathcal{F} \xrightarrow{\sim} R^i\pi_*\mathcal{F}$

We take for granted that Theorem 3 for a given n implies Theorem 4 for n (the proof is in SGA 4.5, Section 7 of the chapter on trigonometric sums). The cases $n = 1$ and $a = 0$ aren't too hard, so we'll omit them. It remains to show that Theorem 4 for n implies Theorem 3 for $n + 1$ for nonzero a.

Let $V_a^n \subset \mathbb{A}^{n+1}$ have coordinates $x_0, \dots, x_n$, and let $g : V_a^n \to \mathbb{G}_m$ be the projection to the first coordinate, and let $h : V_a^n \to \mathbb{G}_m^n$ be the projection to the last n coordinates (h is an isomorphism). By the projection formula, $Rg_*\mathcal{F}(\psi\sigma_{n+1}) \simeq Rg_*(g^*\mathcal{L}(\psi) \otimes h^*\mathcal{F}(\psi\sigma_n)) \simeq \mathcal{L}(\psi) \otimes Rg_*h^*\mathcal{F}(\psi\sigma_n)$. Note that g factors as $V_a^n \xrightarrow[\sim]{h} \mathbb{G}_m^n \xrightarrow{\pi_n} \mathbb{G}_m \xrightarrow[\sim]{\tau} \mathbb{G}_m$, where $\tau(x) = ax^{-1}$. Thus, $Rg_*\mathcal{F}(\psi\sigma_{n+1}) \simeq \mathcal{L}(\psi) \otimes \tau_*R\pi_*\mathcal{F}(\psi\sigma_n)$. The same thing is true for lower shriek: $Rg_!\mathcal{F}(\psi\sigma_{n+1}) \simeq \tau_*R\pi_!\mathcal{F}(\psi\sigma_n)$. We have Leray spectral sequences

$$\begin{aligned} {}^!E_2^{pq} &= H_c^p(\mathbb{G}_m, \mathcal{L}(\psi) \otimes \tau_*R^q\pi_!\mathcal{F}(\psi\sigma_n)) \implies H_c^{p+q}(V_a, \mathcal{F}(\psi\sigma_{n+1})) \\ {}^*E_2^{pq} &= H^p(\mathbb{G}_m, \mathcal{L}(\psi) \otimes \tau_*R^q\pi_*\mathcal{F}(\psi\sigma_n)) \implies H^{p+q}(V_a, \mathcal{F}(\psi\sigma_{n+1})) \end{aligned}$$

By the inductive hypothesis, $R^q\pi_!\psi\mathcal{F}(\psi\sigma_n) \cong 0$ unless $q = n-1$, so the only possible nonzero terms are ${}^!E_2^{p(n-1)}$ for $p \in \{0, 1, 2\}$. Moreover, the inductive hypothesis implies that $R^q\pi_!\psi\mathcal{F}(\psi\sigma_n) \to R^q\pi_*\psi\mathcal{F}(\psi\sigma_n)$ is an isomorphism, so the same thing applies to ${}^*E_2^{pq}$. Since $\mathbb{G}_m$ is non-proper, ${}^!E_2^{0q} \cong 0$, so ${}^*E_2^{0q} \cong 0$. By Poincaré duality (apply everything we've said to ψ^{-1}), ${}^!E_2^{2(n-1)} \cong 0$. Thus, the only possible nonzero term is ${}^!E_2^{1(n-1)} \cong {}^*E_2^{1(n-1)}$.

We can compute $\mathrm{Swan}_0(\mathcal{L}(\psi)\otimes\tau_*R^{n-1}\pi_!\mathcal{F}(\psi\sigma_n)) = 1$ and $\mathrm{Swan}_\infty(\mathcal{L}(\psi)\otimes\tau_*R^{n-1}\pi_!\mathcal{F}(\psi\sigma_n)) = n$. Thus, GOS tells us that $\chi_c(\mathbb{G}_m, \mathcal{L}(\psi)\otimes\tau_*R^{n-1}\pi_!\mathcal{F}(\psi\sigma_n)) = -n-1$. Since there is only one nonzero cohomology group, we see that $H_c^n(V_a^n, \mathcal{F}(\psi\sigma_n)) \cong H_c^1(\mathbb{G}_m, \mathcal{L}(\psi)\otimes\tau_*R^{n-1}\pi_!\mathcal{F}(\psi\sigma_n))$ has dimension n, as desired.