Functional Analysis and Its Applications, Vol. 41, *No.* 4, *pp.* 284–294, 2007
Translated from Funktsional′nyi Analiz i Ego Prilozheniya, Vol. 41, *No.* 4, *pp.* 46–59, 2007
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Lax Operator Algebras*

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Received February 28, 2007

Abstract. In this paper, we develop the general approach, introduced in [1], to Lax operators on algebraic curves. We observe that the space of Lax operators is closed with respect to their usual multiplication as matrix-valued functions. We construct orthogonal and symplectic analogs of Lax operators, prove that they form almost graded Lie algebras, and construct local central extensions of these Lie algebras.

Key words: Lax operator, current algebra, Tyurin data, almost graded structure, local central extension.

1. Introduction

A general approach to Lax operators on algebraic curves was proposed by one of the authors in [1], where the conventional theory of Lax and zero curvature representations with a *rational spectral parameter* was generalized to the case of algebraic curves Γ of arbitrary genus g. The linear space of such operators associated with an effective divisor $D = \sum_k n_k P_k$, $P_k \in \Gamma$, was defined as the space of meromorphic $(n \times n)$ matrix functions on Γ having poles of multiplicity at most n_k at the points P_k and at most simple poles at ng more points γ_s. The coefficients of the Laurent expansion of these matrix functions in a neighborhood of each point γ_s had to obey certain linear constrains parametrized by the point α_s of a projective space (see relations (2.1)–(2.3) below).

According to [12], generic sets (γ_s, α_s) parameterize stable rank n degree ng framed holomorphic vector bundles B on Γ. It was noted in [1] that the requirements on the form of Lax operators at the points γ_s mean that these operators can be viewed as meromorphic sections of the bundle $\mathrm{End}(B)$ with pole divisor D. It is an easy consequence of this remark that Lax operators having poles of arbitrary orders at the points P_k form an algebra with respect to the usual pointwise multiplication.

In the simplest case of two marked points, $D = P_+ + P_-$, this enables us to equip the algebra of the corresponding operators with an almost graded structure generalizing the graded structure of the classical affine algebra $\widehat{\mathfrak{gl}(n)}$. Recall that a Lie algebra $\mathscr{V}$ is said to be *almost graded* if $\mathscr{V} = \bigoplus \mathscr{V}_i$, where $\dim \mathscr{V}_i < \infty$, $[\mathscr{V}_i, \mathscr{V}_j] \subseteq \bigoplus_{k=i+j-k_0}^{k=i+j+k_1} \mathscr{V}_k$, and k_0 and k_1 are independent of i and j.

The general notion of almost graded algebras and modules over them was introduced in [3]-[5], where generalizations of the Heisenberg and Virasoro algebras were introduced. In a number of papers, whose survey can be found in [11], the almost graded analogs of classical affine Lie algebras, called Krichever–Novikov current algebras, were investigated. It is natural to treat the algebra of Lax operators having poles at two points as a generalization of the Krichever–Novikov $\mathfrak{gl}(n)$-algebra.

A central extension of $\mathscr{V}$ is said to be *local* if it is an almost graded Lie algebra itself. Local central extensions are given by local 2-cocycles. A 2-cocycle γ is said to be local if there exists a $K \in \mathbb{Z}$ such that $\gamma(\mathscr{V}_i, \mathscr{V}_j) = 0$ for $|i + j| > K$. The notion of local cocycle is introduced in [3], where, in addition, its uniqueness was conjectured and the proof for Virasoro type algebras

*Research is supported in part by the National Science Foundation (USA) under grant DMS-04-05519 (I. M. Krichever), by the Russian Foundation for Basic Research under grant no. 05-01-00170, and by the program "Mathematical Methods in Nonlinear Dynamics" of the Russian Academy of Sciences (O. K. Sheinman).

 0016–2663/07/4104–0284

was outlined; a complete proof is given in [9] and [10]. The locality condition is important when considering analogs of highest weight representations.

To construct the orthogonal and symplectic analogs of Lax operators is the main goal of the present paper. In these cases, the Lax operators do not form an associative algebra; they only form a Lie algebra. For all classical Lie algebras $\mathfrak{g}$, the corresponding Lax operator algebras can be viewed as a “twisted” version of the Krichever-Novikov current algebras and loop algebras.

In Sec. 2, we prove the multiplicative properties of $\mathfrak{gl}(n)$-valued Lax operators, introduce $\mathfrak{g}$-valued Lax operators for $\mathfrak{g}=\mathfrak{so}(n)$ and $\mathfrak{g}=\mathfrak{sp}(2n)$, and prove that they are closed with respect to the pointwise bracket.

In Sec. 3, we define an almost graded structure on Lax operator algebras and show that $\dim \mathscr{V}_i = \dim \mathfrak{g}$, as is the case for Krichever–Novikov algebras.

In Sec. 4, on every type of a Lax operator algebra we define a 2-cocycle and prove its locality.

The authors are grateful to M. Schlichenmaier for fruitful criticism.

2. Lax Operators and Their Lie Bracket

2.1. Lax operator algebras for $\mathfrak{gl}(n)$ and $\mathfrak{sl}(n)$. Following [1], we define a *Lax operator* with Tyurin parameters $\{\alpha_s,\gamma_s \mid s=1,\dots,gr\}$ as a $\mathfrak{gl}(n)$-valued function L on Γ that is holomorphic outside $P_\pm$ and $\{\gamma_s \mid s=1,\dots,gr\}$ and has at most simple poles at the points γ_s, i.e., satisfies

$$L=\frac{L_{s,-1}}{z-z_s}+L_{s0}+O(z-z_s),\qquad z_s=z(\gamma_s); \tag{2.1}$$

moreover,

(i) $L_{s,-1}=\alpha_s\beta_s^t$ and

$$\operatorname{tr} L_{s,-1}=\beta_s^t\alpha_s=0, \tag{2.2}$$

where $\alpha_s\in\mathbb{C}^n$ is fixed, $\beta_s\in\mathbb{C}^n$ is arbitrary, and the superscript t indicates transposition of a matrix. In particular, $L_{s,-1}$ has rank 1.

(ii) α_s is an eigenvector of the matrix L_{s0},

$$L_{s0}\alpha_s=k_s\alpha_s. \tag{2.3}$$

Lemma 2.1. *Let L' and L'' satisfy conditions* (2.1)–(2.3). *Then $L=L'L''$ satisfies these conditions as well.*

Proof. From (2.1), we have

$$L=\frac{L'_{s,-1}L''_{s,-1}}{(z-z_s)^2}+\frac{L'_{s,-1}L''_{s0}+L'_{s0}L''_{s,-1}}{(z-z_s)}+L'_{s,-1}L''_{s1}+L'_{s0}L''_{s0}+L'_{s1}L''_{s,-1}+O(1). \tag{2.4}$$

It follows from (2.2) for L' that the first term is zero,

$$L'_{s,-1}L''_{s,-1}=\alpha_s({\beta'_s}^t\alpha_s){\beta''_s}^t=0.$$

For the second term, we have $L_{s,-1}=L'_{s,-1}L''_{s0}+L'_{s0}L''_{s,-1}=\alpha_s({\beta'_s}^tL''_{s0})+(L'_{s0}\alpha_s){\beta''_s}^t$. It follows by (2.2) for L' that $L'_{s0}\alpha_s=k'_s\alpha_s$, and hence $L_{s,-1}=\alpha_s\beta_s^t$, where $\beta_s^t={\beta'_s}^tL''_{s0}+k'_s{\beta''_s}^t$. Further, $\operatorname{tr}L_{s,-1}=({\beta'_s}^tL''_{s0}+k'_s{\beta''_s}^t)\alpha_s=k''_s{\beta'_s}^t\alpha_s+k'_s{\beta''_s}^t\alpha_s=0$.

Consider the expression $L_{s,0}\alpha_s$, where $L_{s,0}=L'_{s,-1}L''_{s1}+L'_{s0}L''_{s0}+L'_{s1}L''_{s,-1}$. It follows from the definition of Lax operators that $L''_{s,-1}\alpha_s=0$ and $L'_{s0}L''_{s0}\alpha_s=k'_sk''_s\alpha_s$. We also have $L'_{s,-1}L''_{s1}\alpha_s=\alpha_s({\beta'_s}^tL''_{s1}\alpha_s)$. Hence α_s is an eigenvector of the matrix $L_{s,0}$ with eigenvalue $k_s={\beta'_s}^tL''_{s1}\alpha_s+k'_sk''_s$. □

Since conditions (2.1) and (2.2) are linear, we see that Lax operators form an associative algebra and hence the corresponding Lie algebra. The latter is called the *Lax operator algebra*.

If, along with conditions (2.1)–(2.3), the function L satisfies the condition $\operatorname{tr}L=0$, it is called an $\mathfrak{sl}(n)$*-valued Lax operator*. Such Lax operators form a Lie algebra.

2.2. Lax operator algebras for $\mathfrak{so}(n)$. For the elements of this Lie algebra, we have $X^t = -X$. We introduce a matrix function L ranging in $\mathfrak{so}(n)$ by the same expansion as in Sec. 2.1 but change condition (i), because there is no rank one skew-symmetric matrix, and accordingly change condition (ii). We omit the index s for brevity and write out the expression (2.1) in the form

$$L = \frac{L_{-1}}{z} + L_0 + O(z), \tag{2.5}$$

where $L_0, L_1, \dots$ are skew-symmetric. Instead of condition (i) in Sec. 2.1, we require that

$$L_{-1} = \alpha\beta^t - \beta\alpha^t, \tag{2.6}$$

where $\alpha \in \mathbb{C}^n$ is fixed, $\beta \in \mathbb{C}^n$ is arbitrary, and

$$\alpha^t\alpha = \beta^t\alpha (= \alpha^t\beta) = 0. \tag{2.7}$$

By analogy with (2.3), we require that

$$L_0\alpha = k\alpha \tag{2.8}$$

for some complex number k.

Let us prove that the space of Lax operators is closed with respect to the Lie bracket in the case $\mathfrak{g} = \mathfrak{so}(n)$. We point out that there is no structure of associative algebra in this case.

Lemma 2.2. *Properties* (2.5)–(2.8) *are invariant with respect to the Lie bracket.*

Proof. 1. First, let us prove the absence of the term with z^{-2}. The corresponding coefficient is equal to

$$\begin{aligned}[L_{-1}, L'_{-1}] &= [\alpha\beta^t - \beta\alpha^t, \alpha\beta'^t - \beta'\alpha^t] \\ &= (\beta^t\alpha)(\alpha\beta'^t - \beta'\alpha^t) - (\alpha^t\alpha)(\beta'\beta^t - \beta\beta'^t) - (\alpha^t\beta')(\alpha\beta^t - \beta\alpha^t).\end{aligned}$$

It vanishes by virtue of relation (2.7) (applied to both β and β'). We point out that the term with z^{-2} in the product $L_{-1}L'_{-1}$ does not vanish.

2. Now let us compute the term with z^{-1} in the product LL'. The coefficient is equal to

$$\begin{aligned}L_{-1}L'_0 + L_0L'_{-1} &= \alpha(\beta^tL'_0) - \beta(\alpha^tL'_0) + (L_0\alpha)\beta'^t - (L_0\beta')\alpha^t \\ &= \alpha(\beta^tL'_0) - \beta(-k'\alpha^t) + k\alpha\beta'^t - (L_0\beta')\alpha^t \\ &= \alpha(\beta^tL'_0 + k\beta'^t) - (L_0\beta' - k'\beta)\alpha^t.\end{aligned}$$

(Here we have used relation (2.8).) We see that it does not have the required form (2.6). Now consider the corresponding coefficient in the expansion of the commutator:

$$\begin{aligned}[L, L']_{-1} &= \alpha(\beta^tL'_0 - \beta'^tL_0 + k\beta'^t - k'\beta^t) - (L_0\beta' - L'_0\beta - k'\beta + k\beta')\alpha^t \\ &= \alpha\beta''^t - \beta''\alpha^t,\end{aligned}$$

where $\beta''^t = \beta^tL'_0 - \beta'^tL_0 + k\beta'^t - k'\beta^t$.

One can readily verify that β'' satisfies (2.7).

3. Let us verify condition (2.8) on the eigenvalue of the degree 0 matrix coefficient $(LL')_0 = L_{-1}L'_1 + L_0L'_0 + L_1L'_{-1}$.

By (2.7), $L'_{-1}\alpha = 0$; hence for the third term we have $L_1L'_{-1}\alpha = 0$.

For the first term, we have

$$L_{-1}L'_1\alpha = (\alpha\beta^t - \beta\alpha^t)L'_1\alpha = \alpha(\beta^tL'_1\alpha) - \beta(\alpha^tL'_1\alpha).$$

The last term on the right-hand side in this equation vanishes owing to the skew-symmetry of L'_1.

Thus,

$$(LL')_0\alpha = k''\alpha, \quad \text{where } k'' = \beta^tL'_1\alpha + kk'. \tag{2.9}$$

□

2.3. Lax operator algebras for $\mathfrak{sp}(2n)$. For the elements of the symplectic algebra, we have $X^t = -\sigma X\sigma^{-1}$, where σ is a nondegenerate skew-symmetric matrix.

Take the expansion for L in the form

$$L = \frac{L_{-2}}{z^2} + \frac{L_{-1}}{z} + L_0 + L_1 z + L_2 z^2 + O(z^3) \tag{2.10}$$

(again, we omit the subscript s for brevity), where $L_{-2}, L_{-1}, L_0, L_1, \dots$ are symplectic matrices and

$$L_{-2} = \nu\alpha\alpha^t, \quad L_{-1} = (\alpha\beta^t + \beta\alpha^t)\sigma \qquad (\nu \in \mathbb{C},\ \beta \in \mathbb{C}^{2n}). \tag{2.11}$$

By analogy with (2.7), we require that

$$\beta^t\sigma\alpha = 0. \tag{2.12}$$

Note that $\alpha^t\sigma\alpha = 0$ owing to the skew-symmetry of the matrix σ.

Next, we require that

$$L_0\alpha = k\alpha \tag{2.13}$$

for some complex number k.

We impose the new relation

$$\alpha^t\sigma L_1\alpha = 0. \tag{2.14}$$

Now let us prove that the space of Lax operators is closed with respect to the Lie bracket in the case $\mathfrak{g} = \mathfrak{sp}(2n)$. We point out that there is no structure of associative algebra in this case either.

Lemma 2.3. *Properties* (2.10)–(2.14) *are invariant with respect to the Lie bracket.*

Proof. Let $L'' = [L, L']$.

1. The absence of the terms of orders -4 and -3 in z in L'' follows from the relations $\beta^t\sigma\alpha = 0$ and $\alpha^t\sigma\alpha = 0$ alone.

2. For the term of order -2, we have

$$L''_{-2} = (\nu' k - \nu k' + \beta^t\sigma\beta')\alpha\alpha^t\sigma;$$

hence it has the form required by (2.11). (Here and below, ν', β', and L'_i have the same meaning for L' as ν, β, and L_i do for L.)

3. For the term of order -1, a straightforward computation using (2.10) and (2.11) gives

$$\begin{aligned} L''_{-1} = {} & \alpha(\nu\cdot\alpha^t\sigma L'_1 - \nu'\cdot\alpha^t\sigma L_1 + \beta^t\sigma L'_0 - \beta'^t\sigma L_0 + k\beta'^t\sigma - k'\beta^t\sigma) \\ & + (-\nu L'_1\alpha + \nu' L_1\alpha - L'_0\beta + L_0\beta' + k\beta' - k'\beta)\alpha^t\sigma. \end{aligned}$$

Denote the second bracket by β''. Then the relations $L_1^t = -\sigma L_1\sigma^{-1}$ and $L_0^t = -\sigma L_0\sigma^{-1}$ (which hold for symplectic matrices) imply that the first bracket is equal to $\beta''^t\sigma$. Hence

$$L''_{-1} = (\alpha\beta''^t + \beta''\alpha^t)\sigma,$$

where

$$\beta'' = -\nu L'_1\alpha + \nu' L_1\alpha - L'_0\beta + L_0\beta' + k\beta' - k'\beta.$$

Let us show that $\beta''^t\sigma\alpha = 0$. Making use of the above expression for $\beta''^t\sigma$, we find

$$\beta''^t\sigma\alpha = \nu\cdot\alpha^t\sigma L'_1\alpha - \nu'\cdot\alpha^t\sigma L_1\alpha + \beta^t\sigma L'_0\alpha - \beta'^t\sigma L_0\alpha + k\beta'^t\sigma\alpha - k'\beta^t\sigma\alpha.$$

The first two terms in this expression vanish by virtue of relation (2.14) applied to L and L'. To the second pair of terms, we apply the relations $L_0\alpha = k\alpha$ and $L'_0\alpha = k'\alpha$; after that, all remaining terms vanish by virtue of relations (2.12).

4. Let us verify relation (2.13) on the eigenvalues of the term of degree zero. By definition,

$$(LL')_0 = \nu\alpha\alpha^t\sigma L'_2 + (\alpha\beta^t + \beta\alpha^t)\sigma L'_1 + L_0L'_0 + L_1(\alpha\beta'^t + \beta'\alpha^t)\sigma + \nu' L_2\alpha\alpha^t\sigma.$$

The last pair of terms obviously vanishes after the multiplication by α on the right. We obtain

$$(LL')_0\alpha = \nu\alpha\alpha^t\sigma L'_2\alpha + \alpha\beta^t\sigma L'_1\alpha + \beta\alpha^t\sigma L'_1\alpha + kk'\alpha.$$

The third summand is zero by (2.14). Thus, α is an eigenvector of the zero degree term in the product LL',

$$(LL')_0\alpha = \alpha(\nu\cdot\alpha^t\sigma L'_2\alpha + \beta^t\sigma L'_1\alpha + kk'),$$

and in the commutator we obtain

$$L_0''\alpha = \alpha(\nu \cdot \alpha^t\sigma L_2'\alpha - \nu' \cdot \alpha^t\sigma L_2\alpha + \beta^t\sigma L_1'\alpha - \beta'^t\sigma L_1\alpha). \quad (2.15)$$

5. Let us verify the conservation of the relation $\alpha^t\sigma L_1\alpha = 0$ in the product and the commutator. For the product, by definition,

$$(LL')_0 = L_{-2}L_3' + L_{-1}L_2' + L_0L_1' + L_1L_0' + L_2L_{-1}' + L_3L_{-2}'.$$

Replacing L_{-2}, L_{-2}', L_{-1}, and L_{-1}' by known expressions, we obtain

$$\alpha^t\sigma(LL')_0\alpha = \nu(\alpha^t\sigma\alpha)\alpha^t\sigma L_3'\alpha + ((\alpha^t\sigma\alpha)\beta^t + (\alpha^t\sigma\beta)\alpha^t)\sigma L_2'\alpha - k(\alpha^t\sigma L_1'\alpha)$$
$$+ \tilde{k}(\alpha^t\sigma L_1\alpha) + \alpha^t\sigma L_2(\alpha(\beta'^t\sigma\alpha) + \beta'(\alpha^t\sigma\alpha)) + \nu'\alpha^t\sigma L_3\alpha(\alpha^t\sigma\alpha).$$

By virtue of relations (2.11)–(2.14), this expression vanishes. □

3. Almost Graded Structure

In this section, we consider the following cases: $\mathfrak{g} = \mathfrak{sl}(n)$, $\mathfrak{g} = \mathfrak{so}(n)$, $\mathfrak{g} = \mathfrak{sp}(2n)$, and, to conclude with, $\mathfrak{g} = \mathfrak{gl}(n)$, which is a somewhat special case with respect to the almost gradeness. The general definition of almost graded structure was given in the introduction. Denote by $\overline{\mathfrak{g}}$ the Lax operator algebra corresponding to $\mathfrak{g}$.

For all algebras $\mathfrak{g}$ listed above except for $\mathfrak{gl}(n)$ and for each $m \in \mathbb{Z}$, let

$$\mathfrak{g}_m = \{L \in \overline{\mathfrak{g}} \mid (L) + D \geqslant 0\},$$

where (L) is the divisor of a $\mathfrak{g}$-valued function L,

$$D = -mP_+ + (m+g)P_- + \sum_{s=1}^{ng} \gamma_s$$

for $\mathfrak{g} = \mathfrak{sl}(n)$ and $\mathfrak{g} = \mathfrak{so}(n)$, and

$$D = -mP_+ + (m+g)P_- + 2\sum_{s=1}^{ng} \gamma_s$$

for $\mathfrak{g} = \mathfrak{sp}(2n)$.

We refer to $\mathfrak{g}_m$ as the (*homogeneous*) *subspace of degree* m of the Lie algebra $\overline{\mathfrak{g}}$.

Theorem 3.1. *For* $\mathfrak{g} = \mathfrak{sl}(n)$, $\mathfrak{so}(n)$, *and* $\mathfrak{sp}(2n)$, *the following assertions hold*:

1. $\dim \mathfrak{g}_m = \dim \mathfrak{g}$.
2. $\overline{\mathfrak{g}} = \bigoplus_{m=-\infty}^{\infty} \mathfrak{g}_m$.
3. $[\mathfrak{g}_k, \mathfrak{g}_l] \subseteq \bigoplus_{m=k+l}^{k+l+g} \mathfrak{g}_m$.

Proof. First, let us prove 1. By the Riemann–Roch theorem, the dimension of the space of all $\mathfrak{g}$-valued functions L satisfying the relation $(L) + D \geqslant 0$ is equal to $(\dim \mathfrak{g})(ng+1)$ for $\mathfrak{g} = \mathfrak{sl}(n)$ and $\mathfrak{g} = \mathfrak{so}(n)$. For $\mathfrak{g} = \mathfrak{sp}(2n)$, it is equal to $(\dim \mathfrak{g})(2ng+1)$. We shall prove that for $\mathfrak{g} = \mathfrak{sl}(n)$ or $\mathfrak{g} = \mathfrak{so}(n)$ and for an arbitrary $m \in \mathbb{Z}$, there are exactly $\dim \mathfrak{g}$ relations at every pole γ_s, while for $\mathfrak{g} = \mathfrak{sp}(2n)$ the number of these relations is $2\dim \mathfrak{g}$. This will mean that $\dim \mathfrak{g}_m = \dim \mathfrak{g}$.

First, consider the case $\mathfrak{g} = \mathfrak{sl}(n)$. The elements of the subspace $\mathfrak{g}_m$ satisfy certain conditions of three kinds coming from (2.2) and (2.3); these are the following conditions on the residues, eigenvalues, and traces of the matrix $L \in \mathfrak{g}_m$:

1. At every weak singularity, one has $L_{-1} = \alpha\beta^t$, which would give $\dim \mathfrak{g}$ relations (since $L_{-1} \in \mathfrak{g}$) if the right-hand side were fixed. But the right-hand side depends on the free n-dimensional vector β. Hence we have $\dim \mathfrak{g} - n$ conditions at each of the ng simple poles γ_s.

2. At every weak singularity, one has $L_0\alpha = k\alpha$, which gives n conditions. Taking into account one free parameter k, we obtain $n-1$ conditions at each γ_s.

3. We also have $\operatorname{tr} L = 0$, i.e., one more relation at every weak singularity.

Thus, we have $(\dim \mathfrak{g} - n) + (n-1) + 1 = \dim \mathfrak{g}$ relations at every γ_s, as desired.

For $\mathfrak{g} = \mathfrak{so}(n)$, we follow the same line of argument. Again, the relation $L_{-1} = \alpha\beta^t - \beta\alpha^t$ (2.6) gives $\dim \mathfrak{g} - n$ equations, the relation $L_0\alpha = k\alpha$ (2.8) gives $n-1$ equations, and $\beta^t\alpha = 0$ (2.7) gives one more equation. All in all, we obtain $\dim \mathfrak{g}$ equations at every point γ_s.

For $\mathfrak{g} = \mathfrak{sp}(2n)$, at every point γ_s we have the following conditions:

$L_{-2} = \nu\alpha\alpha^t$:	$\dim \mathfrak{g} - 1$ conditions (one free parameter ν);
$L_{-1} = (\alpha\beta^t + \beta\alpha^t)\sigma$:	$\dim \mathfrak{g} - 2n$ conditions ($2n$ free parameters β);
$L_0\alpha = k\alpha$:	$2n-1$ condition (one free parameter k);
$\beta^t\sigma\alpha = 0,\ \alpha^t\sigma L_1\alpha = 0$:	2 conditions;

i.e., there are $2\dim \mathfrak{g}$ conditions at each of the ng points, as desired.

For $m > 0$ and $m < -g$, the subspaces $\mathfrak{g}_m$ are linearly independent for the obvious reason: the orders at $P_\pm$ of the elements of these subspaces are different for different m.

For $-g \le m \le 0$, the linear independence of $\mathfrak{g}_m$ follows from the fact that there are no *everywhere holomorphic* Lax operators. We point out that the last argument applies only to the case of simple Lie algebras. That explains why the case of reductive Lie algebra $\mathfrak{gl}(n)$ requires some modification (see below).

Assertion 3 of the theorem follows from the consideration of orders at the points $P_\pm$. □

Theorem 3.1 defines an *almost graded structure* on $\overline{\mathfrak{g}}$.

Now consider the case $\mathfrak{g} = \mathfrak{gl}(n)$. In this case, $\overline{\mathfrak{g}}$ contains a subspace of functions ranging in the 1-dimensional space of scalar matrices. Let L be such a function. By (2.2), we obtain $\operatorname{tr} L_{-1} = 0$. Since L_{-1} is a scalar matrix, we obtain $L_{-1} = 0$. Hence L is holomorphic everywhere except at $P_\pm$. Let $\mathscr{A}$ be the algebra of meromorphic functions on Γ holomorphic everywhere except at $P_\pm$. Then $L \in \mathscr{A} \cdot \mathrm{id}$, where id is the identity matrix. Therefore,

$$\overline{\mathfrak{gl}(n)} = \overline{\mathfrak{sl}(n)} \oplus \mathscr{A} \cdot \mathrm{id}\,. \tag{3.1}$$

In [3], a certain base $\{A_m\}$ (later called the *Krichever-Novikov base*) was introduced in the space of such functions. Denote $\mathscr{A}_m = \mathbb{C}A_m$ and set $\mathfrak{gl}(n)_m = \mathfrak{sl}(n)_m \oplus (\mathscr{A}_m \cdot \mathrm{id})$. For $m > 0$ and $m < -g$, this definition is equivalent to the above definition of $\mathfrak{g}_m$ (with $\mathfrak{g} = \mathfrak{gl}(n)$).

As follows from (3.1), Theorem 3.1 with $\mathfrak{g}_m = \mathfrak{gl}(n)_m$ remains valid for $\mathfrak{g} = \mathfrak{gl}(n)$. Only relation 3 in the theorem holds with a different upper limit of summation, which is determined by the algebra $\mathscr{A}$ (see [3]).

4. Central Extensions of Lax Operator Algebras

4.1. Central extensions of Lax operator algebras over $\mathfrak{gl}(n)$. The 2-cocycle defining a central extension for Krichever-Novikov current algebras (in particular, loop algebras) is given by the conventional expression $\operatorname{tr} \operatorname{res}_{P_+} L\,dL'$. For these algebras, the cocycle is local. A cocycle χ is said to be *local* [3] if there exist constants μ' and μ'' such that $\chi(\mathfrak{g}_m, \mathfrak{g}_{m'}) = 0$ unless $\mu' \leqslant m + m' \leqslant \mu''$, where $\mathfrak{g}_m$ and $\mathfrak{g}_{m'}$ are the homogeneous subspaces introduced in the previous section. In the case of Lax operator algebras, the above cocycle is no longer local. In this section, we improve it so as to obtain a local cocycle.

The eigenvalue k of the zero degree component L_0 of an operator L (see (2.3)) can be treated as a linear functional of L. We denote this functional by $k(L)$.

Lemma 4.1. *At every weak singularity, the* 1*-form* $\operatorname{tr} L\,dL'$ *has an at most simple pole, and*

$$\operatorname{res} \operatorname{tr} L\,dL' = k([L, L']). \tag{4.1}$$

Proof. Let us compute both parts of the relation explicitly.

1. Using the relation

$$dL' = -\frac{\alpha\beta'^t}{z^2} + L'_1 + \cdots$$

and (2.1), we obtain

$$L\,dL' = -\frac{\alpha\beta^t\alpha\beta'^t}{z^3} - \frac{L_0\alpha\beta'^t}{z^2} - \frac{L_1\alpha\beta'^t - \alpha\beta^t L'_1}{z} + \cdots .$$

The first term vanishes, since $\beta^t\alpha = 0$. The second term vanishes when we take the trace, since $L_0\alpha = k\alpha$ and $\operatorname{tr}\alpha\beta'^t = \beta'^t\alpha = 0$. The third term gives the desired residue. We have

$$\operatorname{res}\operatorname{tr} L\,dL' = \operatorname{tr}(\alpha\beta^t L'_1 - L_1\alpha\beta'^t) = \beta^t L'_1\alpha - \beta'^t L_1\alpha.$$

2. Now let us compute the right-hand side of the relation in question. Denote by $[L, L']_0$ the zero degree coefficient in the expansion (2.1) for the commutator $[L, L']$. We have

$$[L, L']_0 = \alpha\beta^t L'_1 + L_0 L'_0 + L_1\alpha\beta'^t - \alpha\beta'^t L_1 - L'_0 L_0 - L'_1\alpha\beta^t.$$

Multiply both sides of this relation by α on the right. Then the third terms in both rows vanish, since they contain the factors $\beta'^t\alpha$ and $\beta^t\alpha$, which are zero by (2.2). The second terms cancel each other, since $L_0L'_0\alpha = k'k\alpha$ and $L'_0L_0\alpha = kk'\alpha$. Hence

$$[L, L']_0\alpha = \alpha\beta^t L'_1\alpha - \alpha\beta'^t L_1\alpha = \alpha(\beta^t L'_1\alpha - \beta'^t L_1\alpha).$$

The expression in brackets is a 1×1-matrix, that is, a complex number. Certainly, this complex number is the eigenvalue of $[L, L']_0$ on the vector α, i.e., $k([L, L'])$. Its value exactly coincides with the expression computed above for $\operatorname{tr}\operatorname{res} L\,dL'$, which completes the proof. □

We wish to eliminate the singularities of the 1-form $\operatorname{tr} L dL'$ at γ_s by subtracting another expression for the eigenvalue on the right-hand side in (4.1). Remarkably, we can give this new expression in terms of connections in holomorphic vector bundles on Γ whose explicit form is given in [2]. In what follows, $\mathscr{L}$ denotes the 1-form of such a connection.

Lemma 4.2. *Let $\mathscr{L}$ be a matrix-valued 1-form such that locally, near a weak singularity,*

$$\mathscr{L} = \mathscr{L}_{-1}\frac{dz}{z} + \mathscr{L}_0\,dz + \cdots ,$$

where $\mathscr{L}$ satisfies the same assumptions as L (see (2.1)–(2.3)) with the only modification: $\tilde\beta^t\alpha = 1$, where $\mathscr{L}_{-1} = \alpha\tilde\beta^t$. Then the 1-form $\operatorname{tr} L\mathscr{L}$ has an at most simple pole at $z = 0$, and

$$\operatorname{res}\operatorname{tr} L\mathscr{L} = k(L).$$

Proof. We have

$$L = \frac{\alpha\beta^t}{z} + L_0 + \cdots , \qquad \mathscr{L} = \left(\frac{\alpha\tilde\beta^t}{z} + \mathscr{L}_0 + \cdots\right)dz;$$

hence

$$L\mathscr{L} = \left(\frac{\alpha\beta^t\alpha\tilde\beta^t}{z^2} + \frac{\alpha\beta^t\mathscr{L}_0 + L_0\alpha\tilde\beta^t}{z} + \cdots\right)dz.$$

Just as above, the first term is zero. For the second term, we have

$$\operatorname{res}\operatorname{tr} L\mathscr{L} = \operatorname{tr}(\alpha\beta^t\mathscr{L}_0 + L_0\alpha\tilde\beta^t) = \beta^t\mathscr{L}_0\alpha + \tilde\beta^t L_0\alpha.$$

Next, $\mathscr{L}_0\alpha = \tilde k\alpha$ and $L_0\alpha = k\alpha$; hence $\beta^t\mathscr{L}_0\alpha = \tilde k\beta^t\alpha = 0$ and $\tilde\beta^t L_0\alpha = k\tilde\beta^t\alpha = k$. This completes the proof. □

Theorem 4.3. *For every 1-form $\mathscr{L}$ satisfying the assumptions of Lemma 4.2, the 1-form $\operatorname{tr}(L\,dL' - [L, L']\mathscr{L})$ is regular except at the points $P_\pm$, and the expression*

$$\gamma(L, L') = \operatorname{res}_{P_+}\operatorname{tr}(L\,dL' - [L, L']\mathscr{L})$$

gives a local cocycle on the Lax operator algebra.

Proof. In the course of proof of Lemmas 4.1 and 4.2, we have seen that the 1-forms $\operatorname{tr} L\,dL'$ and $\operatorname{tr}[L,L']\mathscr{L}$ have simple poles at each point γ_s and their residues are equal to the same quantity $k_s([L,L'])$. Hence, their difference is regular at every γ_s.

Assume that at the point P_+ we have the expansions

$$L(z)=\sum_{i=m}^{\infty} a_i z^i, \quad L'(z)=\sum_{j=m'}^{\infty} b_j z^j, \quad \mathscr{L}(z)=\sum_{k=m_+}^{\infty} c_k z^k dz. \tag{4.2}$$

Then

$$L(z)\,dL'(z)=\sum_{p=m+m'}^{\infty}\Bigg(\sum_{i+j=p} j a_i b_j\Bigg) z^{p-1}\,dz$$

and

$$[L(z),L'(z)]\mathscr{L}=\sum_{p=m+m'+m_+}^{\infty}\Bigg(\sum_{i+j+k=p}[a_i,b_j]c_k\Bigg) z^p\,dz.$$

For one of these 1-forms to have a nontrivial residue at the point P_+, it is necessary that either $m+m'\leqslant 0$ or $m+m'+m_+\leqslant -1$; in other words,

$$m+m'\leqslant \max\{0,-1-m_+\}.$$

If L and L' are homogeneous of degrees m and m', respectively, then their expansions (similar to (4.2)) at the point P_- start from $i=-m-g$ and $j=-m'-g$, respectively. The expansion of $\mathscr{L}$ starts from some integer m_-. Hence the condition at P_- reads

$$-m-m'-2g\leqslant \max\{0,-1+m_-\}.$$

Finally, we obtain

$$\min\{0,1-m_-\}-2g\leqslant m+m'\leqslant \max\{0,-1-m_+\}.$$

Since $m_\pm$ are fixed (because $\mathscr{L}$ is fixed), the latter exactly means that the cocycle is local. □

4.2. Central extensions of Lax operator algebras over $\mathfrak{so}(n)$. We stick to the same line of argument as in the preceding section.

Lemma 4.4. *At each weak singularity, the* 1*-form* $\operatorname{tr} L\,dL'$ *has an at most simple pole, and*

$$\operatorname{res}\operatorname{tr} L\,dL' = 2k([L,L']). \tag{4.3}$$

Proof. 1) Using (2.5) and the relation

$$dL'=-L'_{-1}z^{-2}+L'_1+\cdots,$$

where L'_{-1} is given by (2.6), we obtain

$$L\,dL'=-\frac{L_{-1}L'_{-1}}{z^3}-\frac{L_0L'_{-1}}{z^2}-\frac{L_1L'_{-1}-L_{-1}L'_1}{z}+\cdots. \tag{4.4}$$

For the first term, we have

$$L_{-1}L'_{-1}=(\alpha\beta^t-\beta\alpha^t)(\alpha\beta'^t-\beta'\alpha^t)=\alpha(\beta^t\alpha)\beta'^t-\beta(\alpha^t\alpha)\beta'^t-\alpha\beta^t\beta'\alpha^t-\beta\alpha^t\beta'\alpha^t.$$

The first two summands are zero by (2.7). For the remainder, we have

$$\operatorname{tr}(L_{-1}L'_{-1})=\operatorname{tr}(-\alpha\beta^t\beta'\alpha^t-\beta\alpha^t\beta'\alpha^t)=-(\alpha^t\alpha)\beta^t\beta'-(\alpha^t\beta)\alpha^t\beta',$$

which again vanishes by (2.7).

Again (as in Sec. 4.1), the term containing z^{-2} vanishes when we take the trace. By definition,

$$L_0L'_{-1}=L_0(\alpha\beta'^t-\beta'\alpha^t)=k\alpha\beta'^t-L_0\beta'\alpha^t.$$

Now observe that $\operatorname{tr}(L_0\beta'\alpha^t)=\operatorname{tr}(\alpha^tL_0\beta')$ and $\alpha^tL_0=-k\alpha^t$. Hence $\operatorname{tr}(L_0L'_{-1})=2k\alpha^t\beta'$, which is zero by (2.7).

The third term in (4.4) gives the desired residue. We have

$$\operatorname{res} L\,dL'=(L_{-1}L'_1-L_1L'_{-1}).$$

The substitution of L_{-1} and L'_{-1} given by (2.6) results in the relation

$$\operatorname{res} L\, dL' = \alpha\beta^t L'_1 - \beta\alpha^t L'_1 - L_1\alpha\beta'^t + L_1\beta'\alpha^t;$$

hence

$$\operatorname{tr}\operatorname{res} L\, dL' = \beta^t L'_1\alpha - \alpha^t L'_1\beta - \beta'^t L_1\alpha + \alpha^t L_1\beta'.$$

By the skew-symmetry of the matrices L_1 and L'_1, the first two summands in the last relation are equal, and the same is true for the last two summands. Hence

$$\operatorname{tr}\operatorname{res} L\, dL' = 2(\beta^t L'_1\alpha - \beta'^t L_1\alpha).$$

It obviously follows from (2.9) that $[L, L']_0\alpha = \beta^t L'_1\alpha - \beta'^t L_1\alpha$, which proves the lemma. □

Lemma 4.5. *Let $\mathscr{L}$ be a skew-symmetric matrix-valued 1-form such that locally, near the weak singularity,*

$$\mathscr{L} = \mathscr{L}_{-1}\frac{dz}{z} + \mathscr{L}_0\, dz + \cdots ,$$

where $\mathscr{L}_{-1} = \alpha\tilde\beta^t - \tilde\beta\alpha^t$, $\tilde\beta^t\alpha = 1$, and $\mathscr{L}_0\alpha = \tilde k\alpha$. Then the 1-form $\operatorname{tr} L\mathscr{L}$ has at most a simple pole at $z = 0$ and

$$\operatorname{res}\operatorname{tr} L\mathscr{L} = 2k(L).$$

Proof. The coefficient of z^{-2} in the product $L\mathscr{L}$ is, by definition, equal to

$$(\alpha\beta^t - \beta\alpha^t)(\alpha\tilde\beta^t - \tilde\beta\alpha^t) = \alpha(\beta^t\alpha)\tilde\beta^t - \alpha\beta^t\tilde\beta\alpha^t - \beta(\alpha^t\alpha)\tilde\beta^t + \beta\alpha^t\tilde\beta\alpha^t.$$

The first and third terms are zero by (2.7). For the trace of the remaining sum, we have

$$\operatorname{tr}(-\alpha\beta^t\tilde\beta\alpha^t + \beta\alpha^t\tilde\beta\alpha^t) = -(\alpha^t\alpha)\beta^t\tilde\beta + (\alpha^t\beta)\alpha^t\tilde\beta,$$

which is zero for the same reason.

By multiplying the expansions for L and $\mathscr{L}$, we find that

$$\operatorname{tr}\operatorname{res}(L\mathscr{L}) = \operatorname{tr}(L_{-1}\mathscr{L}_0 + L_0\mathscr{L}_{-1}) = \operatorname{tr}(\alpha\beta^t - \beta\alpha^t)\mathscr{L}_0 + \operatorname{tr} L_0(\alpha\tilde\beta^t - \tilde\beta\alpha^t).$$

For the first term, we have

$$\operatorname{tr}(\alpha\beta^t - \beta\alpha^t)\mathscr{L}_0 = \operatorname{tr}(\alpha\beta^t\mathscr{L}_0 - \beta\alpha^t\mathscr{L}_0) = \beta^t\mathscr{L}_0\alpha - \alpha^t\mathscr{L}_0\beta.$$

Observe that, by skew-symmetry, we have $\alpha^t\mathscr{L}_0 = -\beta^t\mathscr{L}_0\alpha$, and hence

$$\operatorname{tr}(\alpha\beta^t - \beta\alpha^t)\mathscr{L}_0 = \beta^t\mathscr{L}_0\alpha - \alpha^t\mathscr{L}_0\beta = 2\beta^t\mathscr{L}_0\alpha = 2\tilde k\beta^t\alpha,$$

which is zero by (2.7).

For the second summand, we have

$$\operatorname{tr} L_0(\alpha\tilde\beta^t - \tilde\beta\alpha^t) = \operatorname{tr}(L_0\alpha\tilde\beta^t - L_0\tilde\beta\alpha^t) = \tilde\beta^t L_0\alpha - \alpha^t L_0\tilde\beta.$$

Since $L_0\alpha = k\alpha$, $\alpha^t L_0 = -k\alpha^t$, and $\tilde\beta^t\alpha = \alpha^t\tilde\beta = 1$, we obtain

$$\operatorname{tr} L_0(\alpha\tilde\beta^t - \tilde\beta\alpha^t) = 2k.$$

□

Theorem 4.6. *For each $\mathscr{L}$ satisfying the assumptions of Lemma 4.5, the 1-form $\operatorname{tr}(L\, dL' - [L, L']\mathscr{L})$ is regular except at the points $P_\pm$, and the expression*

$$\gamma(L, L') = \operatorname{res}_{P_+}\operatorname{tr}(L\, dL' - [L, L']\mathscr{L})$$

gives a local cocycle on the Lax operator algebra.

The proof is similar to that of Theorem 4.3. It relies only on the absence of residues of the 1-form defining the cocycle at the weak singularities.

There is certain ambiguity in the definition of $\mathscr{L}$ in Lemma 4.5. For example, we could require that $\mathscr{L}_{-1} = \alpha\tilde\beta^t$ and take $\operatorname{tr}(L\, dL' - 2[L, L']\mathscr{L})$ in Theorem 4.6.

4.3. Central extensions of Lax operator algebras over $\mathfrak{sp}(2n)$. We stick to the same line of argument as in the preceding sections. First, let us prove the following analog of Lemma 4.4.

Lemma 4.7. *At any weak singularity, the* 1*-form* $\operatorname{tr} L\, dL'$ *has at most a simple pole, and*

$$\operatorname{res} \operatorname{tr} L\, dL' = 2k([L, L']). \tag{4.5}$$

Proof. A straightforward computation of LdL' based on the expansion (2.10) shows that the coefficients of z^{-5} and z^{-4} of this matrix-valued 1-form are equal to 0 by virtue of relations (2.12).

For the term with z^{-3}, we have

$$(L\, dL')_{-3} = -(\beta^t \sigma \beta' - 2\nu')\alpha\alpha^t\sigma.$$

This expression vanishes when we take the trace,

$$\operatorname{tr}(L\, dL')_{-3} = -(\beta^t \sigma \beta' - 2\nu')(\alpha^t\sigma\alpha) = 0.$$

Likewise, using (2.12)–(2.14), we have

$$\operatorname{tr}(L\, dL')_{-2} = \nu(\alpha^t \sigma L_1' \alpha) - 2k(\beta'^t\sigma\alpha) - \nu'(\alpha^t\sigma L_1 \alpha) = 0.$$

Thus, the 1-form $\operatorname{tr} LdL'$ indeed has at most a simple pole at the point in question. A straightforward computation of the residue gives

$$\operatorname{tr}(L\, dL')_{-1} = 2(\nu \cdot \alpha^t\sigma L_2'\alpha - \nu' \cdot \alpha^t\sigma L_2\alpha + \beta^t\sigma L_1'\alpha - \beta'^t\sigma L_1\alpha),$$

which exactly coincides with twice the expression (2.15) for $k([L, L'])$. □

Lemma 4.8. *Let* $\mathscr{L}$ *be a* $\mathfrak{g}$*-valued* 1*-form such that locally, in a neighborhood of each weak singularity,*

$$\mathscr{L} = \mathscr{L}_{-1}\frac{dz}{z} + \mathscr{L}_0\, dz + \cdots,$$

where $\mathscr{L}_{-1} = (\alpha\tilde\beta^t + \tilde\beta\alpha^t)\sigma$, $\tilde\beta^t\sigma\alpha = 1$, $\mathscr{L}_0\alpha = \tilde k\alpha$, *and* $\alpha^t\sigma\mathscr{L}_1\alpha = 0$. *Then the* 1*-form* $\operatorname{tr} L\mathscr{L}$ *has at most a simple pole at* $z = 0$, *and*

$$\operatorname{res}\operatorname{tr} L\mathscr{L} = 2k(L).$$

Proof. The expansion for $L\mathscr{L}$ starts from z^{-3}. We have

$$\operatorname{tr}(L\mathscr{L})_{-3} = \nu(\tilde\beta^t\sigma\alpha)(\alpha^t\sigma\alpha) = 0,$$
$$\operatorname{tr}(L\mathscr{L})_{-2} = \nu(\alpha^t\sigma\tilde\beta)(\alpha^t\sigma\alpha) + (\beta^t\sigma\tilde\beta)(\alpha^t\sigma\alpha) + (\alpha^t\sigma\tilde\beta)(\alpha^t\sigma\beta) = 0.$$

Thus, the 1-form $\operatorname{tr} L\mathscr{L}$ indeed has at most a simple pole at the point in question. The calculation of the residue gives

$$\operatorname{tr}(L\mathscr{L})_{-1} = \nu \cdot \alpha^t\sigma\mathscr{L}_1\alpha + \beta^t\sigma(\mathscr{L}_0\alpha) + (\alpha^t\sigma\mathscr{L}_0)\beta + \tilde\beta^t\sigma(L_0\alpha) + (\alpha^t\sigma L_0)\tilde\beta.$$

By the assumption of the lemma, $\alpha^t\sigma\mathscr{L}_1\alpha = 0$ and $\mathscr{L}_0\alpha = \tilde k\alpha$. The latter relation also implies that $\alpha^t\sigma\mathscr{L}_0 = -\tilde k\alpha^t\sigma$. Consequently,

$$\operatorname{tr}(L\mathscr{L})_{-1} = \tilde\beta^t\sigma(L_0\alpha) + (\alpha^t\sigma L_0)\tilde\beta.$$

By the relations $L_0\alpha = k\alpha$ and $\alpha^t\sigma L_0 = -k\alpha^t\sigma$, we have

$$\operatorname{tr}(L\mathscr{L})_{-1} = 2k(\tilde\beta^t\sigma\alpha) = 2k.$$ □

Just as above, the last two lemmas imply the following assertion:

Theorem 4.9. *For* $\mathfrak{g} = \mathfrak{sp}(2n)$, *the expression*

$$\gamma(L, L') = \operatorname{res}_{P_+}\operatorname{tr}(L\, dL' - [L, L']\mathscr{L})$$

gives a local cocycle on the Lax operator algebra.

References

[1] I. M. Krichever, “Vector bundles and Lax equations on algebraic curves,” Comm. Math. Phys., **229**:2 (2002), 229–269; `http://arxiv.org/abs/hep-th/0108110`.

[2] I. M. Krichever, “Isomonodromy equations on algebraic curves, canonical transformations and Witham equations,” Mosc. Math. J., **2**:4 (2002), 717–752, 806; `http://arxiv.org/abs/hep-th/0112096`.

[3] I. M. Krichever and S. P. Novikov, “Algebras of Virasoro type, Riemann surfaces and structures of the theory of solitons,” Funkts. Anal. Prilozhen., **21**:2 (1987), 46–63.

[4] I. M. Krichever and S. P. Novikov, “Virasoro type algebras, Riemann surfaces and strings in Minkowski space,” Funkts. Anal. Prilozhen., **21**:4 (1987), 47–61.

[5] I. M. Krichever and S. P. Novikov, “Algebras of Virasoro type, energy-momentum tensors and decompositions of operators on Riemann surfaces,” Funkts. Anal. Prilozhen., **23**:1 (1989), 46–63.

[6] I. M. Krichever and S. P. Novikov, “Holomorphic bundles and commuting difference operators. Two-point constructions,” Uspekhi Mat. Nauk, **55**:4 (2000), 181–182.

[7] I. M. Krichever and S. P. Novikov, “Holomorphic bundles on Riemann surfaces and Kadomtsev-Petviashvili equation (KP). I,” Funkts. Anal. Prilozhen., **12**:4 (1978), 41–52.

[8] I. M. Krichever and S. P. Novikov, “Holomorphic bundles on algebraic curves and nonlinear equations,” Uspekhi Math. Nauk, **35**:6 (1980), 47–68.

[9] M. Schlichenmaier, “Local cocycles and central extensions for multi-point algebras of Krichever–Novikov type,” J. Reine Angew. Math., **559** (2003), 53–94.

[10] M. Schlichenmaier, “Higher genus affine algebras of Krichever–Novikov type,” Moscow Math. J., **3**:4 (2003), 1395–1427; `http://arxiv.org/abs/math/0210360`.

[11] O. K. Sheinman, “Affine Krichever-Novikov algebras, their representations and applications,” in: Geometry, Topology and Mathematical Physics. S. P. Novikov’s Seminar 2002–2003, Amer. Soc. Transl. (2), vol. 212 (eds. V. M. Buchstaber, I. M. Krichever), Amer. Math. Soc., Providence, R.I., 2004, 297–316; `http://arxiv.org/abs/Math.RT/0304020`.

[12] A. N. Tyurin, “The classification of vector bundles over an algebraic curve of arbitrary genus,” Izv. Akad. Nauk SSSR Ser. Mat., **29** (1965), 657–688.

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