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Integrable equations, addition theorems, and the Riemann–Schottky problem

V. M. Buchstaber and I. M. Krichever

Abstract. The classical Weierstrass theorem claims that, among the analytic functions, the only functions admitting an algebraic addition theorem are the elliptic functions and their degenerations. This survey is devoted to far-reaching generalizations of this result that are motivated by the theory of integrable systems. The authors discovered a strong form of the addition theorem for theta functions of Jacobian varieties, and this form led to new approaches to known problems in the geometry of Abelian varieties. It is shown that strong forms of addition theorems arise naturally in the theory of the so-called trilinear functional equations. Diverse aspects of the approaches suggested here are discussed, and some important open problems are formulated.

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§ 1. Introduction

Until 1986 the Riemann–Schottky problem remained one of the oldest and most famous unsolved problems in algebraic geometry. Briefly, the Riemann–Schottky problem is the problem of describing the Jacobians of algebraic curves among all principally polarized Abelian varieties, that is, the problem of describing the image of the map

$$B\colon \mathscr{M}_g \longmapsto \mathscr{A}_g = \mathscr{H}_g / \operatorname{Sp}(2g,\mathbb{Z}) \tag{1.1}$$

of the moduli space $\mathscr{M}_g$ of smooth algebraic curves of genus g into the quotient of the Siegel upper half-plane $\mathscr{H}_g$ by some natural action of the group $\operatorname{Sp}(2g,\mathbb{Z})$. By the *Siegel upper half-plane* we mean here the space of symmetric g-dimensional matrices with positive-definite imaginary part. The map (1.1) is induced by a correspondence which, to any smooth algebraic curve Γ of genus g with a fixed basis of one-dimensional cycles (a_i,b_j), $1\leqslant i,j\leqslant g$, with canonical intersection matrix $a_i\cdot a_j=b_i\cdot b_j=0$, $a_i\cdot b_j=\delta_{ij}$, assigns the $g\times g$-matrix of b-periods

$$B_{ij}=\oint_{b_j}\omega_i \tag{1.2}$$

of the basis of normalized holomorphic differentials ω_i that is uniquely determined by the condition $\oint_{a_j}\omega_i=\delta_{ij}$. According to the Torelli theorem, the map B is an *embedding*. At the end of the 1970s, S. P. Novikov expressed the conjecture that the Jacobi varieties are exactly the principally polarized Abelian varieties in whose theta functions the Kadomtsev–Petviashvili (KP) equation is integrated. To be more precise, every symmetric $g\times g$ matrix with positive-definite imaginary part defines a Riemann theta function

$$B\in\mathscr{H}_g\longmapsto\theta(z)=\theta(z\,|\,B),\qquad z=(z_1,\dots,z_g), \tag{1.3}$$

by the formula

$$\theta(z)=\sum_{m\in\mathbb{Z}^g}e^{2\pi i(z,m)+\pi i(Bm,m)},\qquad (z,m)=m_1z_1+\dots+m_gz_g. \tag{1.4}$$

This is an entire function of the variables z_k, $k=1,\dots,g$. It readily follows from (1.4) that $\theta(z)$ admits the following monodromy property:

$$\theta(z+e_k)=\theta(z),\qquad \theta(z+B_k)=e^{-2\pi i z_k-\pi i B_{kk}}\theta(z), \tag{1.5}$$

where e_k are the basis vectors in $\mathbb{C}^g$ and B_k is the vector equal to the kth column of the matrix B. The relations (1.5) mean that θ is a section of the canonical vector bundle $\mathscr{L}$ on the Abelian variety $T^g = \mathbb{C}^g/(\mathbb{Z}^g + B\mathbb{Z}^g)$.

Novikov conjecture. *A symmetric matrix B with positive-definite imaginary part is the matrix of b-periods of a basis of normalized holomorphic differentials on some smooth algebraic curve Γ if and only if there are three g-dimensional vectors U, V, W such that the function*

$$u(x,y,t) = 2\partial_x^2 \log\theta(Ux + Vy + Wt + Z \,|\, B) \tag{1.6}$$

for any $Z \in \mathbb{C}^g$ is a solution of the KP equation

$$3u_{yy} = (4u_t - 6uu_x + u_{xxx})_x. \tag{1.7}$$

This conjecture was expressed by Novikov in the framework of his *problem of effectivization of theta function formulae* of the theory of finite-gap integration. The starting point of this theory is the following result of one of the authors of the present paper.

Theorem 1.1 (Krichever [1]). *If B is the period matrix of a basis of normalized holomorphic differentials on some algebraic curve Γ, then the function $u(x,y,t)$ given by the formula (1.6) satisfies the KP equation (1.7). Here U, V, and W are the vectors of b-periods of the normalized meromorphic differentials with poles at some point $P_0 \in \Gamma$ of orders* 2, 3, *and* 4, *respectively.*

This theorem is a special case of a general algebro-geometric construction that assigns a solution of some soliton equation to a family of algebro-geometric data $\{\Gamma, P_\alpha, z_\alpha, S^{g+k-1}(\Gamma)\}$. Here Γ stands for a non-singular algebraic curve of genus g with the distinguished points P_α in whose neighbourhoods some local coordinates z_α are fixed, and $S^{g+k-1}(\Gamma)$ stands for the symmetric power of the curve. This construction was proposed by one of the authors (see [1], [2]) and is based on the notion of Baker–Akhiezer functions that are defined on the corresponding algebraic curve by their analytic properties. In fact, these analytic properties are an axiomatization of the analytic properties of Bloch functions of finite-gap Schrödinger operators and were established at the initial period of development of the theory of finite-gap integration of the Korteweg–de Vries equation [3]–[6].

The first advance in the proof of the Novikov conjecture was made by Dubrovin. Before formulating his result, we must present a series of standard results of theta function theory and the corresponding notation. First of all, we need the notion of theta function with characteristic; this function is defined for any pair of real numbers $\alpha, \beta \in \mathbb{R}$ by a formula similar to (1.4):

$$\theta[\alpha,\beta](z \,|\, B) = \sum_{m \in \mathbb{Z}^g} \exp\bigl(2\pi i(z+\beta, m+\alpha) + \pi i(B(m+\alpha), m+\alpha)\bigr). \tag{1.8}$$

As is known, the theta functions

$$\Theta[\varepsilon, 0](z) = \theta[\varepsilon, 0](2z \,|\, 2B) \tag{1.9}$$

corresponding to all half-integer characteristics ε form a basis in the space of sections of the bundle $\mathscr{L}^2$, that is, a basis in the space of theta functions of weight 2 satisfying the following monodromy properties:

$$\Theta(z+e_k)=\Theta(z), \qquad \Theta(z+B_k)=e^{-4\pi i z_k-2\pi i B_{kk}}\Theta(z). \tag{1.10}$$

Theorem 1.2 (Dubrovin [7]). *The function $u(x,y,t)$ given by the formula (1.6) is a solution of the KP equation if and only if the equality*

$$\partial_U^4\Theta[\varepsilon,0]-\partial_U\partial_W\Theta[\varepsilon,0]+\partial_V^2\Theta[\varepsilon,0]+c\,\Theta[\varepsilon,0]=0, \qquad c=\text{const}, \tag{1.11}$$

holds for any half-integer characteristic $\varepsilon\in\frac{1}{2}\mathbb{Z}^g/\mathbb{Z}^g$.

Here the symbol $\Theta[\varepsilon,0]=\Theta[\varepsilon,0](0)$ is the standard notation for the so-called theta constants. Similarly, the absence of an argument of the derivative of a theta function means that the value of this derivative at the origin is taken. For instance, $\partial_U^4\Theta[\varepsilon,0]=\partial_U^4\Theta[\varepsilon,0](0)$, where ∂_U stands for the derivative in the direction of the vector U.

Dubrovin also proved that the compatibility conditions for the equations (1.11) single out a variety of dimension $3g-3=\dim\mathscr{M}_g$. It was thus proved that the KP equation solves the Riemann–Schottky problem, at least locally (modulo possible additional components). The proof of the Novikov conjecture was completed by Shiota in 1986 [8]. For a detailed survey of works concerning the Riemann–Schottky problem and the works about algebro-geometric integration of non-linear equations of mathematical physics which finally led to the solution of the problem, see [9].

To give an impression of the role played by the new ideas introduced by the Novikov conjecture into the solution of the Riemann–Schottky problem, we now sketch the history of the problem and briefly describe the results obtained in the framework of classical algebro-geometric approaches to this solution (for details, see [9]).

For $g=2$ and 3 the dimensions of the spaces $\mathscr{M}_g$ and $\mathscr{A}_g$ coincide, and therefore it follows immediately from the Torelli theorem that, in this case, every generic matrix $B\in\mathscr{H}_g$ is the period matrix of a Riemann surface. The only condition is given by the Martens theorem claiming that the Jacobi variety of a Riemann surface is indecomposable, that is, cannot be represented as the direct product of Abelian varieties of positive dimension. This condition can be effectively described in the language of theta constants.

A non-trivial identity for the period matrix of a Riemann surface of genus 4 was obtained by Schottky in 1888. Since for $g=4$ the space $\mathscr{M}_g$ is of codimension 1, the corresponding *Schottky relation* gave a solution (at least local) of the characterization problem for the corresponding Jacobi varieties. The proof of the fact that the variety singled out by the Schottky relation is irreducible was obtained by Igusa only in 1981 [10]. Generalizations of this relation to the case of curves of arbitrary genus were formulated as a conjecture in 1909 in the joint paper [11] of Schottky and Jung and proved by Farkas and Rauch [12]. Van Geemen [13] proved later on that the Schottky–Jung relations give a local solution of the Riemann–Schottky problem. As is well known, these relations certainly give no complete solution of the problem, because they distinguish a subvariety containing additional components already for $g=5$ (Donagi [14]).

Among other results giving a local solution of the Riemann–Schottky problem we note the Andreotti–Mayer theorem [15] according to which the condition that the dimension of the set $\operatorname{Sing}\Theta$ of singular points of the theta divisor is not less than $g-4$ distinguishes a subvariety of dimension $3g-3$, and one of components of this subvariety coincides with the closure of $\mathscr{M}_g$. This subvariety is reducible already for $g=4$ (Beauville [16]).

Necessary and sufficient conditions characterizing the Jacobi varieties of non-hyperelliptic surfaces were obtained by Lie and Wirtinger. These conditions are equivalent to the condition that the theta divisor has two distinct representations in a neighbourhood of some non-singular point of the divisor, in the form of a shifted sum of $g-1$ copies of the surface C embedded in an Abelian variety, that is,

$$\Theta = C + \cdots + C + \kappa. \tag{1.12}$$

The problem of an effective description of geometric Lie–Wirtinger conditions in terms of equations is by no means trivial and does not yet have a final solution (see the survey [17]).

The characterization of Jacobians proposed by Gunning ([18], [19]) was also geometric in its original formulation. This characterization is based on Fay's trisecant formula [20].

Let us consider a map of a principally polarized Abelian variety X into a complex projective space $\mathbb{CP}^{2^g-1}$, defined by a basis family (1.9) of theta functions of weight two:

$$\phi_2(z) = \Theta[\varepsilon_1, 0](z) : \cdots : \Theta[\varepsilon_{2^g}, 0](z). \tag{1.13}$$

These functions are even, and therefore the map ϕ_2 can be decomposed into a composition

$$X \xrightarrow{\pi} X/\sigma \xrightarrow{K} \mathbb{CP}^{2^g-1}, \tag{1.14}$$

where $\sigma(z) = -z$ is the involution of the Abelian variety and π is the projection onto the quotient space. The map K is referred to as the *Kummer map* and its image $K(X)$ is called the *Kummer variety*. As is known, the Kummer map is an embedding of a variety with singularities. By an *N-secant* of the Kummer variety we mean an $(N-2)$-dimensional plane in $\mathbb{CP}^{2^g-1}$ meeting $K(X)$ at N points. The existence of an N-secant passing through points $K(A_i)$, $i = 0, 1, \dots, N-1$, is equivalent to the condition that these points are linearly dependent, that is, the condition that there are constants c_i for which $\sum_{i=0}^{N-1} c_i K(A_i) = 0$. An immediate corollary to Fay's trisecant formula is the following assertion: if X is the Jacobian of an algebraic curve Γ, then any three distinct points of Γ determine a *one-parameter* family of trisecants. In a somewhat rougher form, the main result of Gunning is that the existence of such a one-parameter family of trisecants is not only necessary but also sufficient for a given principally polarized Abelian variety to be the Jacobian of some algebraic curve.

The problem of describing the Gunning geometric criterion in terms of equations turned out to be quite non-trivial and required a series of serious steps in its solution. The first steps were made by Welters in the papers [21] and [22] whose starting point was seemingly Mumford's remark [23] that the passage to the limit in Fay's trisecant formula gives the theta function formula (1.6) for algebro-geometric solutions of the

KP equation (Theorem 1.1). An infinitesimal analogue of a trisecant is an inflection point of the Kummer variety, that is, a point A at which there is a line in $\mathbb{CP}^{2^g-1}$ containing the image of the formal 2-germ of some curve in X. By definition, the last condition is equivalent to the existence of g-dimensional vectors $U \neq 0$ and V for which the 2^g-dimensional vectors $K(A)$, $\partial_U K(A)$, $(2\partial_V + \partial_U^2)K(A)$ are linearly dependent. By [22], the existence of vectors U and V such that the set of the corresponding inflection points contains a *formal infinite* germ of some curve in X is a characteristic property of the Jacobians.

A fundamental fact of the theory of soliton equations is that corresponding to each of these equations is an entire system of simultaneous equations, the so-called hierarchy of equations. Algebro-geometric solutions of the KP hierarchy are given by the formula

$$u(t_1, t_2, \dots) = 2\partial_x^2 \log \theta\biggl(\sum_i U_i t_i + Z \,|\, B\biggr), \qquad t_1 = x, \quad t_2 = y, \quad t_3 = t. \quad (1.15)$$

Using the results of Welters [22], Arbarello and De Concini [24] proved that the Gunning theorem implies that a function $u(t)$, $t = \{t_i\}$, of the form (1.15) satisfies the first $N = N(g) = [(\frac{3}{2})^g g!]$ equations of the KP hierarchy if and only if the matrix B is the b-period matrix of some algebraic curve. We note that the bound obtained in [24] for the number of equations in the KP hierarchy that are needed to characterize the Jacobians is certainly overestimated. A trivial consequence of results of one of the authors about commuting ordinary differential operators is the assertion that it suffices to take the first $N = g+1$ equations of the hierarchy. The Novikov conjecture was that the number of equations *does not depend* on g and is equal to $N = 1$.

The crucial step in the argument proposed by Shiota to prove the Novikov conjecture was the proof of the following statement: if a function $u(x, y, t)$ given by the formula (1.6) satisfies the KP equation, then one can find vectors U_i such that the function $u(t)$ given by the formulae (1.15) is a solution of the KP hierarchy. The main problem faced by Shiota was that the possibility of the above extension of a solution of the KP equation to a solution of the KP hierarchy is controlled by some *a priori* non-trivial cohomological obstruction. A sufficient condition for this obstruction to be trivial is that the theta divisor Θ contain no complex line parallel to the vector $U = U_1$. The proof of the last property was the most technically complicated part of Shiota's work, whose meaning was clarified in the paper [25].

The interest in topics connected with the Riemann–Schottky problem did not disappear after the proof of the Novikov conjecture. First of all, this is related to a series of other problems in the geometry of Abelian varieties, among which we note the problem of characterizing the principally polarized Abelian varieties that are Prymians of two-fold coverings of algebraic curves and also the remarkable conjecture of Welters that the existence of a *single* trisecant is sufficient to characterize the Jacobians. A simple comparison of the Welters conjecture with the Gunning theorem requiring the existence of such a one-parameter family of secants shows how strong the assertion of the conjecture is. We note that, at present, it is proved that the existence of a trisecant distinguishes among the principally polarized Abelian varieties a subvariety such that one of its components coincides with the variety of Jacobians [26]. Moreover, it is known that the corresponding subvariety is irreducible for $g = 4, 5$ [27].

The objective of this paper is to present and analyze some new approaches to problems of Riemann–Schottky type connected with integrable *functional* and *linear* differential equations. Let us begin with the multidimensional vector analogue of the Cauchy equation proposed by the authors in [28] and [29].

The classical Cauchy functional equation ([30], Chap. V, § 1, pp. 98–105)

$$\psi(x+y) = \psi(x)\psi(y), \tag{1.16}$$

which arises in enormously many problems (see, for instance, [31]), completely characterizes the exponential function $\psi(x) = e^{kx}$, where k is a parameter. The equation (1.16) is one example of the so-called addition theorems of the form

$$F(f(x), f(y), f(x+y)) = 0. \tag{1.17}$$

Until recently, there were few examples of addition theorems of this kind. For instance, by the Weierstrass theorem, if F is a polynomial in three variables, then the only functions in the class of analytic functions $f(x)$ admitting an addition theorem of the above form are the elliptic functions (that is, the functions connected with algebraic curves of genus $g = 1$) and their degenerations.

It is natural to refer to an equation of the form

$$F(f(x), \phi(y), \psi(x+y)) = 0, \tag{1.18}$$

where $f(x) = \bigl(f_1(x), \ldots, f_N(x)\bigr)$, $\phi(y) = (\phi_1(y), \ldots, \phi_N(y))$, $\psi(z) = (\psi_1(z), \ldots, \psi_N(z))$ are vector functions of the g-dimensional arguments $x = (x_1, \ldots, x_g)$, $y = (y_1, \ldots, y_g)$, $z = (z_1, \ldots, z_g)$, and F is a function of $3N$ variables, as a *vector multidimensional addition theorem*.

It should be noted that versions of theorems of this kind can be found already in the classical paper [32] of Abel, in which the following problem is treated: *determine three functions ϕ, f, and ψ satisfying the equation*

$$\psi(\alpha(x,y)) = F(x, y, \phi(x), \phi'(x), \ldots, f(y), f'(y), \ldots), \tag{1.19}$$

where α and F are given functions of the corresponding number of variables. In particular, in [32] this problem was solved for the equation

$$\psi(x+y) = \phi(x)f'(y) + f(y)\phi'(x). \tag{1.20}$$

Vector addition theorems important for modern applications can be found in the paper [33] of Frobenius and Stickelberger, where, for instance, it was shown that the Weierstrass zeta function satisfies the functional equation

$$\bigl(\zeta(x) + \zeta(y) + \zeta(z)\bigr)^2 + \zeta'(x) + \zeta'(y) + \zeta'(z) = 0, \tag{1.21}$$

where $x + y + z = 0$.

As was noted above, the concept of Baker–Akhiezer functions plays the crucial role in the algebro-geometric scheme for integrating soliton equations.

The Baker–Akhiezer functions are uniquely determined by their analytic properties on an auxiliary algebraic curve. These properties enable one to interpret the Baker–Akhiezer functions as analogues of the exponential functions on algebraic curves. The initial aim of the papers [28] and [29] was to try to find functional equations *characterizing* the Baker–Akhiezer functions to the same extent to which the Cauchy equation characterizes the ordinary exponential function. It turned out that the Baker–Akhiezer functions satisfy a functional equation which is a 'vector analogue' of the Cauchy equation (1.16).

The Cauchy equation is 'rigid' in a sense. The class of functions determined by it remains practically the same if one weakens the equation (1.16) and considers the equation

$$c(x+y)\psi(x)\psi(y)=1,$$

whose solutions are given by the formulae $\psi(x)=e^{k(x+x_0)}$, $c(x)=e^{-k(x+2x_0)}$. By the vector analogue of (1.16) we mean the functional equation

$$\sum_{k=1}^{N+1} c_k(x+y)\psi_k(x)\psi_k(y)=1 \tag{1.22}$$

for vector functions $c(x)=\bigl(c_1(x),\dots,c_{N+1}(x)\bigr)$ and $\psi(x)=\bigl(\psi_1(x),\dots,\psi_{N+1}(x)\bigr)$.

Important remark. In what follows we distinguish between the *strong* and *weak* forms of the functional equation (1.22). The point is that, when assuming a kind of genericity, the functions c_k in this equation can be explicitly expressed by using the functions ψ_k and their derivatives in the form of ratios of certain $(N+1)$-dimensional determinants. Therefore, by a solution of the weak form of (1.22) we mean a vector function $\psi=\bigl(\psi_1,\dots,\psi_{N+1}\bigr)$ satisfying (1.22) for some family of functions c_k. By a solution of the strong form of (1.22) we mean a set of functions c_k, ψ_k in which the expressions for c_k do not explicitly reduce to general determinantal formulae.

As was proved in [28], the Baker–Akhiezer functions corresponding to algebraic curves of genus g give a solution of the weak form of the equation (1.22) for $N=g$. Explicit expressions of these functions in terms of theta functions of Riemann surfaces show that the corresponding solutions of the equation (1.22) have the following special form up to an exponential factor:

$$\psi_k(x)=\frac{\tau(x+A_k)}{\tau(x+A_0)}\,, \tag{1.23}$$

where $\tau(z)$ is a function of the vector argument $z=(z_1,\dots,z_g)$ which is a theta function of an algebraic curve, $A_k=(A_{k,1},\dots,A_{k,g})$ are g-dimensional vectors, and $k=0,1,\dots,N+1$. Explicit expressions for the functions c_k in terms of theta functions were proposed in the paper [29] of the authors, and these expressions do not reduce to general determinantal formulae. Thus, the *strong form* of the addition theorem for theta functions of algebraic curves was established.

The formula (1.23) enables one to introduce the notion of *g-dimensional Cauchy functional equation of rank N* as a vector addition theorem of the following special form.

Definition. A function $\tau(z)$ of a g-dimensional vector argument z is said to be a *solution of the Cauchy equation of rank* N if there are g-dimensional vectors A_0, $A_1, \dots, A_{N+1}$ and functions c_k such that the following equation holds:

$$\sum_{k=1}^{N+1} c_k(x+y)\tau(x+A_k)\tau(y+A_k) = \tau(x+A_0)\tau(y+A_0). \tag{1.24}$$

Remark-definition. In general, one and the same function $\tau(z)$ can be a solution of Cauchy functional equations of different ranks (due to special choices of the vectors A_k). By the *rank* of such a function $\tau(z)$ we mean the minimal number N_* of the possible ranks N of the equations (1.24) satisfied by the function.

One of the results of the papers [28] and [29] is the assertion that *all* solutions of the equation (1.22) for $N=2$ are Baker–Akhiezer functions of genus 2. This shows that the vector analogue of the Cauchy equation (1.22) for $N=2$ is equivalent to the Cauchy functional equation of rank 2.

In this connection, we recall the question posed by us in [28]: *are the functional equations* (1.22) *and* (1.24) *equivalent for* $N>2$?

It follows from the classical addition theorem for the theta function $\tau=\theta(z\,|\,B)$ corresponding to a generic Abelian variety of dimension g (see the formula (1.25) below) that these functions are solutions of the Cauchy functional equation (1.24) of rank $N \leqslant 2^g-1$. This, when compared with the fact that *the rank of the theta functions does not exceed* g *for the Jacobians of curves*, motivated the authors to state the following conjecture.

Conjecture (Buchstaber [Bukhshtaber]–Krichever [28]). *The rank of a given theta function does not exceed* g *if and only if it is constructed from the matrix of* b*-periods of holomorphic differentials on a Riemann surface of genus* g.

Under the assumption that the vectors A_k are generic, this conjecture was recently proved by Grushevskii [34], [35]. Before formulating the main idea of his proof, we now present an approach, which is standard in the theory of theta functions, to the reduction of certain quadratic relations for these functions to the conditions of linearly dependence of the corresponding points of the Kummer variety.

The theta function $\theta(z\,|\,B)$ corresponding to a generic Abelian variety of dimension g admits the classical addition theorem

$$\theta(z_1+z_2\,|\,B)\,\theta(z_1-z_2\,|\,B) = \sum_{\varepsilon} \theta[\varepsilon,0](2z_1\,|\,2B)\,\theta[\varepsilon,0](2z_2\,|\,2B), \quad \varepsilon \in \frac{1}{2}\mathbb{Z}^g/\mathbb{Z}^g. \tag{1.25}$$

The weak form of our addition theorem for these functions becomes

$$\sum_{k=0}^{N+1} c_k(x+y)\theta(x+A_k)\theta(y+A_k) = 0, \tag{1.26}$$

where $c_0(x) = -1$. According to (1.25), by setting $z_1 + z_2 = x + A_k$ and $z_1 - z_2 = y + A_k$, we obtain

$$\theta(x + A_k \,|\, B)\,\theta(y + A_k \,|\, B) = \sum_{\varepsilon} \theta[\varepsilon, 0](x + y + 2A_k \,|\, 2B)\,\theta[\varepsilon, 0](x - y \,|\, 2B), \quad \varepsilon \in \frac{1}{2}\mathbb{Z}^g/\mathbb{Z}^g. \tag{1.27}$$

Making the change $x + y = 2u$ and $x - y = 2v$, we substitute this expression into (1.26) and, using the notation (1.9), obtain

$$\sum_{\varepsilon}\left[\sum_{k=0}^{N+1} c_k(2u)\Theta[\varepsilon, 0](u + A_k)\right]\Theta[\varepsilon, 0](v) = 0. \tag{1.28}$$

The family of functions $\Theta[\varepsilon, 0](v)$, where ε ranges over $\frac{1}{2}\mathbb{Z}^g/\mathbb{Z}^g$, is linearly independent. Hence, it follows from (1.28) that the $(N+2)$ points A_k, $k = 0, 1, \ldots, N+1$, must satisfy the relations

$$\sum_{k=0}^{N+1} c_k(2u)K(u + A_k) = 0 \tag{1.29}$$

for any $u \in \mathbb{C}^g$, where K is the Kummer map.

The above arguments establish the equivalence between the weak form (1.26) of our addition theorem for theta functions of the Jacobians and a special case of a more general result of the paper [19] (see Corollary 2.1 below). The main idea of Grushevskii's proof of our conjecture is that, under certain non-degeneracy conditions on the vectors A_i, one can show that the functions $c_i(2u)$ in the equation (1.29) for $N = g$ are well defined as global meromorphic functions of the variable $u \in \mathbb{C}^g$. In this case the $(g-1)$ equations $c_i(2u) = 0$, $i > 2$, single out a one-parameter family of trisecants. The proof of our conjecture was thus reduced to the Gunning theorem.

We stress that the necessity of additional non-degeneracy conditions on the vectors A_i is related to the fact that the equivalence of strong and weak forms of the addition theorems for theta functions is by no means trivial. The assertion that the equation (1.29) for $N = g$ characterizes the Jacobians of algebraic curves under genericity assumptions which differ from those used in [35] was recently obtained in [36]. The starting point of the latter paper, whose authors were seemingly not acquainted with our conjecture, was a very interesting attempt to construct an analogue of the Castelnuovo theory (see [37]) for Abelian varieties.

As was already mentioned above, in the paper [29] the authors proposed explicit expressions for the functions c_k in terms of theta functions, and these expressions do not reduce to determinantal formulae. This *strong form* of the addition theorem has a remarkably simple form of *cubic* relations for theta functions of Jacobian varieties.

Theorem 1.3. *For any family of $(g+2)$ pairwise distinct points $A_k \in \Gamma \subset J(\Gamma)$,*

$$\sum_{k=0}^{g+1} h_k\,\theta(A_k + x)\theta(A_k + y)\theta(A_k + z)\big|_{x+y+z=R} = 0. \tag{1.30}$$

The constants h_k and the constant vector R in the formula (1.30) depend on the family of points of the curve and are given by the formula (2.20). The expressions take an especially simple form if all the points A_k are Weierstrass points of hyperelliptic curves. This observation was made in [34] and used there to solve the well-known problem on characterizing hyperelliptic Jacobians.

In § 2.4 we consider the addition theorem for theta functions of Jacobians that corresponds to the limit case of coinciding points A_k. The weak form of this addition theorem claims that the equality

$$\sum_{k=0}^{g+1} c_k(u)\mathscr{D}^{(k)}_{U_1,\dots,U_k} K(A_0+u)=0, \qquad u\in\mathbb{C}^g, \tag{1.31}$$

holds for any point $A\in\Gamma\subset J(\Gamma)$, where the vectors U_i are determined by the $(g+1)$-germ of the curve Γ at the point A_0, that is, by the congruence $A_0-\sum_{s=1}^{g+1}\frac{1}{s}U_s z^s\in\Gamma \pmod{z^{g+2}}\subset J(\Gamma)$, and the differential operators $\mathscr{D}^{(k)}_{U_1,\dots,U_k}$ are determined by the equality

$$\exp\biggl(-\sum_{s=1}^{g+1}\frac{z^s}{s}\partial_{U_s}\biggr)=\sum_{k=0}^{g+1} z^k\mathscr{D}^{(k)}_{U_1,\dots,U_k}+O(z^{g+2}). \tag{1.32}$$

The equality (1.31) can be regarded as the definition of the notion of $(g+2)$-*multiple flattening point*. We note that, under this definition, the 3-multiple flattening points are just the inflection points whose definition was given above. Under genericity assumptions, when the coefficients c_k can be uniquely recovered up to a common factor, the equations $c_i(u)=0$, $i>1$, define at least a one-parameter family of inflection points. Hence, by the results of Welters, the equation (1.31) in general position is characteristic for the Jacobians. In the opinion of the authors, the following conjecture is natural.

Conjecture. *The equation* (1.31) *characterizes the Jacobians without the genericity assumption.*

In principle, the proof of (1.31) presented in § 2.4 enables one to obtain formulae for the coefficients c_k, that is, to establish the strong form of the addition theorem. However, the corresponding expressions are extremely awkward.

Problem. Find an effective form of the strong addition theorem (1.31).

The same problem is important for the intermediate addition theorems corresponding to the limit cases of the equality (1.29) for which only some of the points A_k coincide. For instance, if the points $A_0,\dots,A_g$ are merged, then the weak form of the corresponding addition theorem claims that the equality

$$K(A+u)=\sum_{k=0}^{g} c_k(u)\mathscr{D}^{(k)}_{U_1,\dots,U_k} K(A_0+u)=0, \qquad u\in\mathbb{C}^g, \tag{1.33}$$

holds for any pair of distinct points $A_0, A\in\Gamma\subset J(\Gamma)$, where the vectors U_i are determined by the g-germ of Γ at the point A_0, that is, by the condition $A_0-\sum_{s=1}^{g}U_s\frac{z^s}{s}\in\Gamma \pmod{z^{g+1}}\subset J(\Gamma)$. The problem of finding an effective version for the strong form of the equality (1.33) is of special interest because it is related to the theory of continuous analogues of the so-called Krichever–Novikov bases (KN-bases).

The KN-bases were introduced in [38] to solve the problem of operator quantization for closed bosonic strings. These bases are analogues of the Laurent bases in the case of algebraic curves of arbitrary genus with a pair of distinguished points. In essence, these bases $\psi_n(A)$, $A \in \Gamma$, are a special case of the Baker–Akhiezer functions of a discrete argument $n \in \mathbb{Z}$. As was noted in [38], in this case the discrete Baker–Akhiezer functions ψ_n satisfy the remarkable relation

$$\psi_n(A)\psi_m(A) = \sum_{k=0}^{g} c_{n,m}^k \psi_{n+m+k}(A), \tag{1.34}$$

which was a foundation of the notions of *almost graded* algebras and modules over them.[1]

The notion of continuous analogue of KN-bases was introduced in the paper [39] of Grinevich and Novikov, where it was proved that in a special case the corresponding Baker–Akhiezer functions $\psi(t_1, A)$ of a continuous argument satisfy the equation

$$\psi(t_1, A)\psi(t_1', A) = \mathscr{L}\psi(t_1 + t_1', A), \tag{1.35}$$

where $\mathscr{L}$ is a differential operator of order g with respect to the variable t_1 and replaces the difference operator on the right-hand side of (1.34).

The solution of the problem of explicit effective recovery of the coefficients of the operator $\mathscr{L}$ (which is equivalent to the problem of finding the strong form of the addition theorem (1.33)) is presented in the third and fourth sections of the survey, which are devoted mainly to addition theorems derived from *trilinear equations* recently introduced by one of the authors of this survey together with D. V. Leikin (see [40] and [41]).

The role played by the Hirota bilinear equations [42] in the contemporary theory of soliton equations is well known. The Hirota equations can be regarded as the limit case of the weak form of the addition theorem (1.24). The authors have no doubt that the trilinear equations will play the same role in the study of the strong forms of addition theorems of the form (1.30) and their degenerations. The formalism of the theory of trilinear equations turns out to be also extremely useful when studying addition theorems of more general form than that in (1.22), namely,

$$\sum_{k=1}^{N+1} c_k(x+y)\varphi_k(x)\psi_k(y) = 1. \tag{1.36}$$

The equation (1.36) is an equation for three vector functions, namely, $c(x)$, $\varphi(x)$, and $\psi(x)$. If $N = 0$, then the solution of (1.36) is again given by exponential functions. The case $N = 1$ was studied in detail in [43]. As above, the solution in the class of analytic functions is expressed in terms of Baker–Akhiezer functions of genus 1. Explicit solutions of this equation in the case of arbitrary $N = g$ were constructed in [29] in terms of theta functions of Jacobian varieties.

[1] For brevity we use in (1.34) an indexing that differs by the shift $n \to n - g/2$ from the notation of the paper [38].

Let us note that the classical form of the addition theorem (1.17), where F is a polynomial in three variables, is obviously a special case of the equation (1.36). There are many reasons to regard the following assertion as an analogue of the famous Weierstrass theorem for the equation (1.36).

Conjecture. *Up to exponential factors, every solution of an equation of the form*

$$\sum_{k=1}^{N+1} c_k(x+y)\varphi_k(x)\psi_k(y) = \varphi_0(x)\psi_0(y), \tag{1.37}$$

in the class of analytic functions on $\mathbb{C}^g$ is given by a family of theta functions on an Abelian variety.

Additional conditions, like a dependence of N on g and an assumption about the form of the functions $c_k(x)$, $k = 1, \dots, N+1$, yield in particular some conditions on the Abelian varieties similar to the conditions which had led to the characterization of the Jacobians. An addition theorem of the form

$$F(\mathbf{f}_1(x), \mathbf{f}_2(y), \mathbf{f}_3(x+y)) = 0, \tag{1.38}$$

where $\mathbf{f}_k(x)$, $x \in \mathbb{C}^g$, is a vector function consisting of an N-tuple of partial derivatives of the functions f_k, $k = 1, 2, 3$, and F is a polynomial in $3N$ variables, gives important equations of class (1.36).

The fifth and last section is devoted to the application of *integrable linear equations* to problems related to the Riemann–Schottky problem. The KP equation (1.7) is the compatibility condition for the overdetermined system of auxiliary linear problems

$$(\partial_y - L)\,\psi = 0, \qquad (\partial_t - A)\,\psi = 0 \longmapsto [\partial_y - L, \partial_t - A] = 0, \tag{1.39}$$

where L and A are differential operators of the form

$$L = -\partial_x^2 + u(x,y,t), \qquad A = \partial_x^3 - \frac{3}{2}\,u\,\partial_x + w(x,y,t). \tag{1.40}$$

The equivalence between the non-linear equation and the compatibility condition for the overdetermined system of linear problems can be regarded to some extent as a definition of the soliton equations. Here the role of one of the auxiliary equations is distinguished when constructing spectral transformations linearizing the corresponding non-linear equation. The primary role of the linear problem is clearly manifested in the scheme for constructing elliptic solutions of the KP equation proposed in [44] and extended later to some other soliton equations in [45]–[47].

In a recent paper of one of the authors it was shown that the characterization of Jacobians by using the KP equation contains superfluous information, and the Riemann–Schottky problem can be solved with the help of only one of the auxiliary linear problems (1.40).

Theorem 1.4 (Krichever [48]). *A symmetric matrix B with positive-definite imaginary part is the b-period matrix of a basis of normalized holomorphic differentials* *on some smooth algebraic curve* Γ *if and only if there are two constants* p *and* E *and three* g*-dimensional vectors* U, V, A *such that the linear equation*

$$(\partial_y - \partial_x^2 + u(x,y))\psi = 0 \tag{1.41}$$

with the potential

$$u = 2\partial_x^2 \log\theta(Ux + Vy + Z \,|\, B) \tag{1.42}$$

has a solution of the form

$$\psi = \frac{\theta(A + Ux + Vy + Z \,|\, B)}{\theta(Ux + Vy + Z \,|\, B)} \exp(px + Ey). \tag{1.43}$$

Using the addition theorem (1.25), one can readily show that the equation (1.41) with u and ψ defined by (1.42) and (1.43) is equivalent to the system of equations

$$\partial_y\Theta[\varepsilon,0](A/2) - \partial_x^2\Theta[\varepsilon,0](A/2) - 2p\,\partial_x\Theta[\varepsilon,0](A/2) + (E-p^2)\Theta[\varepsilon,0](A/2) = 0, \tag{1.44}$$

which must hold for any half-integer characteristic ε.

The description of Jacobians of algebraic curves by means of the system of equations (1.44) is stronger than the characterization using the system of equations (1.11). In terms of the Kummer map, the equations (1.44) are equivalent to the condition that the corresponding Kummer variety has an inflection point. This condition is a special case of the Welters trisecant conjecture [22].

In the opinion of the authors, the possibilities of the approaches described in the present survey for diverse problems in the geometry of Abelian varieties are far from being exhausted. As an acknowledgement, we note the recent paper [49] in which the characterization problem for the Prymians was solved by using the integrable ansatz for the two-dimensional Schrödinger operator.

§ 2. Baker–Akhiezer functions and addition theorems

As was already noted in the Introduction, the starting point of the papers [28] and [29] in which the vector analogues (1.22), (1.24), and (1.36) of the Cauchy equation were introduced was an attempt to find functional equations characterizing the Baker–Akhiezer functions. Before presenting the results of these papers, we formulate necessary facts about the Baker–Akhiezer functions.

2.1. Baker–Akhiezer functions. Let Γ be a non-singular algebraic curve of genus g with distinguished points P_α and fixed local coordinates $k_\alpha^{-1}(Q)$ in neighbourhoods of these distinguished points, where $k_\alpha^{-1}(P_\alpha) = 0$ for $\alpha = 1,\dots,l$. Let us fix a family $\mathbf{q} = \{q_\alpha(k)\}$ of polynomials,

$$q_\alpha(k) = \sum_i t_{\alpha,i} k^i. \tag{2.1}$$

As was proved in [1], [2], for any generic family of points $\gamma_1,\dots,\gamma_g$ there is a function $\psi(t,Q)$, $t = \{t_{\alpha,i}\}$, unique up to proportionality, such that:

(a) ψ *is meromorphic on* Γ *outside the points* P_α *and has at most simple poles at the points* γ_s (*if they are distinct*);

(b) ψ *can be represented in a neighbourhood of* P_α *as*

$$\psi(t,Q) = \exp\bigl(q_\alpha(k_\alpha)\bigr)\left(\sum_{s=0}^{\infty} \xi_{s,\alpha}(t)k^{-s}\right), \qquad k_\alpha = k_\alpha(Q). \tag{2.2}$$

We choose a point P_0 and normalize the function ψ by the condition

$$\psi(t,P_0) = 1. \tag{2.3}$$

It should be stressed that the Baker–Akhiezer function $\psi(t,Q)$ is determined by its analytic properties with respect to the variable Q. It depends on the coefficients $t_{\alpha,i}$ of the polynomials q_α as on *external parameters.*

The *existence* of the Baker–Akhiezer function is proved by presenting an explicit theta function formula [1]. We introduce some necessary notation. First of all, we recall that fixing a basis a_i, b_i of cycles on Γ with canonical intersection matrix enables us to define a basis of normalized holomorphic differentials ω_i, the matrix B of their b-periods, and the corresponding theta function $\theta(z) = \theta(z \mid B)$. The basis vectors e_k and the vectors $B_k = \{B_{kj}\}$ span a lattice in $\mathbb{C}^g$, and the quotient by this lattice is a g-dimensional torus $J(\Gamma)$, the so-called *Jacobian* of the curve Γ. The map $A\colon \Gamma \longmapsto J(\Gamma)$ given by the formula

$$A_k(Q) = \int_{Q_0}^{Q} \omega_k \tag{2.4}$$

is referred to as the *Abel map.* If a vector Z is defined by the formula

$$Z = \kappa - \sum_{s=1}^{g} A(\gamma_s), \tag{2.5}$$

where κ is a vector of Riemann constants depending on the choice of basis cycles and on the initial point of the Abel map Q_0, then the function $\theta(A(Q)+z)$ (if it is not identically zero) has exactly g zeros on Γ coinciding with the points γ_s,

$$\theta(A(\gamma_s) + Z) = 0. \tag{2.6}$$

We note that the function $\theta(A(Q) + Z)$ itself is multivalued on Γ, but the zeros of this function are well defined. Let us introduce differentials $d\Omega_\alpha$ such that the differential $d\Omega_{\alpha,i}$ is holomorphic outside P_α, has a pole at P_α of the form $d\Omega_\alpha = d(k_\alpha^i + O(k_\alpha^{-1}))$, and is normalized by the condition $\oint_{a_i} d\Omega_\alpha = 0$. We denote by $2\pi i U_{\alpha,i}$ the vector of its b-periods with coordinates

$$2\pi i U_{\alpha,i,k} = \oint_{b_k} d\Omega_{\alpha,i}. \tag{2.7}$$

Using the translation properties of the theta function, one can show that the function $\psi(t,Q)$ given by the formula

$$\psi(t,Q) = \Phi(x,Q)\exp\left(\sum_{\alpha,i} t_{\alpha,i}\int_{P_0}^{Q} d\Omega_{\alpha,i}\right), \tag{2.8}$$

where

$$\Phi(x,Q)=\frac{\theta(A(Q)+x+Z)\,\theta(A(P_0)+Z)}{\theta(A(Q)+Z)\,\theta(A(P_0)+x+Z)} \tag{2.9}$$

and

$$x=\sum_{\alpha,i} t_{\alpha,i}U_{\alpha,i}, \tag{2.10}$$

is a *single-valued* function of the variable Q on Γ. It follows from the definition of the differentials $d\Omega_{\alpha,i}$ that this function has the desired form of essential singularity at the points P_α, and it follows from (2.6) that the function ψ has poles outside these points at the points γ_s.

The *uniqueness* of the Baker–Akhiezer function reduces to the Riemann–Roch theorem, according to which a meromorphic function on Γ having at most g generic poles is constant. Indeed, suppose that the analytic properties of a function $\widetilde{\psi}$ coincide with those of ψ. Then the function $\widetilde{\psi}\psi^{-1}$ is a meromorphic function on Γ whose number of poles does not exceed g. Hence, this is a constant which is equal to 1 due to the normalization condition (2.3).

We present another assertion which is needed in what follows and whose proof also reduces to the Riemann–Roch theorem. For any positive divisor $D=\sum n_iQ_i$ we consider the linear space $L(\mathbf{q},D)$ of functions having poles outside the points P_α at the points Q_i and of multiplicities at most n_i and having the form (2.1) in a neighbourhood of P_α. As was proved in [1], the dimension of this space is equal to $\dim L(\mathbf{q},D)=d-g+1$ for generic divisors of degree $d=\sum n_i\geqslant g$.

The function $\Phi(x,Q)$ given by formula (2.9) belongs to the class of the so-called *factorial* functions [50]. This is a multivalued function on Γ. A single-valued branch of it can be singled out if one draws cuts on Γ along a-cycles. In what follows we use the notation Γ^* for the curve Γ with such cuts.

Lemma 2.1. *For any generic family of points $\gamma_1,\dots,\gamma_g$ the formula (2.9) with the vector Z defined by (2.5) defines a unique function $\Phi(x,Q)$, where $x=(x_1,\dots,x_g)$ and $Q\in\Gamma$, having the following properties*:

1) *Φ is a single-valued meromorphic function of Q on Γ^* having at most simple poles at the points γ_s (if they are all distinct)*;
2) *the boundary values $\Phi_j^{\pm}(x,Q)$, $Q\in a_j$, on the different sides of the cuts satisfy the conditions*

$$\Phi_j^+(x,Q)=e^{-2\pi i x_j}\Phi_j^-(x,Q), \qquad Q\in a_j; \tag{2.11}$$

3) *Φ is normalized by the condition*

$$\Phi(x,P_0)=1. \tag{2.12}$$

As in the case of Baker–Akhiezer functions, for any effective divisor $D=\sum n_iQ_i$ one can introduce the notion of associated linear space $\mathscr{L}(x,D)$ as the space of meromorphic functions on Γ^* having poles at the points Q_i of multiplicities not exceeding n_i and such that the boundary values on the sides of the cuts satisfy (2.11). It follows from the Riemann–Roch theorem that for a generic divisor of degree $d=g$ the dimension of this space is equal to

$$\dim\mathscr{L}(x,D)=d-g+1. \tag{2.13}$$

According to (2.13), the dimension of the space of factorial functions with poles of multiplicity at most two at the points γ_s is equal to $g+1$ for any generic family of points $\gamma_1, \ldots, \gamma_g$. The following assertion can be regarded as an explicit representation of a set of g functions which, together with the function Φ, define a basis in this space.

Lemma 2.2. *For any generic family of points $\gamma_1, \ldots, \gamma_g$ the function $C_k(x, Q)$ given by the formula*

$$C_k(x,Q) = \frac{\theta(A(Q)+Z+A(\gamma_k)-A(P_0))\,\theta(A(Q)+x+Z-A(\gamma_k)+A(P_0))}{\theta^2(A(Q)+Z)\,\theta(A(P_0)+x+Z)\,\theta(2A(\gamma_k)-A(P_0)+Z)}, \tag{2.14}$$

where Z is defined by (2.5), is a unique function such that:

1) *C_k, as a function of the variable $Q \in \Gamma^*$, is a meromorphic function with at most simple poles at the points γ_s, $s \neq k$, and a second-order pole of the form*

$$C_k(x,Q) = \theta^{-2}(A(Q)+Z) + O(\theta^{-1}(A(Q)+Z)) \tag{2.15}$$

at the point γ_k;

2) *the boundary values $C^{\pm}_{k,j}(x,Q)$, $Q \in a_j$, of the function C_k on the different sides of the cuts satisfy the equations*

$$C^{+}_{k,j}(x,Q) = e^{-2\pi i x_j} C^{-}_{k,j}(x,Q), \qquad Q \in a_j; \tag{2.16}$$

3)

$$C_k(x, P_0) = 0. \tag{2.17}$$

The uniqueness of a function C_k having the above properties is an immediate corollary to the Riemann–Roch theorem. The fact that the function C_k given by the formula (2.14) satisfies these conditions can be verified immediately. Indeed, the function $\theta(A(Q)+Z)$ vanishes at the point γ_k, and hence the equality (2.14) means that C_k has a second-order pole with normalized leading coefficient at this point. It follows from the equality (2.5) that the first factor in the numerator vanishes at the point P_0 and at the points γ_s, $s \neq k$. Since the first factor of the denominator has a second-order pole at all points γ_s, the function C_k has first-order poles at the points γ_s, $s \neq k$, and a second-order pole at the point γ_k. The difference between the arguments of the theta functions containing $A(Q)$ in the numerator and the denominator is equal to x. This, together with the monodromy relations (1.10), implies (2.16).

2.2. Strong form of addition theorem for theta functions of Jacobians. Let us consider the function $\Phi(x,Q)\Phi(y,Q)$. It follows from the definition of Φ that this product belongs to the space $\mathscr{L}(x+y, D)$, where $D = 2\gamma_1 + \cdots + 2\gamma_g$. By (2.13), the dimension of this space is equal to $g+1$. The functions $\Phi(x+y,Q)$ and $C_k(x+y,Q)$ are linearly independent. Hence, they form a basis of the space $\mathscr{L}(x+y, D)$. The coefficients of the expansion of $\Phi(x,Q)\Phi(y,Q)$ in this basis can be found by comparing the second-order poles at the points γ_k and by using the normalization condition (2.12). These simple considerations lead to the following assertion, which is in essence the main result of this subsection.

Theorem 2.1. *The following identity holds*:

$$\Phi(x,Q)\Phi(y,Q) = \Phi(x+y,Q) + \sum_{k=1}^{g} C_k(x+y,Q)\phi_k(x)\phi_k(y), \tag{2.18}$$

in which the functions Φ *and* C_k *are defined by the formulae* (2.9) *and* (2.14), *respectively, and the functions* $\phi_k(x)$ *are given by the formula*

$$\phi_k(x) = \frac{\theta(A(\gamma_k)+x+Z)\theta(A(P_0)+Z)}{\theta(A(P_0)+x+Z)}\,. \tag{2.19}$$

In what follows we shall identify the algebraic curve Γ with its image under the Abel map. Correspondingly, the points P_0, γ_s, $Q \in \Gamma$ are identified with the vectors $A_0 = A(P_0)$, $A_s = A(\gamma_s)$, $s = 1,\dots,g$, and $A_{g+1} = A(Q)$.

Direct substitution of the formulae (2.9), (2.14), (2.19) into the equality (2.18), determination of a common denominator, and then cancellation of some factors leads to the following identity:

$$\begin{aligned}&\theta(A_0+x+y+Z)\theta(A_{g+1}+x+Z)\theta(A_{g+1}+y+Z)\\ &\quad= \sum_{k=0}^{g} \frac{\theta(A_{g+1}+x+y+Z-A_k+A_0)\theta(A_{g+1}+Z+A_k-A_0)}{\theta(2A_k+Z-A_0)}\\ &\qquad\times \theta(A_k+x+Z)\theta(A_k+y+Z).\end{aligned} \tag{2.20}$$

The standard application of the bilinear addition formula (1.25) (as described in the Introduction in the proof of the formula (1.29)) shows that the equality (2.20) is equivalent to the equality

$$\sum_{k=0}^{g+1}(-1)^{\delta_{k,g+1}} \frac{\theta(A_{g+1}+2u-Z-A_k+A_0)\,\theta(A_{g+1}+Z+A_k-A_0)}{\theta(2A_k+Z-A_0)}\,K(A_k+u) = 0, \tag{2.21}$$

where $\delta_{k,g+1}$ is the Kronecker delta, which is non-zero only for $k = g+1$, and the variable u is defined by the equality $2u = x+y+2Z$.

The formula (2.20) (or the equivalent formula (2.21)) is a strong form of our addition theorem for theta functions of Jacobian varieties. We stress once again that these formulae hold for any family of points $A_i \in \Gamma \subset J(\Gamma)$, $i = 0,1,\dots,g+1$, and for an arbitrary vector $u \in \mathbb{C}^g$. Let us show that the formula (2.21) readily implies both Fay's trisecant formula and the following more general assertion of Gunning.

Corollary 2.1 (Gunning, Theorem 2 in [19]). *For any non-singular algebraic curve* Γ *of genus* g, *for any number* m, $1 \leqslant m \leqslant g$, *and for any family of* $(2m+1)$ *points of the curve, which is identified with its image under the Abel map* (*that is, for any* $A_0,\dots,A_{m+1},Q_1,\dots,Q_m \in A(\Gamma) \subset J(\Gamma)$), *the* $(m+2)$ *points of the form* $K(A_i+u)$ *of the Kummer variety, where*

$$2u = \sum_{s=1}^{m} Q_s - \sum_{k=0}^{m+1} A_k, \tag{2.22}$$

belong to an m*-dimensional plane.*

We note first that if the vectors Z and u are given by the formulae (2.5) and (2.22), then since the theta function is even, the first factor in the numerator of the formula (2.21) has the form $\theta(A_k + Z')$, where

$$Z' = \kappa - \sum_{s=1}^{m} Q_s - \sum_{k=m+2}^{g+1} A_k. \tag{2.23}$$

It follows from the formulae (2.5) and (2.6), applied to the family $Q_1, \dots, Q_m, A_{m+2}, \dots, A_{g+1}$, that $\theta(A_k + Z') = 0$ for $k > m + 1$. The last equality means that if the vector u is given by the formula (2.22), then only the first $(m+2)$ terms in (2.21) are non-zero. This proves the assertion of the corollary. We note that for $m = g$ this assertion is equivalent to the weak form (1.29) of our addition theorem and for $m = 1$ it coincides with Fay's trisecant formula [20].

2.3. Riemann–Schottky problem. Simple dimensional considerations applied to the equation (1.29) show that the equivalent equation (1.22) holds for $N = 2^g - 1$ for an arbitrary Abelian variety. In its complete form, the conjecture that the equation (1.22) with $N = g$ characterizes the Jacobians of algebraic curves remains unproved. Diverse degenerate situations cause fundamental difficulties. For instance, the equation (1.29) for $N = g$ does not exclude the possibility that the points $K(A_k + u)$ belong to a plane of dimension m with $m < g$. In this case it is impossible to define the coefficients c_k of the linear dependence as single-valued functions of the variable u. The arguments used in the paper [34], where the first attempt to prove the conjecture was made, turned out to be insufficient to exclude such degeneracy. In [35] additional conditions were presented that exclude any possible degeneracy and under which the remaining part of the proof of the conjecture can be preserved.

Theorem 2.2 ([34], [35]). *Suppose that for an irreducible principally polarized Abelian variety X of dimension g there is a family of $(g + 2)$ pairwise distinct points $A_k \in X$ such that the vectors $K(A_k + u)$ are linearly dependent for any $u \in X$. Assume also that there is a pair of indices k and l such that the vectors $K(A_i + v)$, where $2v = -(A_k + A_l)$, span a linear subspace whose dimension is exactly equal to $g + 1$. In this case, X is the Jacobian of some non-singular algebraic curve Γ, and the points A_k belong to the image of Γ in $J(\Gamma)$ under the Abel map.*

The equation (1.29) for given A_k and u is an overdetermined system of linear equations for the coefficients c_k. Under the assumptions of the theorem, the rank of this system for $u = v$ is exactly equal to $g + 1$. Hence, there is a family of characteristics $\varepsilon_0, \dots, \varepsilon_g$ such that the $(g+1)\times(g+2)$ matrix with matrix elements $\{\Theta[\varepsilon_s](A_k + u)\}$ has rank $g+1$ for any u in a small neighbourhood of the vector v. This enables one to uniquely define the coefficients c_i for any $u \in \mathbb{C}^g$ by the formula $c_i(u) = (-1)^i \det\{\Theta[\varepsilon_s](A_k + u),\ k \neq i\}$.

The functions $c_i(u)$, regarded up to proportionality, define a map $c\colon \mathbb{C}^g \to \mathbb{CP}^{g+1}$. Our next objective is to prove that the rank of the differential of this map at the point $u = v$ is maximal. Without loss of generality we can assume that the indices k and l in the statement of the theorem are equal to 0 and 1, respectively. In this case, $K(A_0 + v) = K(A_1 + v)$ by the definition of v and because the theta functions are even. Thus, $c_i(v)=0$ for $i>1$, since the first two columns of the matrix whose determinant defines c_i coincide. By the definition of c_i, we also have $c_0(v)=c_1(v)\neq 0$.

Suppose that the rank of the differential dc at the point v is not maximal. This means that there is a vector $V\in\mathbb{C}^g$ such that the derivative of the map $c(u)$ at v in the direction of V vanishes. The last assertion, with the fact that $c(u)$ is a point of the projective space, is equivalent to the equalities $\partial_V c_i\big|_{u=v}=\lambda c_i(v)$, where λ is a constant independent of i. Differentiating the equality (1.29) using what was said above and the fact that the derivative of the theta function is odd, we obtain

$$\sum_{k=0}^{g+1}\lambda c_i(2u)\big(\lambda K(A_k+u)+\partial_V K(A_k+u)\big)\big|_{u=v}=2c_0\partial_V K(A_0+v)=0. \tag{2.24}$$

Since the Kummer map is an embedding of the quotient space $X/(z=-z)$, it follows from (2.24) that the vector A_0+v must be a point of second order in X. This is possible only if $A_0-A_1=0\in X$, which contradicts the assumption of the theorem that the points must be pairwise distinct.

An immediate consequence of the maximality of the rank of dc thus established is the assertion that the equations $c_i=0$, $i>2$, define a one-parameter family of trisecants, which is a characteristic property of Jacobians by the Gunning theorem. This completes the proof.

As was already noted in the Introduction, the assertion that the equation (1.29) with $N=g$ is characteristic for the Jacobians was proved in the recent paper [36] under additional assumptions which differ from those formulated in [35]. Namely, this assertion was proved under the assumption that the points A_k are in *theta general position*, which means by definition that for any subset of $(g+1)$ points $A_{i_0},A_{i_1},\dots,A_{i_g}$ there is a vector z such that $\theta(A_{i_k}+z)=0$, $k=1,\dots,g$, and $\theta(A_{i_0}+z)\neq 0$.

2.4. Degenerate cases of addition theorems. In this subsection we present addition theorems obtained when considering a special degenerate case of the Baker–Akhiezer function. Namely, let us consider the Baker–Akhiezer function $\psi(t,A;A_0)$ which, regarded as a function of a point $A\in\Gamma\subset J(\Gamma)$ of the curve, is *holomorphic everywhere except for the distinguished point* A_0, at which it is of the form

$$\psi(t,A;A_0)=\biggl(\sum_{s=-g}^{\infty}\xi_s(t,A_0)\,z^s\biggr)\exp\biggl(\sum_{i=1}^{\infty}t_iz^{-i}\biggr), \tag{2.25}$$

where $z=z(A)$ is a local coordinate in a neighbourhood of A_0.[2]

A theta function expression for $\psi(t,A;A_0)$ is given by formulae that are practically identical to the formulae (2.8)–(2.10) in which the vector Z must be set equal to $Z_0=\kappa-gA_0$. By shifting the origin if necessary, one can always assume that the vector of Riemann constants vanishes, $\kappa=0$, and we do assume this in what

[2] We note that the symbols used in this subsection contain an explicit indication of the dependence of ψ on the choice of a distinguished point, in contrast to the general case in which any explicit indication of the dependence of the Baker–Akhiezer function on the choice of the distinguished points and the divisor of the poles was omitted for brevity.

follows. Due to what was said above, the explicit formula for $\psi(t,A;A_0)$ becomes

$$\psi(t,A;A_0)=\Phi(x,A;A_0)\exp\biggl(\sum_{j=1}^{\infty}t_j\Omega_j(A)\biggr), \tag{2.26}$$

where

$$\Phi(x,A;A_0)=\frac{\theta(A+x-gA_0)}{\theta(A-gA_0)\,\theta(x+(g-1)A(P_0))}, \qquad x=\sum_j t_jU_j, \tag{2.27}$$

and $\Omega_j(A)$ is a normalized Abelian integral holomorphic outside the distinguished point A_0 in whose neighbourhood this integral has the form $\Omega_j=z^{-j}+O(1)$.

The classical bilinear Riemann relations establish a relationship between the b-periods $2\pi iU_j$ of the differential $d\Omega_j$ and the coefficients of the expansion of the curve $A(z)\in\Gamma\subset J(\Gamma)$ in a neighbourhood of the point A_0:

$$A(z)=A_0-\sum_{k=1}^{\infty}\frac{1}{k}U_kz^k. \tag{2.28}$$

The addition theorems related to the function $\psi(t,A;A_0)$ and concerning theta functions of the Jacobians are a simple corollary to the assertion that *the following equality holds*:

$$\begin{aligned}\psi(t,A;A_0)\psi(t',A;A_0)&=v_0(t,t',A_0)\psi(t+t',A;A_0)\\&\qquad+\sum_{k=1}^{g}v_k(t,t',A_0)\partial_{t_k}\psi(t+t',A;A_0).\end{aligned} \tag{2.29}$$

The proof of this assertion, a substantive part of which is that the coefficients v_k do not depend on A, is standard. The product on the left-hand side of (2.29) is the Baker–Akhiezer function corresponding to the parameters $(t+t')$ and the divisor $D=2gA_0$. The space of such functions is of dimension $g+1$. The functions $\psi(t+t',A;A_0)$ and $\partial_{t_k}\psi(t+t',A;A_0)$, $k=1,\dots,g$, are linearly independent, and hence form a basis in this space. Thus, the product of Baker–Akhiezer functions can be expanded in this basis. The coefficients of the expansion can be found recursively by comparing the leading coefficients of the expansion of right- and left-hand sides with respect to the parameter z.

Let us now use the explicit theta function formulae for ψ. Substitution of them into (2.29) leads to an equivalent equality of the form

$$\begin{aligned}\theta(A+x-gA_0)\,\theta(A+y-gA_0)&=\theta(A+x+y-gA_0)\,\theta(A-gA_0)\\\times\biggl(w_0(x,y,A_0)&+\sum_{k=1}^{g}\bigl(\Omega_k(A)+\partial_{U_k}\log\theta(A+x+y-gA_0)\bigr)w_k(x,y,A_0)\biggr).\end{aligned} \tag{2.30}$$

Using the addition theorem (1.25) just as in the proof of the formula (1.29), we see that the coefficients w_j in the equality (2.30) have the form

$$\begin{gathered}w_j(x,y,A_0)=\sum_{\varepsilon}W_{j,\varepsilon}(u,A_0)\Theta[\varepsilon,0](v),\\2u=x+y-2gA_0,\qquad 2v=x-y,\end{gathered} \tag{2.31}$$

and the equality itself is equivalent to

$$K(A+u)=\theta(A+u+gA_0)\,\theta(A-gA_0)\times\biggl(W_0(u,A_0)+\sum_{k=1}^{g}\bigl(\Omega_k(A)+\partial_{U_k}\log\theta(A+u+gA_0)\bigr)W_k(u,A_0)\biggr),\tag{2.32}$$

where $W_j\in\mathbb{CP}^{2^g-1}$ are points of the projective space with the homogeneous coordinates $W_{j,\varepsilon}$.

Expanding the equality (2.32) in a neighbourhood of A_0, we see that the vector W_j is a linear combination of the vectors $\mathscr{D}^{(k)}K(A_0+u)$, $k=0,1,\dots,g-j$, where the operators $\mathscr{D}^{(k)}=\mathscr{D}^{(k)}_{U_1,\dots,U_k}$ are given by the formula (1.32). If the local coordinate at the point A_0 is given by $z^g=\theta(A-gA_0)$, then the system of equations determining W_j acquires the especially simple form

$$\mathscr{D}^{(j)}K(A_0+u)=\sum_{k=0}^{j}W_{g-k}\mathscr{D}^{(j-k)}\theta(u+(g+1)A_0),\qquad j=0,\dots,g.\tag{2.33}$$

The formulae (2.32) and (2.33) represent a *strong form of the addition theorem given by* (1.33).

Remark. The equation (2.32) is equivalent to the original equality (2.29), which can be represented in another form. Namely, the functions $\partial^k_{t_1}\psi(t+t',A;A_0)$, $k=0,1,\dots,g$, form a basis in the space $L(t+t',2gA_0)$ of the corresponding Baker–Akhiezer functions. Expanding the left-hand side of (2.29) in this basis, we arrive at the following assertion: *there is a unique differential operator* $\mathscr{L}=\sum_{i=0}^{g}\tilde v(t,t',A_0)\partial^i_{t_1}$ *for which*

$$\psi(t,A;A_0)\psi(t',A;A_0)=\mathscr{L}\psi(t+t',A;A_0).\tag{2.34}$$

As was already mentioned in the Introduction, the relation (2.34) can be regarded as a continuous analogue of the almost graded structure of KN-bases. We shall return to a more detailed consideration of the equality (2.34) in the next section.

In the conclusion of the present section, we give a *strong form* of the equality (1.33). This form is obtained by equating the $(g+1)$th coefficients of the expansions of the right- and left-hand sides of the equality (2.32) in powers of z.

Theorem 2.3. *The equality*

$$\mathscr{D}^{(g+1)}K(A_0+u)=\biggl(\sum_{k=0}^{g}W_{g-k}\mathscr{D}^{(g-k+1)}\theta(u+(g+1)A_0)\biggr)+\theta(u+(g+1)A_0)\biggl(\sum_{k=1}^{g}W_k\bigl(d_k+\partial_{U_1}\partial_{U_k}\log\theta(u+(g+1)A_0)\bigr)\biggr)\tag{2.35}$$

holds for any point $A_0\in\Gamma\subset J(\Gamma)$. *In this equality the vectors* $W_s=W_s(u,A_0)$ *are given by the equations* (2.33) *and the constants* $d_k=d_k(A_0)$ *are determined by the expansion of the Abelian integrals* Ω_k *at the point* A_0, *namely,* $\Omega_k=z^{-k}+d_kz+O(z^2)$.

§ 3. Trilinear equations

3.1. Definition of multilinear operator. Comparison with the Hirota formalism. Let $t_1,\dots,t_{k-1},z\in\mathbb{C}^g$ and let L be a linear differential operator with respect to z with constant coefficients. To the operator L we assign a *k-linear operator* $L(f_1,\dots,f_k)$ by the formula

$$L(f_1,\dots,f_k)(t_1,\dots,t_{k-1})=L\bigl[f_1(t_1+z)\cdots f_{k-1}(t_{k-1}+z)f_k(t_1+\dots+t_{k-1}-z)\bigr]\big|_{z=0}.$$

Any equation of the form $L(f_1,\dots,f_k)=0$ is called a *k-linear equation.*

For $k=2$ this construction gives bilinear differential operators that have long been used in the theory of Abelian functions and invariant theory ([50], [51]). The method of constructing solutions of non-linear evolution equations that was discovered by Hirota [42] has made these bilinear operators widely known as *Hirota operators* among experts in the theory of integrable systems. The Hirota bilinear formalism has a natural generalization which is usually called the *multilinear formalism* (see [52]). If H is a linear differential operator with respect to $t_1,\dots,t_{k-1}$ with constant coefficients, then the *Hirota k-linear operator* $H(f_1,\dots,f_k)$ can be written out as follows:

$$H(f_1,\dots,f_k)(z)=H\Biggl[\prod_{j=1}^{k}f_j\biggl(z+\sum_{m=1}^{k-1}t_m\exp\Bigl\{jm\frac{2\pi\imath}{k}\Bigr\}\biggr)\Biggr]\Bigg|_{t_1=\dots=t_{k-1}=0},$$

where $\imath^2=-1$. *Non-linear partial differential equations* of the form

$$H(f_1,\dots,f_k)(z)=0$$

are studied in the framework of this approach, and it is natural to refer to them as *Hirota k-linear equations.*

For $k\geqslant 3$ the equation $L(f_1,\dots,f_k)=0$ is a *functional equation.* The relationship between our k-linear functional equations and the Hirota k-linear differential equations is the topic of a separate and promising investigation.

In this section our attention is focused on bilinear (differential) equations and trilinear (functional) equations.

3.2. Formalism of bilinear and trilinear operators. Let $u,v,w\in\mathbb{C}^g$. We introduce the operators

$$D_i=\frac{\partial}{\partial u_i}+\frac{\partial}{\partial v_i}-\frac{\partial}{\partial w_i}\,,\qquad i=1,\dots,g.$$

For every tuple of non-negative integers (that is, for any multi-index) of the form $\omega=(\omega_1,\dots,\omega_g)$ we set $D^\omega=D_1^{\omega_1}\cdots D_g^{\omega_g}$ and $\partial^\omega=\partial_1^{\omega_1}\cdots\partial_g^{\omega_g}$, where $\partial_i=\dfrac{\partial}{\partial z_i}$. We define linear differential operators with constant coefficients

$$\mathscr{D}=\sum_{|\omega|\geqslant 0}\alpha_\omega D^\omega\quad\text{and}\quad L=\sum_{|\omega|\geqslant 0}\alpha_\omega\partial^\omega,$$

where $|\omega|=\omega_1+\dots+\omega_g$.

The following relations hold:

$$L(f_1,f_2)(u)=\mathscr{D}(f_1(u)f_2(w))\big|_{w=u},$$
$$L(f_1,f_2,f_3)(u,v)=\mathscr{D}(f_1(u)f_2(v)f_3(w))\big|_{w=u+v}.$$

For the cases in which the triple of functions $f_i(z)$, $i=1,2,3$, is clear, we shall use the brief notation

$$B(\mathscr{D})=\frac{\mathscr{D}(f_1(u)f_2(w))}{f_1(u)f_2(w)}\bigg|_{w=u},\qquad Q(\mathscr{D})=\frac{\mathscr{D}(f_1(u)f_2(v)f_3(w))}{f_1(u)f_2(v)f_3(w)}\bigg|_{w=u+v}.$$

In this notation the following formulae hold:

$$B(\mathscr{D})=\sum_{|\omega|\geqslant 0}\alpha_\omega B(D^\omega),\qquad Q(\mathscr{D})=\sum_{|\omega|\geqslant 0}\alpha_\omega Q(D^\omega).$$

The trilinear equation

$$\mathscr{D}(f_1(u)f_2(v)f_3(w))\big|_{w=u+v}=0 \tag{3.1}$$

reduces to a polynomial relation between the logarithmic derivatives of the functions $f_1(u)$, $f_2(v)$, $f_3(u+v)$, and hence is a strong form of a functional equation of the type (1.36),

$$\sum_{k=1}^{N+1}c_k(x+y)\varphi_k(x)\psi_k(y)=1.$$

We note that if entire functions $f_i(z)$, $i=1,2,3$, give a solution of the functional equation (3.1) for a given operator $\mathscr{D}$, then the functions $e^{\beta_i z}f_i(z)$, $i=1,2,3$, where the β_i are constants and $\beta_3=\beta_1+\beta_2$, also give a solution of this equation.

Example 3.1. Let $g=1$. We set $\rho_i(z)=(\log f_i(z))'$. Then

$$Q(D_1)=\rho_1(u)+\rho_2(v)-\rho_3(u+v),$$
$$Q(D_1^2)=(\rho_1(u)+\rho_2(v)-\rho_3(u+v))^2+\rho_1'(u)+\rho_2'(v)+\rho_3'(u+v).$$

The general solution of the functional equation $Q(D_1)=0$ is given by the linear functions ρ_i, that is, $f_i(z)=\exp(\alpha(z-\beta_i)^2+\gamma_i)$, where α, β_i, and γ_i are constants and $\beta_3=\beta_1+\beta_2$.

A famous particular solution (see (1.21)) of the functional equation $Q(D_1^2)=0$ was obtained by Frobenius and Stickelberger [33]. This particular solution was applied to exactly soluble problems in quantum mechanics ([53], [54]). The general analytic solution of this functional equation was described in [55]. The functions f_i give the wave function for the ground state for the quantum three-body problem. In the case of general position, the solution is given by functions of the form $\rho_i(z)=\zeta(z-\delta_i)-\beta_i$, that is, $f_i(z)=e^{-\beta_i z+\gamma_i}\sigma(z-\delta_i)$, where $\zeta(z)$ and $\sigma(z)$ are the Weierstrass functions, β_i, γ_i, and δ_i are constants, and $\beta_3=\beta_1+\beta_2$ and $\delta_3=\delta_1+\delta_2$. We also note the normalized particular solution

$$f_i(z)=\frac{1}{\sqrt{2\pi}\Delta}\exp\biggl(-\frac{(z-q_i)^2}{2\Delta^2}\biggr), \tag{3.2}$$

where q_i are constants and $\Delta=(q_1+q_2-q_3)/\sqrt{3}$.

Let $g=2$. We set

$$\rho_i^{(j,k)}(z)=\frac{\partial^{j+k}\log f_i(z)}{\partial z_1^j\,\partial z_2^k}\,. \tag{3.3}$$

Then

$$Q(2D_2+D_1^3)=2r_{(0,1)}+r_{(3,0)}+3r_{(2,0)}r_{(1,0)}+r_{(1,0)}^3,$$

where $r_{(j,k)}=\rho_1^{(j,k)}(u)+\rho_2^{(j,k)}(v)+(-1)^{j+k}\rho_3^{(j,k)}(u+v)$.

Theorem 3.1 (Buchstaber–Leikin [56]). *The sigma function of genus 2 is subject to the trilinear law of addition*

$$[2D_2+D_1^3]\sigma(u)\sigma(v)\sigma(w)\big|_{u+v+w=0}=0,$$

where $D_j=\partial_{u_j}+\partial_{v_j}+\partial_{w_j}$.

3.3. Construction of generating functions for bilinear and trilinear operators. The extension of the above results to the case of higher genera is based on the following general construction.

Let ξ be the coordinate in $\mathbb{C}$. We choose a smooth map $\phi\colon\mathbb{C}\to\mathbb{C}^g$ that is regular at $\xi=0$ and functions $G_i\colon\mathbb{C}\to\mathbb{C}$, $i=1,2$, meromorphic in a neighbourhood of $\xi=0$, and we form the functions

$$\begin{aligned}F_2(u,\xi)&=G_1(\xi)\frac{f_1(u+\phi(\xi))}{f_1(u)}\,\frac{f_2(u-\phi(\xi))}{f_2(u)}\,,\\ F_3(u,v,\xi)&=G_2(\xi)\frac{f_1(u+\phi(\xi))}{f_1(u)}\,\frac{f_2(v+\phi(\xi))}{f_2(v)}\,\frac{f_3(u+v-\phi(\xi))}{f_3(u+v)}\,.\end{aligned} \tag{3.4}$$

The functions $F_2(u,\xi)$ and $F_3(u,v,\xi)$ are generating functions of bilinear and trilinear operators. Indeed,

$$\left.\begin{aligned}\frac{f_1(u+z)}{f_1(u)}\,\frac{f_2(u-z)}{f_2(u)}&=\sum_{|\omega|\geqslant0}\frac{z^\omega}{\omega!}B(D^\omega),\\ \frac{f_1(u+z)}{f_1(u)}\,\frac{f_2(v+z)}{f_2(v)}\,\frac{f_3(u+v-z)}{f_3(u+v)}&=\sum_{|\omega|\geqslant0}\frac{z^\omega}{\omega!}Q(D^\omega),\end{aligned}\right\}\quad z\in\mathbb{C}^g.$$

Multiplication by the function $G_i(\xi)=\xi^{k_i}(1+O(\xi))$, $k_i\in\mathbb{Z}$, and the substitution $z=\phi(\xi)$ lead to the generating functions

$$F_2(u,\xi)=\xi^{k_1}\sum_{i\geqslant0}\xi^iB(\mathscr{D}_i^{\mathrm{bi}}),\qquad F_3(u,v,\xi)=\xi^{k_2}\sum_{i\geqslant0}\xi^iQ(\mathscr{D}_i^{\mathrm{tri}}).$$

On the other hand, for *given* functions f_i, G_j, and ϕ the series expansions of $F_2(u,\xi)$ and $F_3(u,v,\xi)$ in powers of ξ can be computed immediately. We arrive at the following result.

Lemma 3.1 (Buchstaber–Leikin [41]). *If the coefficient* $[F_2]_k$ *in the expansion* $F_2(u,\xi)=\sum[F_2]_i\xi^i$ *does not depend on the variable* u *for some* k, *then the entire functions* $f_i(z)$, $i=1,2$, *satisfy the* ***bilinear*** *differential equation* $B(\mathscr{D}_{k-k_1}^{\mathrm{bi}}-[F_2]_k)=0$.

If in the expansion $F_3(u,v,\xi)=\sum[F_3]_i\xi^i$ *the coefficient* $[F_3]_k$ *does not depend on the variables* u *and* v *for some* k, *then the entire functions* $f_i(z)$, $i=1,2,3$, *satisfy the* ***trilinear*** *functional equation* $Q(\mathscr{D}_{k-k_2}^{\mathrm{tri}}-[F_3]_k)=0$.

Below in this section we show how this construction can be used to find bilinear and trilinear equations satisfied by σ-functions of planar algebraic curves.

3.4. Necessary information. We begin with fundamental notions.

Let V be a representative of a family of planar algebraic curves, the so-called (n,s)-*curves*,

$$V=\{(x,y)\in\mathbb{C}^2 \mid f(x,y)=0\},\quad f(x,y)=y^n-x^s-\sum_{(n-j)(s-i)>ij}\lambda_{(n-j)(s-i)-ij}x^iy^j,$$

where n and s are coprime positive integers with $s>n>1$. The restriction of our consideration to this class of curves leads to no loss of generality, because every planar algebraic curve has an (n,s)-model.

Corresponding to a curve V is a *Weierstrass sequence* $(w_1,w_2,\dots)$. This is a set of positive integers arranged in increasing order and not representable in the form $an+bs$ with non-negative integers a and b. Let $w(\xi)=\sum_i \xi^{w_i}$. We have

$$w(\xi)=\frac{1}{1-\xi}-\frac{1-\xi^{ns}}{(1-\xi^n)(1-\xi^s)}.$$

The genus of a non-singular (n,s)-curve is equal to the length $g=w(1)=(n-1)(s-1)/2$ of the Weierstrass sequence. We introduce a grading by setting $\deg x=n$, $\deg y=s$, and $\deg\lambda_k=k$. Then $\deg f(x,y)=ns$. Let $m_k(x,y)$ be a vector formed by the monomials (x^iy^j) and arranged in decreasing order of deg for $0\leqslant in+js<2g+k$, where $i\geqslant 0$ and $0\leqslant j<n$. The length of the vector $m_k(x,y)$ is equal to $g+k$.

We define the vector of basis holomorphic differentials by the formula

$$dA(x,y)=m_0(x,y)\frac{dx}{f_y(x,y)},\quad f_y(x,y)=\frac{\partial}{\partial y}f(x,y),\qquad (x,y)\in V.$$

For a base point $A_0\in V$ we take the point at infinity on V. We define the Abel map by the formula

$$A(x,y)=\int_{A_0}^{(x,y)}dA(x,y).$$

We also introduce a local parametrization of the curve V in a neighbourhood of the point A_0. Namely, we set

$$\bigl(x(\xi),y(\xi)\bigr)=\bigl(\xi^{-n},\xi^{-s}\rho(\xi)\bigr),\quad\text{where}\quad \rho(\xi)=1+\sum_{i>0}\rho_i(\lambda)\xi^i,\quad \rho_i(\lambda)\in\mathbb{Q}[\lambda].$$

The coefficients $\rho_i(\lambda)$ are determined from the equation $f\bigl(x(\xi),y(\xi)\bigr)=0$, that is,

$$\rho(\xi)^n=1+\sum_{(n-j)(s-i)>ij}\rho(\xi)^j\xi^{(n-j)(s-i)-ij}\lambda_{(n-j)(s-i)-ij}.$$

Then in a neighbourhood of A_0 the Abel map has the expansion

$$A\bigl(x(\xi),y(\xi)\bigr)=\biggl(-\frac{\xi^{w_1}}{w_1}(1+a_1(\xi)),\dots,-\frac{\xi^{w_g}}{w_g}(1+a_g(\xi))\biggr)^t,\tag{3.5}$$

where the $a_i(\xi)$ are series in positive powers of ξ with coefficients in $\mathbb{Q}[\lambda]$.

Let W be the universal Abelian covering over V. The points of the space W are pairs $((x,y);[\gamma])$, where $(x,y)\in V$ and γ is a representative of the equivalence class of paths from the base point to the point (x,y). Paths γ_1 and γ_2 belong to $[\gamma]$ if the contour $\gamma_1\circ\gamma_2^{-1}$ is homologous to zero. The integration of the vector of basis holomorphic differentials along the path γ defines a *single-valued* map $A\colon W\to\mathbb{C}^g$. For clear reasons, we denote this map in just the same way as the Abel map.

We write

$$\psi(x,y;[\gamma])=\exp\biggl\{-\int_{[\gamma]}\bigl\langle A^*((x',y'),[\gamma']),dA(x',y')\bigr\rangle\biggr\},\tag{3.6}$$

where the path γ' is a part of the path γ from the base point to the point (x',y'), $\langle a,b\rangle=\sum a_ib_i$, and $A^*((x,y),[\gamma])$ is the map dual to the Abel map and given by basis integrals of the second kind with poles only at the base point, where the following expansion with respect to the local parameter holds:

$$A^*\bigl(x(\xi),y(\xi)\bigr)=\bigl(\xi^{-w_1}(1+a_1^*(\xi)),\dots,\xi^{-w_g}(1+a_g^*(\xi))\bigr)^t,$$

the $a_i^*(\xi)$ being series in positive powers of ξ with coefficients in $\mathbb{Q}[\lambda]$.

The matrices of half-periods η, η' and ω, ω' of the maps A^* and A are related by the Legendre identity

$$\Omega J\Omega^t=\frac{\pi\imath}{2}J,\quad\text{where}\quad \Omega=\begin{pmatrix}\omega&\omega'\\ \eta&\eta'\end{pmatrix},\quad J=\begin{pmatrix}0&-1_g\\ 1_g&0\end{pmatrix},\quad \imath^2=-1.\tag{3.7}$$

The function $\psi(x,y;[\gamma])$ is a single-valued entire function on W having an isolated zero of order $g=(n-1)(s-1)/2$ at the base point:

$$\psi\bigl(x(\xi),y(\xi);[\gamma]\bigr)=\xi^g\exp\{G(\xi,\lambda)\},\tag{3.8}$$

where $G(\xi,\lambda)$ is an entire function such that $G(0,\lambda)=G(\xi,0)=0$. If the genus is $g=1$, we have $\psi(x,y;[\gamma])=\sigma\bigl(-A(x,y;[\gamma])\bigr)$.

When a cycle of the form $\chi=\sum_{j=1}^g(k_ja_j+k_j'b_j)$ is added to the contour γ (recall that a_i and b_i denote the basis cycles; see p. 20), the function $\psi(x,y;[\gamma])$ is transformed as follows:

$$\frac{\psi(x,y;[\chi+\gamma])}{\psi(x,y;[\gamma])}=\exp\biggl\{-\Bigl\langle A^*[\chi],A((x,y),[\gamma])+\frac{1}{2}A[\chi]\Bigr\rangle+\pi\imath((k+\ell)^t(k'+\ell')-\ell^t\ell')\biggr\},\tag{3.9}$$

where $A[\chi]$ and $A^*[\chi]$ are the periods of A and A^* for a circuit of the cycle χ. The vectors ℓ and ℓ' are defined as follows. To a Weierstrass sequence we assign the *Weierstrass partition* $(\pi_1,\dots,\pi_g)$ by the formula

$$\pi_k=w_{g-k+1}-(g-k),\qquad k=1,\dots,g.$$

(In particular, $w_g = 2g-1$ and $\pi_1 = g$.) Then

$$\ell = (1,\dots,1) \quad \text{and} \quad \ell' = (\pi_1,\dots,\pi_g).$$

By a *sigma function* we mean an entire function on $\mathbb{C}^g$ having the following property:

$$\sigma(u+A[\chi]) = \sigma(u)\exp\biggl\{-\Bigl\langle A^*[\chi], u+\frac{1}{2}A[\chi]\Bigr\rangle + \pi\imath((k+\ell)^t(k'+\ell')-\ell^t\ell')\biggr\}. \tag{3.10}$$

As is known from the theory of Fourier series, it follows from (3.7) that the translation property (3.10) determines a σ-function up to factor constant with respect to u.

The following formula relates the functions σ and θ:

$$\sigma(u) = \Delta^{-1/8}\sqrt{\frac{\pi^g}{|\omega|}}\exp\biggl\{\frac{1}{2}u^T\eta\omega^{-1}u\biggr\}\theta[\varepsilon_R](\omega^{-1}u\,|\,\omega^{-1}\omega'), \tag{3.11}$$

where Δ stands for the discriminant of the curve V and $\theta[\varepsilon_R]$ for the theta function with the characteristic (see (1.8)) of the vector of Riemann constants. We recall that the discriminant Δ of the curve given by an equation $f(x,y)=0$ is an irreducible polynomial in the parameters λ_k whose vanishing means that the given curve has double points, that is, points at which the gradient of the function $f(x,y)$ with respect to (x,y) vanishes.

The formula (3.11) shows that if one characterizes an entire function by its behaviour under translations by periods, using methods of the theory of Fourier series, then the passage from σ to θ and conversely is a linear transformation of the argument of the function and multiplication of the function by an exponential of a quadratic form. Thus, the theories of the functions σ and θ are the theories of the same object, though in different realizations, namely:

(θ) Fourier expansion with respect to z using the period matrix B (see (1.4));

(σ) power series with rational coefficients with respect to u and λ that are parameters of the curve.

For instance, for the Weierstrass function $\sigma(u)$ of a curve given by the equation $y^2 = 4x^3 - g_2x - g_3$ we have the expansion

$$\sigma(u) = u\sum_{i,j\geqslant 0}\frac{a_{i,j}}{(4i+6j+1)!}\left(\frac{g_2u^4}{2}\right)^i(2g_3u^6)^j$$

with *integral* coefficients $a_{i,j}$ that are defined from the initial conditions

$$\{a_{0,0}=1;\ a_{i,j}=0,\ \min(i,j)<0\}$$

by the recursion

$$a_{i,j} = 3(i+1)a_{i+1,j-1} + \frac{16}{3}(j+1)a_{i-2,j+1} - \frac{1}{3}(2i+3j-1)(4i+6j-1)a_{i-1,j}.$$

The series for a sigma function of genus 2 is also defined by a linear recursion [41]. The initial segment of the series for the σ-function for the curve $y^2 = 4x^5 + \sum_{k=0}^{3} \lambda_k x^k$ is of the form

$$\sigma(u) = u_1 - \frac{u_2^3}{3} - \left(\frac{u_1 u_2^4}{48} + \frac{u_2^7}{5\,040}\right)\lambda_3 + \left(\frac{u_1^3}{24} - \frac{u_1^2 u_2^3}{24} - \frac{u_1 u_2^6}{360} + \frac{u_2^9}{22\,680}\right)\lambda_2 + \left(-\frac{u_1^3 u_2^2}{24} - \frac{u_1^2 u_2^5}{120} - \frac{u_1 u_2^8}{5\,040} + \frac{u_2^{11}}{99\,792}\right)\lambda_1 + \left(-\frac{u_1^4 u_2}{12} - \frac{u_1^3 u_2^4}{72} - \frac{u_1^2 u_2^7}{504} + \frac{u_1 u_2^{10}}{22\,680} + \frac{u_2^{13}}{1\,389\,960}\right)\lambda_0 + \cdots .$$

The Riemann vanishing theorem is an important property which is common for the σ-functions and θ-functions of Jacobians. Using the parametrization (3.5) of the Abel map, we can write the following consequence of the Riemann vanishing theorem:

$$\sigma\left(\sum_{i=1}^{g} A(\xi_i)\right) = \frac{\det\big(A(\xi_1), \dots, A(\xi_g)\big)}{\operatorname{vand}(\xi_1, \dots, \xi_g)} \exp\{H(\xi_1, \dots, \xi_g)\},$$

where $A(\xi) = A\big(x(\xi), y(\xi)\big)$, $\operatorname{vand}(\xi_1, \dots, \xi_g)$ is the Vandermonde determinant of the variables ξ_i, and $H(\xi_1, \dots, \xi_g; \lambda)$ is an entire function of λ and ξ_i which is symmetric with respect to ξ_i and such that $H(0, \dots, 0; \lambda) = H(\xi_1, \dots, \xi_g; 0) = 0$.

Even ten years ago the θ-function served as a main tool of applications of Abelian functions in problems of the theory of integrable systems. After the papers [57] and [58], the development of approaches based on the theory of σ-functions was started. At present, the circle of investigations in this direction is already rather broad (see [41]).

3.5. Polynomials defining an algebraic law of addition. An arbitrary polynomial $\phi(x, y)$ defines an entire rational function on a curve V. By a zero of the function $\phi(x, y)$ on the curve V we mean a point (ξ, η) such that

$$\{f(\xi, \eta) = 0, \phi(\xi, \eta) = 0\}.$$

The number of zeros of the function $\phi(x, y)$ is called its *order*.

The subsequent construction is based on the following fact.

Let the order of an entire rational function $\phi(x, y)$ on a curve V be equal to $2g + k$, where $k \geqslant 0$. Then $\phi(x, y)$ is completely determined (*up to a constant factor independent of* (x, y)) *by any family of $g + k$ zeros of it.*

The converse assertion also holds.

Every family of $g + k$ points on a curve V, $k \geqslant 0$, can be realized as a subset of zeros of an entire rational function $\phi(x, y)$ of order $2g + k$ on V, and such a function $\phi(x, y)$ is unique up to constant factor.

This fact follows from the classical Weierstrass gap theorem. In particular, ordinary polynomials in one variable are entire rational functions on any curve of genus $g = 0$ and are completely determined by all their zeros.

Let X be the gth symmetric power of the curve V, that is, $\mathsf{X} = V^g/\Sigma_g$, where the symmetric group Σ_g acts on the direct product by the permutations of coordinates. A point U of the space X is an unordered family of g points $(x_i, y_i) \in V$, which we denote by $U = [(x_i, y_j)]$. Let $U' \in \mathsf{X}$ and $U' = [(x_{g+i}, y_{g+i})]$. We define entire rational functions $R^U_{2g}(x,y)$ and $R^{U,U'}_{3g}(x,y)$ of orders $2g$ and $3g$ by the formulae

$$R^U_{2g}(x,y) = \frac{\det(m_1(x,y), m_1(x_1,y_1), \dots, m_1(x_g,y_g))}{\det((1,0,\dots,0)^t, m_1(x_1,y_1), \dots, m_1(x_g,y_g))}, \tag{3.12}$$

$$R^{U,U'}_{3g}(x,y) = \frac{\det(m_{g+1}(x,y), m_{g+1}(x_1,y_1), \dots, m_{g+1}(x_{2g},y_{2g}))}{\det((1,0,\dots,0)^t, m_{g+1}(x_1,y_1), \dots, m_{g+1}(x_{2g},y_{2g}))}, \tag{3.13}$$

where the $m_k(x,y)$ are the vectors of dimension $g+k$ that were introduced at the beginning of the previous subsection. By construction, R^U_{2g} is completely determined by specifying the point U and $R^{U,U'}_{3g}$ is completely determined by specifying the pair of points U and U'. The functions R^U_{2g} and $R^{U,U'}_{3g}$ define a law of addition $U \star U'$ on X as follows: the full family of zeros of an entire rational function on V is assumed to be equivalent to zero. Therefore, the zeros of R^U_{2g} on X are the point U and the point $\overline{U}$ inverse to U; correspondingly, the zeros of $R^{U,U'}_{3g}$ are the points U, U', and $\overline{U \star U'}$.

By the Abel theorem, the Abel map $A\colon \mathsf{X} \to J(V)$ given by the formula

$$A(U) = \sum_{i=1}^{g} A(x_i, y_i)$$

is a homomorphism with respect to the operation $\star$, that is,

$$A(\overline{U}) = -A(U), \qquad A(U \star U') = A(U) + A(U').$$

As was proved in the paper [41], the polynomials $R^u_{2g}(x,y)$ and $R^{u,v}_{3g}(x,y)$ admit a representation in the form (3.4) which can be constructed by using the σ-function, the function $\psi(x,y;[\gamma])$, and the Abel map. Namely, the following formulae hold:

$$\begin{aligned} R^u_{2g}(x,y) &= \frac{\sigma(A((x,y);[\gamma]) - u)}{\psi(x,y;[\gamma])\,\sigma(u)}\,\frac{\sigma(A((x,y),[\gamma]) + u)}{\psi(x,y;[\gamma])\,\sigma(u)}, \\ R^{u,v}_{3g}(x,y) &= \frac{\sigma(A((x,y),[\gamma]) - u)}{\psi(x,y;[\gamma])\,\sigma(u)}\,\frac{\sigma(A((x,y),[\gamma]) - v)}{\psi(x,y;[\gamma])\,\sigma(v)}\,\frac{\sigma(A((x,y),[\gamma]) + u + v)}{\psi(x,y;[\gamma])\,\sigma(u+v)}, \end{aligned} \tag{3.14}$$

where u and v denote points of X and at the same time their images in $J(V)$ under the Abel map. The relations (3.14) can be derived from the Riemann vanishing theorem and the formulae (3.10) and (3.9).

3.6. Trilinear equations for the $\boldsymbol{\sigma}$-functions. We are now ready to realize the construction of Lemma 3.1. Let us first consider an example in which Lemma 3.1 can be applied directly.

Example 3.2. Let $g = 1$. In the classical Weierstrass parametrization we have $(x, y) = (\wp(\xi), \wp'(\xi))$. In this case one can take $\psi(\xi) = \sigma(\xi)$. Then $\phi(\xi) = -\xi$ and $G_i(\xi) = \sigma(\xi)^{-(i+1)}$. Using the classical formulae, we obtain the following equalities:

$$\wp(\xi) - \wp(u) = R_2^u(\wp(\xi), \wp'(\xi)) = F_2(u, \xi) = \frac{1}{\sigma(\xi)^2} \frac{\sigma(u+\xi)}{\sigma(u)} \frac{\sigma(u-\xi)}{\sigma(u)},$$

where the series on the left-hand side is even with respect to ξ with coefficients constant with respect to u at all *even* powers of ξ except for the zero power;

$$\frac{\begin{vmatrix} 1 & \wp(\xi) & \wp'(\xi) \\ 1 & \wp(u) & \wp'(u) \\ 1 & \wp(v) & \wp'(v) \end{vmatrix}}{\begin{vmatrix} 1 & \wp(u) \\ 1 & \wp(v) \end{vmatrix}} = R_3^{u,v}(\wp(\xi), \wp'(\xi)) = F_3(u, v, \xi)$$

$$= \frac{1}{\sigma(\xi)^3} \frac{\sigma(u-\xi)}{\sigma(u)} \frac{\sigma(v-\xi)}{\sigma(v)} \frac{\sigma(u+v+\xi)}{\sigma(u+v)},$$

where the series in powers of ξ on the left-hand side has coefficients that are constant with respect to u and v at *all odd* powers of ξ. The coefficient of ξ^{-1} vanishes, which gives a representation of the classical relation (1.21) in the trilinear form $Q(D^2) = 0$.

We now return to the discussion of bilinear and trilinear operators annihilating the σ-function of an (n, s)-curve V.

Let us consider a topological linear space $\mathscr{L}_{\mathscr{M}}$ in which a topological basis is formed by the set $\mathscr{M} = \{x^i y^j \mid i \in \mathbb{Z},\ 0 \leqslant j < n\}$ of monomials arranged in decreasing order of $\deg(x^i y^j) = in + js$ and a topological linear space $\mathscr{L}_{\Xi}$ in which a topological basis is formed by the set $\Xi = \{\xi^k \mid k \in \mathbb{Z}\}$ arranged in increasing order of the degree.

A local parametrization in a neighbourhood of the base point, $(x, y) = (\xi^{-n}, \xi^{-s}\rho(\xi))$, where $\rho(\xi) = 1 + o(\xi)$, defines a continuous linear map $T\colon \mathscr{L}_{\mathscr{M}} \to \mathscr{L}_{\Xi}$ by the formula $T(x^i y^j) = \xi^{-(in+js)}\rho(\xi)^j$. We show that this map is a homeomorphism, that is, there is a continuous inverse map. Indeed, since $\gcd(n, s) = 1$, it follows that for $\delta \in \{0, 1, \dots, n-1\}$ one can find an integer $k(\delta)$ and an index $\ell(\delta) \in \{0, 1, \dots, n-1\}$ which are uniquely defined by the condition $\delta = k(\delta)n + \ell(\delta)s$. Thus, for any integer m the set $\mathscr{M}$ contains one and only one monomial $x^{i(m)} y^{j(m)}$ such that $i(m)n + j(m)s = m$, namely, $i(m) = [m/n] + k(m - n[m/n])$ and $j(m) = \ell(m - n[m/n])$. Hence, the matrix of the transformation T is upper triangular and with 1s on the main diagonal. Therefore, a continuous inverse linear map $T^{-1}\colon \mathscr{L}_{\Xi} \to \mathscr{L}_{\mathscr{M}}$ with lower triangular matrix is defined.

We denote by $\mathscr{M}_-$ (by $\mathscr{M}_+$) the subset of $\mathscr{M}$ formed by the monomials containing x to a negative (non-negative) power, and we denote by π_- (by π_+) the canonical projection of $\mathscr{L}_{\mathscr{M}}$ onto $\mathscr{L}_{\mathscr{M}_-}$ (onto $\mathscr{L}_{\mathscr{M}_+}$, respectively). The monomial of highest weight in $\mathscr{M}_-$ is $y^{n-1}x^{-1}$.

Thus, let us expand the right-hand sides of the relations (3.14) in series of powers of the local parameter ξ, that is, in the basis Ξ. We obtain the generating functions

$$\begin{aligned} F_2(\xi) &= \frac{\exp\{\langle A(\xi),\partial_z\rangle - 2G(\xi,\lambda)\}\sigma(z+u)\sigma(z-u)}{\xi^{2g}\sigma(u)^2}\bigg|_{z=0} \\ &= \sum_{i\in\mathbb{Z}} \xi^i B(\mathscr{D}_i^{\mathrm{bi}}), \\ F_3(\xi) &= \frac{\exp\{\langle A(\xi),\partial_z\rangle - 3G(\xi,\lambda)\}\sigma(z-u)\sigma(z-v)\sigma(z+v+u)}{\xi^{3g}\sigma(u)\sigma(v)\sigma(v+u)}\bigg|_{z=0} \\ &= \sum_{i\in\mathbb{Z}} \xi^i Q(\mathscr{D}_i^{\mathrm{tri}}), \end{aligned}$$

where $A(\xi) = A\bigl(x(\xi), y(\xi)\bigr)$ (see (3.5)), the function $G(\xi,\lambda)$ is the same as in (3.8), and $\partial_z = (\partial_{z_1},\dots,\partial_{z_g})^t$. It is convenient to represent the function $F_2(\xi)$ in the form $F_2(\xi) = \langle B(\mathscr{D}^{\mathrm{bi}}),\Xi\rangle$, where $\mathscr{D}^{\mathrm{bi}} = (\mathscr{D}_i^{\mathrm{bi}})^t$, $i\in\mathbb{Z}$. Let us apply the transformation T^{-1} and pass to the basis $\mathscr{M}$. We obtain

$$R_{2g}^u(x,y) = \langle B(\mathscr{D}^{\mathrm{bi}}), T^{-1}\mathscr{M}\rangle = \langle (T^{-1})^t B(\mathscr{D}^{\mathrm{bi}}),\mathscr{M}\rangle,$$

and, since $R_{2g}^{(u)}(x,y)$ is a polynomial, this implies that

$$R_{2g}^{(u)}(x,y) = \langle (\pi_+\circ T^{-1})^t B(\mathscr{D}^{\mathrm{bi}}),\mathscr{M}_+\rangle, \qquad \langle (\pi_-\circ T^{-1})^t B(\mathscr{D}^{\mathrm{bi}}),\mathscr{M}_-\rangle = 0.$$

Thus, on the one hand, *the σ-function satisfies the system of bilinear equations* $\pi_-\circ(T^{-1})^t B(\mathscr{D}^{\mathrm{bi}}) = 0$, and on the other hand, the *coefficients of the polynomial* $R_{2g}^{(u)}(x,y)$ *are values of the bilinear operators* $\pi_+\circ(T^{-1})^t B(\mathscr{D}^{\mathrm{bi}})$.

Representing the function $F_3(\xi)$ in the form $F_3(\xi) = \langle Q(\mathscr{D}^{\mathrm{tri}}),\Xi\rangle$, where $\mathscr{D}^{\mathrm{tri}} = (\mathscr{D}_i^{\mathrm{tri}})^t$, $i\in\mathbb{Z}$, and passing to the basis $\mathscr{M}$, we obtain, on the one hand, *the system of trilinear equations* $\pi_-\circ(T^{-1})^t Q(\mathscr{D}^{\mathrm{tri}}) = 0$ *whose solution is the σ-function* and, on the other hand, *the family of trilinear operators* $\pi_+\circ(T^{-1})^t Q(\mathscr{D}^{\mathrm{tri}})$ *whose values determine the coefficients of the polynomial* $R_{3g}^{(u,v)}(x,y)$.

In [41] the planned programme was completed for families of hyperelliptic curves, that is, for $(n,s) = (2,2g+1)$. We present the list of trilinear equations obtained in [41] for the hyperelliptic σ-function. For the curve

$$V = \biggl\{(x,y)\in\mathbb{C}^2 \mid y^2 = x^{2g+1} + \sum_{i=2}^{2g+1}\lambda_{2i}x^{2g-i+1}\biggr\}$$

one can readily write out the generating function of the operators $\mathscr{D}_s^{\mathrm{tri}}$ in explicit form (in what follows we speak only of trilinear operators, and thus we omit the symbol $^{\mathrm{tri}}$ indicating 'trilinearity'). We have

$$\begin{aligned} \sum_{s\geqslant 0}\xi^s\mathscr{D}_s = \exp\biggl\{\sum_{q=0}^{g-1}\int_0^{\xi}\biggl(&-D_{q+1} \\ &+ \sum_{p=q+1}^{2g-q} 3(p-q)\,\lambda_{2(p+q+1)}\int_0^{\xi'}\frac{(\xi'')^{2p}\,d\xi''}{\rho(\xi'')}\biggr)\frac{(\xi')^{2q}\,d\xi'}{\rho(\xi')}\biggr\}, \end{aligned}$$

where $\rho^2 = 1 + \Lambda(\xi^2)$ and $\Lambda(t) = \lambda_4 t^2 + \lambda_6 t^3 + \cdots + \lambda_{4g+2} t^{2g+1}$. In particular,

$$\begin{aligned}
&\mathscr{D}_0 = 1,\\
&\mathscr{D}_1 = -D_1,\\
&\mathscr{D}_2 = \frac{1}{2} D_1^2,\\
&\mathscr{D}_3 = -\frac{1}{6}(D_1^3 + 2D_2),\\
&\mathscr{D}_4 = \frac{1}{24}(D_1^4 + 8D_2D_1 + 6\lambda_4),\\
&\mathscr{D}_5 = -\frac{1}{120}(D_1^5 + 20D_2D_1^2 + 24D_3 + 18\lambda_4 D_1),\\
&\mathscr{D}_6 = \frac{1}{720}(D_1^6 + 40D_2D_1^3 + 144D_3D_1 + 40D_3^2 + 18\lambda_4 D_1^2 + 144\lambda_6), \quad \text{and so on.}
\end{aligned}$$

For the operators $\mathscr{D}_s$ and the polynomials $p_{2q}(\lambda)$ given by the generating function

$$\sum_{s\geqslant 0} p_{2s}(\lambda) x^{-s} = \left(1 + \sum_{k=1}^{\infty} \binom{-1/2}{k} \Lambda(x^{-1})^k\right)$$

we have the following equations satisfied by the hyperelliptic σ-function of genus g:

(a) if $g = 2k$, then

$$Q\left(\sum_{q=0}^{s} p_{2q}(\lambda)\mathscr{D}_{2(s-q)+1}\right) = 0, \qquad s \geqslant k,$$
$$Q(\mathscr{D}_{2s}) = 0, \qquad s \geqslant 3k+1;$$

(b) if $g = 2k+1$, then

$$Q(\mathscr{D}_{2s+1}) = 0, \qquad s \geqslant 3k+2,$$
$$Q\left(\sum_{q=0}^{s} p_{2q}(\lambda)\mathscr{D}_{2(s-q)}\right) = 0, \qquad s \geqslant k+1.$$

We present the operators of least weight for which the hyperelliptic σ-function satisfies the equation $Q(\mathscr{D}) = 0$ for small values of g:

$$\begin{aligned}
&g = 1, \quad \mathscr{D} = D_1^2;\\
&g = 2, \quad \mathscr{D} = D_1^3 + 2D_2;\\
&g = 3, \quad \mathscr{D} = D_1^4 + 8D_2D_1 - 6\lambda_4;\\
&g = 4, \quad \mathscr{D} = D_1^5 + 20D_2D_1^2 + 24D_3 - 42\lambda_4 D_1;\\
&g = 5, \quad \mathscr{D} = D_1^6 + 40D_2D_1^3 + 144D_3D_1 + 40D_2^2 - 162\lambda_4 D_1^2 - 216\lambda_6.
\end{aligned}$$

Let us fix an expansion of the function $R_{3g}^{(u,v)}(x,y)$ in the monomials in the set $\mathscr{M}_+$ in the form

$$R_{3g}^{(u,v)}(x,y) = y^1\left(\sum_{i=0}^{[g+1/2]} h_{g+1-2i} x^i\right) + y^0\left(\sum_{i=0}^{[3g/2]} h_{3g-2i} x^i\right).$$

In this case the coefficients $h_i(u,v)$ with $i \in \{0,1,\dots,g-1,g,g+2,\dots,3g-2,3g\}$ have the following form:

(a) if $g=2k$, then

$$h_{2s+1}(u,v) = Q\biggl(\sum_{q=0}^{s} p_{2q}(\lambda)\mathscr{D}_{2(s-q)+1}\biggr), \qquad 0 \leqslant s < k,$$
$$h_{2s}(u,v) = Q(\mathscr{D}_{2s}), \qquad 0 \leqslant s < 3k+1;$$

(b) if $g=2k+1$, then

$$h_{2s+1}(u,v) = Q(\mathscr{D}_{2s+1}), \qquad 0 \leqslant s < 3k+2,$$
$$h_{2s}(u,v) = Q\biggl(\sum_{q=0}^{s} p_{2q}(\lambda)\mathscr{D}_{2(s-q)}\biggr), \qquad 0 \leqslant s < k+1.$$

The expansion of the function $R_{2g}^{(u)}$ in monomials in the set $\mathscr{M}_+$ has a very simple form in the hyperelliptic case, namely,

$$R_{2g}^{(u)}(x,y) = \sum_{i=0}^{g} b_{2(g-i)}(u)x^i.$$

The coefficients $b_i(u)$, $i \in \{0,2,\dots,2g\}$, are expressed in terms of the bilinear operators $\mathscr{B}_i$ generated by the generating function

$$\sum_{s\geqslant 0} \xi^s \mathscr{B}_s = \exp\biggl\{\sum_{q=0}^{g-1} \int_0^{\xi} \biggl(-D_{q+1} + \sum_{p=q+1}^{2g-q} 2(p-q)\,\lambda_{2(p+q+1)} \int_0^{\xi'} \frac{(\xi'')^{2p} d\xi''}{\rho(\xi'')}\biggr) \frac{(\xi')^{2q}\, d\xi'}{\rho(\xi')}\biggr\}$$

as follows:

$$b_{2s}(u) = B(\mathscr{B}_{2s}), \qquad 0 \leqslant s \leqslant g.$$

The operators $\mathscr{B}_{2s+1}$ of odd orders are identically zero for any $s > 0$. The hyperelliptic σ-function of genus g satisfies the system of bilinear equations $\{B(\mathscr{B}_{2s}) = 0 \mid i > g\}$.

Remark 3.1. The operators $\mathscr{D}_{2s+1}^{\mathrm{bi}}$ with $s > 0$ are identically zero for all curves. Therefore, in the general case the σ-function of genus g satisfies the system $\{B(\mathscr{D}_{2s}^{\mathrm{bi}}) = 0 \mid s > g\}$ of bilinear equations. Moreover, for the coefficients $\{b_i(u)\}$ of the expansion of the function $R_{2g}^{(u)}(x,y)$ with respect to the family $\mathscr{M}_+$ the index i indicating the value of the grading of a coefficient takes values in the set $I = \{0, w_1+1, \dots, w_g+1\}$, where the w_i are the elements of the Weierstrass sequence. It follows from considerations related to the solution of the Jacobi inversion problem in terms of the σ-function that

$$b_0(u) = 1 \quad \text{and} \quad b_{w_i+1}(u) = -\frac{\partial^2}{\partial u_1 \partial u_i} \log \sigma(u) = \frac{1}{2} B(D_1 D_i), \quad 0 < i \leqslant g.$$

Hence, the system of bilinear equations $\{B(\mathscr{D}_{2s}^{\rm bi})=0 \mid s>g\}$ of order $>2g$ can be completed by bilinear equations of order $<2g$ for the σ-function,

$$\{B(2\mathscr{D}_{2s}^{\rm bi}-D_1D_s)=0 \mid 0<s\leqslant g\}.$$

In the general case to complete the calculations explicitly, one must find the series expansions in a neighbourhood of the point at infinity on V for the Abel map (3.5) and for the logarithm of the function $\psi(x,y;[\gamma])$ (see (3.8)), and also find the transition matrix T. From considerations related to the grading we arrive at the following result (see [40], [41]).

Theorem 3.2. *Consider a planar curve*

$$V=\biggl\{(x,y)\in\mathbb{C}^2 \mid y^n-x^s-\sum_{(n-j)(s-i)>ij}\lambda_{(n-j)(s-i)-ij}x^iy^j=0\biggr\},$$

where $\gcd(n,s)=1$.

A linear differential operator $\mathscr{D}$ of the least order for which the σ-function associated with V is a solution of the trilinear functional equation $Q(\mathscr{D})=0$ has order $g{+}1$ with respect to the variable u_1. This operator is homogeneous, and $\deg\mathscr{D}=g+1$ *with respect to the grading* $\deg x=n$, $\deg y=s$, $\deg\lambda_k=k$.

3.7. Trilinear analogue of the weak form of the addition theorem. Let us consider a functional equation of the following form:

$$f_1(u+z)f_2(v+z)f_3(u+v-z)=\sum_{k=1}^{N}\varphi_k(u,v)\psi_k(z), \tag{3.15}$$

where all the functions are sought for a given N, namely, f_i, φ_j, ψ_j, $i=1,2,3$, $j=1,\dots,N$.

The following general result holds.

Lemma 3.2. *Let a family of smooth functions f_i, φ_j, ψ_j, $i=1,2,3$, $j=1,\dots,N$, give a solution of the equation (3.15). Then for any family I, $\#I>N$, of pairwise distinct multi-indices there is a non-trivial linear operator $\mathscr{D}=\sum_{\omega\in I}\alpha_\omega D^\omega$ such that the functions f_1, f_2, and f_3 are solutions of the equation $Q(\mathscr{D})=0$.*

This lemma immediately implies the following assertion.

Corollary 3.1. *The theta functions of g-dimensional Abelian varieties always have trilinear addition theorems.*

Proof. Indeed, the function $\theta(u+z)\theta(v+z)\theta(u+v-z)$ of the variable $z\in\mathbb{C}^g$ is an entire quasi-periodic function of order three. As is known, such functions form a linear space of dimension $N=3^g$. Choosing a basis $(\psi_1(z),\dots,\psi_{3^g}(z))$ in this space, we obtain a solution of the functional equation (3.15) in θ-functions.

§ 4. Continuous Krichever–Novikov basis

A special degenerate case of the Baker–Akhiezer function (see § 2.4), namely, the basis KN-function with parameter $u \in \mathbb{C}^g$, is realized in terms of the σ-function in the form

$$\Psi\bigl(u, (x, y)\bigr) = \frac{\sigma(A(x, y; [\gamma]) - u)}{\psi(x, y; [\gamma])\, \sigma(u)} \exp\bigl\{-\langle A^*(x, y; [\gamma]), u\rangle\bigr\}, \tag{4.1}$$

where $\psi(x, y; [\gamma])$ is the function given by the formula (3.6) and $A^*(x, y; [\gamma])$ is the vector of basis Abelian integrals of second kind with poles at the base point $\infty \in V$ as above. The function Ψ is single-valued on $\mathbb{C}^g \times V$. As a function on V, it has g zeros $A^{-1}(u) \in \mathsf{X}$ and a unique essential singularity $\infty \in V$, at which it has the behaviour $\Psi \sim \xi^{-g} \exp\{p(\xi^{-1})\}(1 + O(\xi))$, where p is a polynomial of degree not exceeding $2g - 1$. For instance, in the hyperelliptic case we have the following expansion with respect to the local parameter ξ at this singular point:

$$\Psi\bigl(u, (x(\xi), y(\xi))\bigr) = \xi^{-g} \exp\biggl\{-\rho(\xi)\biggl(\sum_{i=1}^{g} u_i \xi^{-2i+1}\biggr)\biggr\}\bigl(1 + O(\xi)\bigr), \tag{4.2}$$

where $\rho(\xi)^2 = 1 + \sum_{i>1} \lambda_{2i} \xi^{2i}$. Thus (see § 2.4), the function Ψ is a degeneration of the Baker–Akhiezer function, corresponding to gluing an essential singularity and g poles at the base point $\infty \in V$. Due to these properties, Ψ is an extremely convenient tool.

We recall (see § 2.4) that there is a unique linear operator

$$\mathscr{L} = \sum_{i=0}^{g} a_i(u, v) \partial_{v_1}^i$$

such that the following identity holds:

$$\Psi\bigl(u, (x, y)\bigr)\Psi\bigl(v, (x, y)\bigr) = \mathscr{L}\Psi\bigl(u + v, (x, y)\bigr).$$

In this section we describe the application of the apparatus of trilinear and bilinear operators in § 3 to the problem of computing the coefficients of the operator $\mathscr{L}$. This application is based on the trilinear strong form of the addition theorem.

4.1. Trilinear strong form of the addition theorem. Let $k > 1$ and let $R_{kg}^{(t_1,\dots,t_{k-1})}(x, y)$ be an entire rational function of order kg on V with $k - 1$ given zeros at the points $A^{-1}(t_i) \in \mathsf{X}$, $i = 1, \dots, k - 1$ (see the cases $k = 2, 3$ in the previous section). In this case, using the function Ψ, we obtain a factorization

$$R_{kg}^{(t_1,\dots,t_{k-1})}(x, y) = \biggl(\prod_{i=1}^{k} \Psi\bigl(t_i, (x, y)\bigr)\biggr)\bigg|_{t_k = -\sum_{j=1}^{k-1} t_j}. \tag{4.3}$$

The formula (4.3) immediately implies the following result that connects the multiplication in the continuous KN basis with the algebraic law of addition on X (see § 3.5).

Lemma 4.1. *The following identity holds*:

$$\Psi\bigl(u,(x,y)\bigr)\Psi\bigl(v,(x,y)\bigr) = \frac{R_{3g}^{(u,v)}(x,y)}{R_{2g}^{(u+v)}(x,y)}\,\Psi\bigl(u+v,(x,y)\bigr), \qquad u,v\in\mathbb{C}^g.$$

We introduce the following family of functions on V with parameter $w\in\mathbb{C}^g$:

$$G_k^{(w)}(x,y) = \frac{\partial_{w_1}^k\Psi(w,(x,y))}{\Psi(w,(x,y))}\,, \qquad k=0,1,2,\dots\,.$$

The function $G_k^{(w)}(x,y)$ is rational on V with $g+k$ poles $\{k\,\infty, A^{-1}(w)\}$. The coefficients of the function $G_k^{(w)}(x,y)$ are Abelian functions on the Jacobian of the curve V. By definition, we have

$$G_0^{(w)}(x,y)=1,\quad G_1^{(w)}(x,y) = -\bigl(\zeta_1(A(x,y;[\gamma])-w)+\zeta_1(w)+\langle A^*(x,y;[\gamma]),\mathsf{e}_1\rangle\bigr).$$

We note that the formula for $G_1^{(w)}(x,y)$ defines an Abelian function of w and a single-valued function of (x,y), that is, a function independent of the contour γ, because we take the same contour for the maps A and A^*.

For $k>0$ the definition implies the recursion formula

$$G_{k+1}^{(w)}(x,y) = \bigl(\partial_{w_1}+G_1^{(w)}(x,y)\bigr)\,G_k^{(w)}(x,y), \tag{4.4}$$

which results in the explicit expression

$$G_k^{(w)}(x,y) = (\partial_{w_1}+G_1^{(w)}(x,y))^k\,1, \qquad k=0,1,\dots,g.$$

It follows from the Riemann–Roch theorem that the family formed by the functions $G_0^{(w)}(x,y)$, …, $G_\ell^{(w)}(x,y)$ is a basis of the linear space of functions whose sets of poles, counting multiplicities, are subsets of the set $\{\ell\,\infty, A^{-1}(w)\}$.

Since the coefficients of the polynomials $R_{2g}^{(u)}(x,y)$ and $R_{3g}^{(u,v)}(x,y)$ are obtained in the form of bilinear and trilinear operators, we arrive at the following result.

Theorem 4.1. *The coefficients of the linear operator*

$$\mathscr{L} = \sum_{i=0}^{g} a_i(u,v)\partial_{v_1}^i$$

defined by the equation (*see* (1.35))

$$\Psi\bigl(u,(x,y)\bigr)\Psi\bigl(v,(x,y)\bigr) = \mathscr{L}\Psi\bigl(u+v,(x,y)\bigr),$$

are related to bilinear and trilinear operators by the following formula:

$$R_{3g}^{(u,v)}(x,y) = \sum_{k=0}^{g} a_k(u,v)S_k^{(u+v)}(x,y), \tag{4.5}$$

where

$$S_k^{(w)}(x,y) = \Psi(-w,(x,y))\partial_{w_1}^k\Psi(w,(x,y)) = R_{2g}^{(w)}(x,y)G_k^{(w)}(x,y). \tag{4.6}$$

The formula (4.5) is a *trilinear strong form of the addition theorem.*

Remark 4.1. By formula (4.6), the functions $S_k^{(w)}(x,y)$, $k=0,1,2,\dots$, are entire rational functions on V, and moreover, all of them vanish at $A^{-1}(-u-v)\in \mathsf{X}$. Thus, since the function $R_{3g}^{(u,v)}(x,y)$ also vanishes at $A^{-1}(-u-v)\in\mathsf{X}$, there are among the $(2g+1)$ equations following from (4.5) exactly g equations having a simple geometric meaning, namely, they express the fact that *the right- and left-hand sides of* (4.5) *vanish simultaneously and independently of the values of the coefficients* $a_i(u,v)$ *for* $(x,y)\in A^{-1}(-u-v)$. Therefore, in the construction of Theorem 4.1, only $g+1$ equations are essential for determining the values of the $(g+1)$ coefficients $a_i(u,v)$, and the remaining g equations are the *compatibility equations.*

We note an analogy between the function $S_k^{(w)}(x,y)$ given by the formula (4.6) and the ordinary symbol of $\partial_{w_1}^k$ defined by the formula $\operatorname{symb}(\partial_{w_1}^k)=e^{-w_1x}\partial_{w_1}^k e^{w_1x}$. It is reasonable to call the function $S_k^{(w)}(x,y)$ *the symbol of the operator* $\partial_{w_1}^k$ *on the curve* V. Obviously, this definition can be extended to an arbitrary linear operator L by the formula $\operatorname{symb}_V L=\Psi(-w,(x,y))L\Psi(w,(x,y))$. Thus, the trilinear strong form of the addition theorem (4.5) expresses the fact that *the symbol of the operator* $\mathscr{L}$ *on the curve* V *is equal to* $R_{3g}^{(u,v)}(x,y)$.

If a model of the curve V is given, then one can obtain explicit formulae for $G_k^{(w)}(x,y)$ in the form of rational functions in x and y. In the next subsection we give an application of the trilinear strong form of addition theorem in the hyperelliptic case. It is useful to consider this case separately, making it possible to show the main stages of application of the theorem in a form not obscured by the technical details of the general case.

4.2. Hyperelliptic case. We set

$$\wp_{i_1,\dots,i_k}(w)=-\frac{\partial}{\partial w_{i_1}}\cdots\frac{\partial}{\partial w_{i_k}}\log\sigma(w).$$

In the hyperelliptic case the formula (3.12) becomes

$$R_{2g}^{(w)}(x,y)=x^g-\sum_{i=0}^{g-1}\wp_{1,g-i}(w)x^i. \tag{4.7}$$

The formula (3.13) yields the expansion

$$R_{3g}^{(u,v)}(x,y)=yr_1(x)+x^g r_2(x)+r_3(x),$$

where

$$r_1(x)=\sum_{i=0}^{l}h_{g-2i-1}(u,v)x^i,$$

$$r_2(x)=\sum_{i=0}^{g-l-1}h_{g-2i}(u,v)x^i,$$

$$r_3(x)=\sum_{i=0}^{g-1}h_{3g-2i}(u,v)x^i,$$

with $l=\left[\frac{g-1}{2}\right]$ and $h_0=1$.

The following formulae hold:

$$G_1^{(w)}(x,y)=\frac{1}{2}\,\frac{2y+\sum_{i=0}^{g-1}\wp_{1,1,g-i}(w)x^i}{x^g-\sum_{i=0}^{g-1}\wp_{1,g-i}(w)x^i}\,,\quad G_2^{(w)}(x,y)=x+2\wp_{1,1}(w).$$

We set

$$G_k^{(w)}(x,y)=C_k^{(w)}(x,y)+B_k^{(w)}(x,y)G_1^{(w)}(x,y). \tag{4.8}$$

The general recursion formula (4.4) leads to the following recursion for the coefficients $C_k^{(w)}(x,y)$ and $B_k^{(w)}(x,y)$:

$$\begin{aligned} C_{k+1}^{(w)}(x,y)&=\partial_{w_1}C_k^{(w)}(x,y)+G_2^{(w)}(x,y)B_k^{(w)}(x,y),\\ B_{k+1}(w)&=\partial_{w_1}B_k^{(w)}(x,y)+C_k^{(w)}(x,y)\end{aligned} \tag{4.9}$$

with the initial data $(C_0^{(w)}(x,y),B_0^{(w)}(x,y))=(1,0)$. Since $G_2^{(w)}(x,y)=x+2\wp_{1,1}(w)$ is a linear function of x, it follows from this recursion that $C_k^{(w)}(x,y)$ and $B_k^{(w)}(x,y)$ are polynomials in x.

Let us now describe the polynomials $S_k^{(w)}(x,y)=R_{2g}^{(w)}(x,y)G_k^{(w)}(x,y)$. We have

$$\begin{aligned} S_0^{(w)}(x,y)&=R_{2g}^{(w)}(x,y)=x^g-\sum_{i=0}^{g-1}\wp_{1,g-i}(w)x^i,\\ S_1^{(w)}(x,y)&=y+\frac{1}{2}\sum_{i=0}^{g-1}\wp_{1,1,g-i}(w)x^i.\end{aligned}$$

Using the recursion (4.8), we see that the polynomials $S_k^{(w)}(x,y)$, $k>1$, are given by the recursion formula

$$S_k^{(w)}(x,y)=C_k(w)S_0^{(w)}(x,y)+B_k(w)S_1^{(w)}(x,y),$$

where $C_k(w)$ and $B_k(w)$ are defined by the recursion (4.9).

Thus, we have obtained effective formulae for all polynomials $S_k^{(w)}(x,y)$. It follows from these formulae that

$$S_{2s}^{(w)}(x,y)=x^{g+s}+\widetilde{S}_{2s}^{(w)}(x,y)\quad\text{and}\quad S_{2s+1}^{(w)}(x,y)=yx^s+\widetilde{S}_{2s+1}^{(w)}(x,y),$$

where the polynomial $\widetilde{S}_k^{(w)}(x,y)$ is formed by monomials $x^iy^j\in\mathscr{M}$ whose grading is less than $2g+k$ and whose coefficients depend only on w.

Combining the above manipulations, we see that in the hyperelliptic case the formula (4.5) becomes

$$\begin{aligned} y\biggl(\sum_{i=0}^{l}(h_{g-2i-1}-a_{2i+1})x^i\biggr)+x^g\biggl(\sum_{i=0}^{g-l-1}(h_{g-2i}-a_{2i})x^i\biggr)+\sum_{i=0}^{g-1}h_{3g-2i}x^i&\\ =\sum_{k=0}^{g}a_k(u,v)\widetilde{S}_k^{(u+v)}(x,y).&\end{aligned} \tag{4.10}$$

The formula (4.10) corresponds to the system of $(2g+1)$ equations obtained by equating the coefficients of the monomials

$$1, x, \dots, x^{2g-1-l}, y, yx, \dots, yx^l,$$

where $l = \left[\frac{g-1}{2}\right]$, for the polynomials on the left- and right-hand sides of this formula.

We note that the $(g+1)$ equations corresponding to the monomials

$$x^g, \dots, x^{2g-1-l}, y, yx, \dots, yx^l,$$

determine the coefficients $a_k(u,v)$ in the form of functions of $h_i(u,v)$, and the remaining g equations corresponding to the monomials $1, x, \dots, x^{g-1}$ determine relations among the h_i that are compatibility equations in the form of trilinear functional equations for the hyperelliptic σ-function $\sigma(u)$.

Example 4.1 ($g = 1$, $l = 0$). In this example

$$\begin{aligned} L &= a_0(u,v) + a_1(u,v)\partial_{w_1}, \\ S_0^{(w)}(x,y) &= x - \wp_{1,1}(w), \\ S_1^{(w)}(x,y) &= y + \frac{1}{2}\,\wp_{1,1,1}(w). \end{aligned}$$

We obtain the formula

$$y(h_0 - a_1) + x(h_1 - a_0) + h_3 = -a_0(u,v)\wp_{1,1}(u+v) + \frac{1}{2}\,a_1(u,v)\wp_{1,1,1}(u+v).$$

Hence,

$$\begin{gathered} a_1 = h_0 = 1, \qquad a_0 = h_1, \\ h_3 = -a_0(u,v)\wp_{1,1}(u+v) + \frac{1}{2}\,\wp_{1,1,1}(u+v), \\ h_3 + h_1\wp_{1,1}(u+v) = \frac{1}{2}\,\wp_{1,1,1}(u+v). \end{gathered}$$

Using the results of the paper [41] that are presented in the previous section, we see from the first two equations (cf. [39]) that

$$a_1 = Q(1) = 1, \quad a_0(u,v) = Q(-D_1) = \zeta(u+v) - \zeta(u) - \zeta(v),$$

where $\zeta = (\log\sigma)'$ is the Weierstrass function. The third equality leads to the compatibility condition

$$Q(D_1^3) + \wp(u+v)Q(6D_1) + 3\wp'(u+v) = 0,$$

which becomes the following identity in the developed form:

$$\begin{aligned} &(\zeta(u+v) - \zeta(u) - \zeta(v))\bigl\{(\zeta(u+v) - \zeta(u) - \zeta(v))^2 + 3(\wp(u+v) - \wp(u) - \wp(v))\bigr\} \\ &\qquad + 2\wp'(u+v) + \wp'(u) + \wp'(v) = 0 \end{aligned}$$

(this can readily be verified in this classical case by using known addition theorems for elliptic functions).

The recursion (4.9) gives

$$\{B_k^{(u+v)}(x,y)\} = \{0, 1, 0, G_2, \dots\},$$
$$\{C_k^{(u+v)}(x,y)\} = \{1, 0, G_2, \partial_{w_1} G_2, \dots\}.$$

Example 4.2 ($g = 2$, $l = 0$). In this example we have

$$\mathscr{L} = a_0(u,v) + a_1(u,v)\partial_{w_1} + a_2(u,v)\partial_{w_1}^2,$$
$$S_0^{(w)}(x,y) = x^2 - \wp_{1,1}(w)x - \wp_{1,2}(w),$$
$$S_1^{(w)}(x,y) = y + \frac{1}{2}\left(\wp_{1,1,1}(w)x + \wp_{1,1,2}(w)\right),$$
$$S_2^{(w)}(x,y) = C_2(w)S_0^{(w)}(x,y) + B_2(w)S_1^{(w)}(x,y) = (x + 2\wp_{1,1}(w))S_0^{(w)}(x,y).$$

In this case,

$$R_6^{(u,v)}(x,y) = h_0(u,v)x^3 + h_1(u,v)y + h_2(u,v)x^2 + h_4(u,v)x^2 + h_6(u,v).$$

According to the construction presented in the previous section, the coefficients $h_k(u,v)$, $k \in \{0,1,2,4,6\}$, of the function $R_6^{(u,v)}(x,y)$ are of the form $Q(\mathscr{D})$ for the corresponding linear differential operators $\mathscr{D}$. Namely,

$$h_0(u,v) = Q(1) = 1, \quad h_1(u,v) = -Q(D_1), \quad h_2(u,v) = \frac{1}{2}Q(D_1^2),$$
$$h_4(u,v) = \frac{1}{24}Q(D_1^4 + 8D_2D_1 + 6\lambda_4),$$
$$h_6(u,v) = \frac{1}{720}Q(D_1^6 + 40D_2D_1^3 + 18\lambda_4D_1^2 + 144\lambda_6).$$

We carry out the calculations according to (4.5).

The equations corresponding to the monomials x^2, x^3, y give

$$a_2(u,v) = 1, \quad a_1(u,v) = h_1(u,v), \quad a_0(u,v) = h_2(u,v) - \wp_{1,1}(u+v).$$

Thus, we see finally that for $g = 2$ the operator $\mathscr{L}$ is defined by the following coefficients:

$$a_2(u,v) = 1,$$
$$a_1(u,v) = \zeta_1(u+v) - \zeta_1(u) - \zeta_1(v),$$
$$a_0(u,v) = \frac{1}{2}\left\{\left(\zeta_1(u+v) - \zeta_1(u), -\zeta_1(v)\right)^2 - 3\wp_{1,1}(u+v) - \wp_{1,1}(u) - \wp_{1,1}(v)\right\},$$

where $\zeta_1(w) = \partial_{w_1}\sigma(w)$.

The relations between the functions $\{h_i\}$, that is, the compatibility conditions, are given by the equations corresponding to the monomials x and 1,

$$h_4(u,v) + h_2(u,v)\wp_{1,1}(u+v) - h_1(u,v)\wp_{1,1,1}(u+v) + \wp_{1,1}^2(u+v) + \wp_{1,2}(u+v) = 0,$$
$$h_6(u,v) + h_2(u,v)\wp_{1,2}(u+v) - h_1(u,v)\wp_{1,1,2}(u+v) + \wp_{1,1}(u+v)\wp_{1,2}(u+v) = 0,$$

and, as one can immediately see, the validity of these equations ensures (as was already noted above in Remark 4.1) the simultaneous vanishing of the right- and left-hand sides of (4.5) if a point (x,y) is a common zero of the functions $S_0^{(u+v)}(x,y)$ and $S_1^{(u+v)}(x,y)$.

In the general case the manipulations are quite similar to those in the examples. We note that for an (n,s)-curve with $n>2$ the function $\Psi(w,(x,y))$ satisfies a linear differential equation of the form

$$\biggl\{\partial_{w_1}^n-\sum_{i=0}^{n-1}b_i(w)\partial_{w_1}^i\biggr\}\Psi(w,(x,y))=x\Psi(w,(x,y)). \tag{4.11}$$

We present the corresponding analogues of the recursion (4.9) and of the formula (4.8) to compute the functions $G_k^{(w)}(x,y)$. We write

$$G_k^{(w)}(x,y)=\sum_{j=0}^{n-1}C_{j,k}^{(w)}(x,y)G_j^{(w)}(x,y). \tag{4.12}$$

Then the general recursion formula (4.4) and the formula (4.11) imply the recursion

$$\begin{aligned} C_{0,k+1}^{(w)}(x,y)&=\partial_{w_1}C_{0,k}^{(w)}(x,y)+\bigl(x+b_0(w)\bigr)C_{n-1,k}^{(w)}(x,y),\\ C_{i,k+1}(w)(x,y)&=\partial_{w_1}C_{i,k}^{(w)}(x,y)+C_{i-1,k}^{(w)}(x,y)+b_i(w),\quad i=1,\dots,n-1, \end{aligned} \tag{4.13}$$

with the initial data $(C_{0,0}^{(w)}(x,y),C_{1,0}^{(w)}(x,y),\dots,C_{n-1,0}^{(w)}(x,y))=(1,0,\dots,0)$. As in the case of the recursion (4.9), the recursion (4.13) generates a family of polynomials $C_{i,k}^{(w)}(x,y)$, $i=0,1,\dots,n-1$, depending only on the variable x. Thus, in this case, to explicitly compute the coefficients of the operator $\mathscr{L}$ by using Theorem 4.1, one must know not only the functions $R_{2g}^{(u)}(x,y)$ and $R_{3g}^{(u,v)}(x,y)$ but also the explicit form of the functions $G_i^{(w)}(x,y)$ and $b_i(w)$ for $0\leqslant i<n$.

4.3. Algebras associated with continuous Krichever–Novikov bases. We represent the relation (4.6) in the form

$$\Psi(u,P)\Psi(v,P)=\mathscr{L}\Psi(u+v,P)=\sum_{k=0}^{g}a_k(u,v)\partial^k\Psi(u+v,P), \tag{4.14}$$

where $P=(x,y)\in V$ and $\partial=\partial/\partial v_1$. Without loss of generality we can assume that $a_g(u,v)=1$.

For a fixed point P we set

$$\Psi(u,P)=\Psi_{(0,u)}\quad\text{and}\quad\Psi_{(k,u)}=\partial^k\Psi_{(0,u)}. \tag{4.15}$$

We can see from the formulae obtained for $a_k(u,v)$ that the coefficients of the operator $\mathscr{L}$ are obtained by substituting the relation $w=u+v$ into the corresponding functions $\widehat{a}_k(u,v,w)$. It is convenient to stress this fact below by writing

$$a_k(u,v)=C_{(0,u),(0,v)}^{(k,u+v)}. \tag{4.16}$$

In this case (4.14) becomes

$$\Psi_{(0,u)}\Psi_{(0,v)} = \sum_{k=0}^{g} C^{(k,u+v)}_{(0,u),(0,v)} \Psi_{(k,u+v)}. \tag{4.17}$$

Applying the operator $\partial^i_{u_1}\partial^j_{v_1}$ to (4.17), we obtain the formula

$$\Psi_{(i,u)}\Psi_{(j,v)} = \sum_{k=0}^{g+i+j} C^{(k,u+v)}_{(i,u),(j,v)} \Psi_{(k,u+v)}, \tag{4.18}$$

where the functions $C^{(k,u+v)}_{(i,u),(j,v)}$, due to the linear independence of the functions $\Psi_{(k,u)}$, $k = 0, 1, 2, \dots$, are uniquely determined by the family of functions $a_k(u, v)$ and their partial derivatives with respect to u_1 and v_1.

The following recursion formulae hold:

$$\begin{aligned}
C^{(0,u+v)}_{(i+1,u),(j,v)} &= \partial_{u_1} C^{(0,u+v)}_{(i,u),(j,v)};\\
C^{(k,u+v)}_{(i+1,u),(j,v)} &= \partial_{u_1} C^{(k,u+v)}_{(i,u),(j,v)} + C^{(k-1,u+v)}_{(i,u),(j,v)}, \quad k = 1, 2, \dots;\\
C^{(0,u+v)}_{(i,u),(j+1,v)} &= \partial_{v_1} C^{(0,u+v)}_{(i,u),(j,v)};\\
C^{(k,u+v)}_{(i,u),(j+1,v)} &= \partial_{v_1} C^{(k,u+v)}_{(i,u),(j,v)} + C^{(k-1,u+v)}_{(i,u),(j,v)}, \quad k = 1, 2, \dots .
\end{aligned}$$

We note that $C^{(g+i+j,u+v)}_{(i,u),(j,v)} = 1$ for all $i \geqslant 0$, $j \geqslant 0$. Thus, (4.18) describes the multiplication law in the continuous KN-basis of a commutative associative algebra over the field of meromorphic functions on the Jacobian of the curve V. This is the *algebra associated with the continuous KN-basis on the curve V*.

The associativity equations for the multiplication in this algebra form the system of functional equations

$$\sum_{\ell=0}^{Q_1(m)} C^{(\ell,v+w)}_{(j,v),(k,w)} C^{(m,u+v+w)}_{(i,u),(\ell,v+w)} = \sum_{\ell=0}^{Q_2(m)} C^{(\ell,u+v)}_{(i,u),(j,v)} C^{(m,u+v+w)}_{(\ell,u+v),(k,w)}, \quad 0 \leqslant m \leqslant Q, \tag{4.19}$$

where

$$Q = 2g+i+j+k, Q_1(m) = \min(Q-m, g+j+k), \text{ and } Q_2(m) = \min(Q-m, g+i+j),$$

and hence a system of functional-differential equations for the functions $a_k(u, v)$.

According to [59], this system, in the notation

$$L^w_{u,v} = \mathscr{L} = \sum_{k=1}^{g} a_k(u, v)\partial^k_{w_1},$$

is equivalent to the operator equation

$$L^v_{u,v} L^v_{u+v,w} - L^v_{v,w} L^v_{u,v+w} = 0. \tag{4.20}$$

We write $A^j_{i,k}(u, v, w) = a_i(u, v)\partial^j_{v_1} a_k(u + v, w) - a_i(v, w)\partial^j_{v_1} a_k(u, v + w)$.

Corollary 4.1. *The operator equation* (4.20) *is a system* $\{K_i = 0 \mid 0 \leqslant i < 2g\}$ *of functional equations with respect to the functions* $a_0(u,v), \dots, a_{g-1}(u,v)$, *where*

$$K_i = \sum_{j=\max(0,i-g)}^{\min(g,i)} \sum_{k=j}^{g} \binom{k}{j} A^{k-j}_{k,i-j}(u,v,w). \tag{4.21}$$

For instance, for $g = 1$ we obtain the equations $K_1 = 0$ and $K_0 = 0$, where

$$\begin{aligned} K_1 &= K_1(a_0(u,v)) = a_0(u,v) - a_0(v,w) + a_0(u+v,w) - a_0(u,v+w) \\ K_0 &= K_0(a_0(u,v)) = a_0(u,v)a_0(u+v,w) - a_0(v,w)a_0(u,v+w) \\ &\qquad + \partial_v a_0(u+v,w) - \partial_v a_0(u,v+w). \end{aligned}$$

Let δ_1 and δ_2 be differentials in the standard cochain function complex, namely,

$$\begin{aligned} &(\delta_1 a)(u,v) = a(u) + a(v) - a(u+v), \ \text{ where } a = a(u), \\ &(\delta_2 a)(u,v,w) = -a(u+v,w) + a(v,w) + a(u,v+w) - a(u,v), \ \text{ where } a = a(u,v). \end{aligned}$$

We note that $K_1(a_0(u,v)) = (\delta_2 a)(u,v,w)$. Hence, the equation $K_1 = 0$ is exactly the cocycle equation $(\delta_2 a_0)(u,v,w) = 0$. Since the two-dimensional cohomology of the complex in question is zero, we see that there is a unique function $h(u)$ such that $(\delta_1 h)(u,v) = a_0(u,v)$. Thus, the general solution $a_0(u,v)$ of the equation $K_1 = 0$ is of the form

$$a_0(u,v) = h(u) + h(v) - h(u+v).$$

A simple calculation shows that

$$K_0((\delta_1 h)(u,v)) = \delta_2[(\delta_1 h)(u,v)h(u+v) + \partial_{v_1} h(u+v) - h(u)h(v)].$$

Hence, by the equation $K_0 = 0$, there is a unique function $\eta(u)$ such that

$$(\delta_1 \eta)(u,v) = [(\delta_1 h)(u,v)h(u+v) + \partial_{v_1} h(u+v) - h(u)h(v)].$$

Thus, we obtain a functional equation on the function $h(u)$ whose general solution is the Weierstrass function $\zeta(u)$.

The general scheme for constructing a commutative associative algebra with basis $\{E_{(i,u)}\}$ by using a linear differential operator $L^w_{u,v} = \sum_{k=1}^{g} a_k(u,v) D^k_w$ was described in a paper of one of the authors and D. V. Leikin (see [59]). The above manipulations show immediately that if $D_w = \partial_{w_1}$ and $g = 1$, then any such algebra coincides with the algebra associated with the continuous KN-basis on an elliptic curve.

Let us return to the general case for $D_w = \partial_{w_1}$ (see [59]). The equations $K_{g+j} = 0$, where $0 \leqslant j < g$, can be represented in the form

$$(\delta_2 a_j)(u,v,w) = \sum_{k=0}^{g-j-1} \sum_{\ell=0,\, k+\ell>0}^{g-j-k} \binom{j+k+\ell}{j+k} A^{\ell}_{j+k+\ell,g-k}(u,v,w). \tag{4.22}$$

Hence, $(\delta_2 a_{g-1})(u,v,w) = 0$ for any $g \geqslant 1$, and therefore there is a unique function $h_1(u)$ such that $a_{g-1}(u,v) = \delta_1 h_1(u)$.

Theorem 4.2. *Let $h_1(u),\dots,h_g(u)$ be differentiable functions, $u\in\mathbb{C}^g$. Then the recursion formulae*

$$a_{g-k}(u,v)=(\delta_1 h_k)(u,v)+\sum_{\ell=1}^{k-1}h_\ell(u)h_{k-\ell}(v)\\ -\sum_{\ell=0}^{k-1}\sum_{m=1}^{k-\ell-1}\binom{g-\ell}{k-\ell-m}a_{g-\ell}(u,v)D_v^{(k-\ell-m)}h_m(u+v) \tag{4.23}$$

with $k=1,\dots,g$ give the general solution of the system $\{K_{g+j}=0\mid 0\leqslant j<g\}$.

The function $\Psi(u,P)$, which is a degenerate Baker–Akhiezer function (see (4.1)), is completely characterized by its behaviour near a unique singular point of Ψ (see (4.2)),

$$\Psi(u,P)\sim\xi^{-g}\exp\{p(u,\xi^{-1})\}\biggl(1+\sum_{i>0}\eta_i(u)\xi^i\biggr), \tag{4.24}$$

where $p(u,t)=\sum_{j=1}^g u_j\chi_j(t)$ is a function linear with respect to u and a polynomial in t of degree not exceeding $(2g-1)$. Here it is important that $\chi_1(t)=t$. Direct calculations show that the functions $\eta_j(u)$ and the coefficients $a_k(u,v)$, $k=0,1,\dots,g-1$, of the operator $\mathscr{L}$ such that

$$\Psi(\xi,u)\Psi(\xi,v)=\mathscr{L}\Psi(\xi,u+v)$$

are connected by the system $\{B_s=0\mid s=1,2,\dots\}$ of relations, where

$$B_s=\biggl(\sum_{j=q(s)}^{g}\sum_{k=j}^{g}\binom{k}{j}a_k(u,v)D_v^{k-j}\eta_{s+j-g}(u+v)\biggr)-\sum_{\ell=0}^{s}\eta_\ell(u)\eta_{s-\ell}(v),$$

$q(s)=\max(0,g-s)$.

Theorem 4.3 (see [59]). *If the functions $\eta_i(u), i=1,\dots,g$, in the expansion (4.24) are taken as the functions $h_i(u)$ in Theorem 4.2, then the system $\mathscr{B}_{\mathrm{I}}=\{B_s=0\mid s\leqslant g\}$ of equations is transformed into the recursion system (4.23) determining the coefficients $a_k(u,v)$ of the operator $\mathscr{L}$.*

The system $\mathscr{B}_{\mathrm{II}}=\{B_s=0\mid s>g\}$ passes into the system $\{(\delta_1\eta_{g+m})(u,v)=F_m,\ m=1,2,\dots\}$ of equations, where

$$F_m=\biggl(\sum_{j=0}^{g-1}\sum_{k=j}^{g}\binom{k}{j}a_k(u,v)D_v^{k-j}\eta_{m+j}(u+v)\biggr)-\sum_{\ell=1}^{g+m-1}\eta_\ell(u)\eta_{g+m-\ell}(v).$$

Since $\delta_2\delta_1=0$, we arrive at the equations $\{\delta_2F_m=0\}$ of compatibility of the system $\mathscr{B}_{\mathrm{II}}$ in the form of functional-differential recursion equations in $\eta_1,\dots,\eta_g$.

The description of the multiplication in the algebras associated with the continuous KN-basis is completed by the following result.

Theorem 4.4 (see [59]). *With regard to the assertions of Theorems 4.2 and 4.3, the system*

$$\widetilde{K}=\{K_j=0\mid j=0,\dots,g-1\}$$

ensures the compatibility conditions of the system $\mathscr{B}_{\mathrm{II}}=\{B_s=0\mid s>g\}$.

§ 5. Integrable linear problems and applications

In this section we present the main ideas of a new approach to the solution of problems of Riemann–Schottky-type that was suggested in a recent paper of one of the authors [48]. This approach is based on the use of integrable linear problems. It should be noted that we include neither more nor less rigorous sense in the notion of integrable linear problem. In the framework of the present survey this notion will mean linear equations whose solutions are given by explicit theta function formulae.

5.1. Infinite-dimensional analogue of the Calogero–Moser system. As was already noted in the Introduction, an immediate consequence of the addition theorem (1.25) for theta functions is the equivalence of the integrable linear problem (1.41)–(1.43) and the equations (1.44) signifying that the point $K(A/2)$ is an inflection point of the Kummer variety.

The equivalence of the assertion of Theorem 1.4 to another formulation characterizing the Jacobian varieties in terms of some infinite-dimensional analogue of the Calogero–Moser system is by no means trivial.

Let us consider an entire function $\tau(x,y)$ of a complex variable x which depends smoothly on the parameter y. Suppose that this function satisfies the equation

$$\operatorname{res}_x\bigl(\partial_y^2\log\tau+2(\partial_x^2\log\tau)^2\bigr)=0, \tag{5.1}$$

which means that the meromorphic function of x given by the expression on the left-hand side of (5.1) has no *residues*. If $x_i(y)$ is a simple zero of τ, that is, $\tau(x_i(y),y)=0, \partial_x\tau(x_i(y),y)\neq 0$, then it follows from (5.1) that

$$\ddot{x}_i=2w_i, \tag{5.2}$$

where the dots stand for the derivatives with respect to the variable y and w_i is the third coefficient of the Laurent expansion of the function $u(x,y)=-2\partial_x^2\tau(x,y)$ at the point x_i, that is,

$$u(x,y)=\frac{2}{(x-x_i(y))^2}+v_i(y)+w_i(y)(x-x_i(y))+\cdots. \tag{5.3}$$

Formally, if we represent τ in the form of the infinite product

$$\tau(x,y)=c(y)\prod_i(x-x_i(y)), \tag{5.4}$$

then the equation (5.1) turns out to be equivalent to the infinite system of equations

$$\ddot{x}_i=-4\sum_{j\neq i}\frac{1}{(x_i-x_j)^3}, \tag{5.5}$$

which coincides with the equations of motion of a rational, trigonometric, or elliptic Calogero–Moser system if τ is a rational, trigonometric, or elliptic polynomial, respectively.

The equations (5.2) for the zeros of the function $\tau=\theta(Ux+Vy+Z)$ were first introduced in the paper [60] as an immediate corollary to the assumptions of

Theorem 1.4. Expanding the function θ in neighbourhoods of the points $z \in \Theta$ of its divisor such that $\theta(z)=0$, we can readily see that the equation (5.1) is equivalent to the equation

$$[(\partial_2\theta)^2-(\partial_1^2\theta)^2]\partial_1^2\theta+2[\partial_1^2\theta\,\partial_1^3\theta \\ -\partial_2\theta\,\partial_1\partial_2\theta]\partial_1\theta+[\partial_2^2\theta-\partial_1^4\theta](\partial_1\theta)^2=0 \pmod{\theta}, \tag{5.6}$$

which must hold at all points of the divisor Θ. Here and below, ∂_1 and ∂_2 are constant vector fields on $\mathbb{C}^g$ corresponding to the vectors U and V.

Theorem 5.1. *An indecomposable principally polarized Abelian variety* (X,θ) *is the Jacobian variety of a smooth algebraic curve of genus* g *if and only if there are* g*-dimensional vectors* $U \neq 0$ *and* V *such that the equation* (5.6) *holds on the divisor* Θ.

The main idea of the proof of Theorem 1.4 is to show that the linear problem (1.41)–(1.43) admits the introduction of a *spectral* parameter. More precisely, to show that, under the assumptions of the theorem, the equation (1.41) has a formal wave solution, that is, a solution of the form

$$\psi(x,y,k)=e^{kx+(k^2+b)y}\biggl(1+\sum_{s=1}^{\infty}\xi_s(x,y)\,k^{-s}\biggr). \tag{5.7}$$

An attempt to directly construct a formal wave solution whose coefficients are of the form

$$\xi_s=\frac{\tau_s(Ux+Vy+Z)}{\theta(Ux+Vy+Z)}\,, \tag{5.8}$$

where $\tau_s(Z)$ is a holomorphic function of the variable $Z \in \mathbb{C}^g$, leads to problems similar to those faced by Shiota in constructing the τ-function of the KP hierarchy.

The substitution of the series (5.7) in the equation (1.41) gives a recursive system of equations for the coefficients ξ_s of the wave function. The assertions of the following two lemmas show that the equations (5.2) are necessary and sufficient conditions for the *local* solubility of these equations in the class of meromorphic functions.

Lemma 5.1 [60]. *Let* $\tau(x,y)$ *be a holomorphic function of the variable* $x \in D \subset \mathbb{C}$ *and let* τ *depend smoothly on the variable* y. *Suppose that the zeros of* τ *are simple*:

$$\tau(x_i(y),y)=0, \qquad \tau_x(x_i(y),y)\neq 0. \tag{5.9}$$

In this case, if the equation (1.41) *with the potential* $u=-2\partial_x^2\log\tau(x,y)$ *has a meromorphic solution* $\psi_0(x,y)$, *then the equations* (5.2) *are satisfied.*

Let us consider the Laurent expansions of ψ_0 and u in a neighbourhood of one of the zeros x_i of the function τ,

$$u=\frac{2}{(x-x_i)^2}+v_i+w_i(x-x_i)+\cdots\,, \tag{5.10}$$

$$\psi_0=\frac{\alpha_i}{x-x_i}+\beta_i+\gamma_i(x-x_i)+\delta_i(x-x_i)^2+\cdots\,. \tag{5.11}$$

(The coefficients are smooth functions of the variable y.) The substitution of the series (5.10) and (5.11) into (1.41) gives a chain of equations, the first three of which have the form

$$\alpha_i \dot{x}_i + 2\beta_i = 0, \tag{5.12}$$

$$\dot{\alpha}_i + \alpha_i v_i + 2\gamma_i = 0, \tag{5.13}$$

$$\dot{\beta}_i + v_i \beta_i - \gamma_i \dot{x}_i + \alpha_i w_i = 0. \tag{5.14}$$

Differentiating the first of these with respect to y and using the other two, we obtain (5.2).

Lemma 5.2. *Suppose that the equations* (5.2) *hold for the zeros of the function* $\tau(x, y)$. *In this case there is a meromorphic wave solution of the equation* (1.41) *which has only simple poles at the points* x_i.

Proof. The substitution of (5.7) into (1.41) gives the system of equations

$$2\xi'_{s+1} = \partial_y \xi_s + u\xi_s - \xi''_s. \tag{5.15}$$

We prove by induction that, under the assumptions of the lemma, this system has a meromorphic solution with simple poles at the points x_i.

Let us expand the variable ξ_s in a neighbourhood of x_i:

$$\xi_s = \frac{r_s}{x - x_i} + r_{s0} + r_{s1}(x - x_i) \tag{5.16}$$

(for brevity we omit the index i in the notation for the coefficients of this expansion). Suppose that ξ_s is known and the equation (5.15) has a meromorphic solution. This means that the right-hand side of (5.15) has zero residue at the point $x = x_i$:

$$\operatorname{res}_{x_i}(\partial_y \xi_s + u\xi_s - \xi''_s) = \dot{r}_s + v_i r_s + 2r_{s1} = 0. \tag{5.17}$$

We must show that the residue of the next equation is equal to zero. It follows from (5.15) that the coefficients of the expansion for ξ_{s+1} are given by

$$r_{s+1} = -\dot{x}_i r_s - 2r_{s0}, \tag{5.18}$$

$$2r_{s+1,1} = \dot{r}_{s0} - r_{s1} + w_i r_s + v_i r_{s0}. \tag{5.19}$$

Thus,

$$\dot{r}_{s+1} + v_i r_{s+1} + 2r_{s+1,1} = -r_s(\ddot{x}_i - 2w_i) - \dot{x}_i(\dot{r}_s - v_i r_s s + 2r_{s1}) = 0, \tag{5.20}$$

which proves the lemma.

Local solubility of the equations (5.15) does not automatically imply global solubility, that is, the existence of a wave function with the coefficients ξ_s of the form (5.8). Arguments that are quite the same as those used in [8] in constructing the τ-functions of the KP hierarchy show that the existence of a global wave function is controlled by a cohomological obstruction which is an element of the group $H^1(\mathbb{C}^g \setminus \Sigma, M)$, where Σ is the ∂_1-invariant subset of Θ and M is the sheaf of ∂_1-invariant meromorphic functions that are holomorphic outside Θ.

5.2. λ-periodic wave solutions. In contrast to [8], we do not immediately claim that the set Σ is empty. It should be noted that, in order to construct an algebraic curve which turns out in what follows to be the very curve whose Jacobian is isomorphic to X, it suffices to use the global existence of the functions ξ_s along certain affine hyperplanes in $\mathbb{C}^g$.

We denote by $Y_U = \langle Ux\rangle$ the closure of the group Ux in X. Shifting Y_U if necessary, we can assume without loss of generality that Y_U does not belong to the bad subset, $Y_U \notin \Sigma$. In this case, $Y_U + Vy \notin \Sigma$ for sufficiently small y. Let us consider the restriction of the theta function to the affine subspace $\mathbb{C}^d + Vy$, where $\mathbb{C}^d = \pi^{-1}(Y_U)$ and $\pi\colon \mathbb{C}^g \to X = \mathbb{C}^g/\Lambda$ is the universal covering of X:

$$\tau(z,y) = \theta(z+Vy), \qquad z\in\mathbb{C}^d. \tag{5.21}$$

The function $u(z,y) = -2\partial_1^2 \log\tau$ is periodic with respect to $\Lambda_U = \Lambda\cap\mathbb{C}^d$ and for a fixed y it has a second-order pole along the divisor $\Theta^U(y) = (\Theta - Vy)\cap\mathbb{C}^d$.

Lemma 5.3. *Let us fix a vector λ of the lattice $\Lambda_U = \Lambda\cap\mathbb{C}^d\subset\mathbb{C}^g$. Suppose that the equations (5.1) hold for any $\tau(Ux+z,y)$. In this case:*

(i) *the equation (1.41) with the potential $u(Ux+z,y)$ admits a wave solution of the form $\psi = e^{kx+k^2y}\phi(Ux+z,y,k)$ such that the coefficients $\xi_s(z,y)$ of the formal series*

$$\phi(z,y,k) = e^{by}\left(1+\sum_{s=1}^{\infty}\xi_s(z,y)\,k^{-s}\right) \tag{5.22}$$

are λ-periodic meromorphic functions of $z\in\mathbb{C}^d$ with a simple pole along the divisor $\Theta^U(y)$,

$$\xi_s(z+\lambda,y) = \xi_s(z,y) = \frac{\tau_s(z,y)}{\tau(z,y)}\,; \tag{5.23}$$

(ii) *the series $\phi(z,y,k)$ is unique up to a factor $\rho(z,k)$ which is ∂_1-invariant and holomorphic with respect to the variable z,*

$$\phi_1(z,y,k) = \phi(z,y,k)\rho(z,k), \qquad \partial_1\rho = 0. \tag{5.24}$$

Proof. The functions $\xi_s(z)$ are determined by the equations

$$2\partial_1\xi_{s+1} = \partial_y\xi_s + (u+b)\xi_s - \partial_1^2\xi_s. \tag{5.25}$$

A particular solution of the first of these equations $2\partial_1\xi_1 = u+b$ is given by the formula

$$2\xi_1^0 = -2\partial_1\log\tau + (l,z)\,b, \tag{5.26}$$

where (l,z) is a linear form on $\mathbb{C}^d$ given by the inner product of z with a vector $l\in\mathbb{C}^d$ such that $(l,U)=1$. By definition, $\lambda\in Y_U$. Hence, $(l,\lambda)\neq 0$. The periodicity condition for ξ_1^0 determines the constant b, namely,

$$b = (l,\lambda)^{-1}(2\partial_1\log\tau(z+\lambda,y) - 2\partial_1\log\tau(z,y)), \tag{5.27}$$

which depends only on the choice of the vector λ. The addition of a constant to the potential does not influence the result of the previous lemma. Therefore, the equations (5.2) are sufficient for the local solubility of the equations (5.25) in any domain in which the function $\tau(z+Ux,y)$ has simple zeros, that is, outside the set $\Theta_1^U(y)=(\Theta_1-Vy)\cap\mathbb{C}^d$, where $\Theta_1=\Theta\cap\partial_1\Theta$. This set does not contain any ∂_1-invariant line, because every such line is dense in Y_U. Hence, the sheaf of ∂_1-invariant meromorphic functions on $\mathbb{C}^d\setminus\Theta_1^U(y)$ with the poles along $\Theta^U(y)$ coincides with the sheaf of holomorphic ∂_1-invariant functions. Therefore, the group $H^1(C^d\setminus\Theta_1^U(y),M_0)$ is trivial, and global meromorphic solutions ξ_s^0 of the equations (5.25) with simple poles along the divisor $\Theta^U(y)$ exist (for details, see [8], [25]). If a particular solution ξ_s^0 is chosen, then the general global meromorphic solution is given by the formula $\xi_s=\xi_s^0+c_s$, where the constant of integration $c_s(z,y)$ is a holomorphic ∂_1-invariant function of the variable z. Let us show that the condition of λ-periodicity fixes the dependence of this constant of integration on the variable y.

We carry out the proof by induction. Suppose that a λ-periodic solution ξ_{s-1} is known and satisfies the condition that there is a periodic solution ξ_s^0 of the next equation. We denote by ξ_{s+1}^* the solution of the equation (5.25) for a fixed ξ_s^0. One can readily see that the function

$$\xi_{s+1}^0(z,y)=\xi_{s+1}^*(z,y)+c_s(z,y)\,\xi_1^0(z,y)+\frac{(l,z)}{2}\partial_y c_s(z,y) \tag{5.28}$$

is a solution of (5.25) for $\xi_s=\xi_s^0+c_s$. The choice of a λ-periodic ∂_1-invariant function $c_s(z,y)$ does not influence the periodicity of ξ_s but influences the periodicity of ξ_{s+1}^0. For the function ξ_{s+1}^0 to be periodic it is necessary that the function $c_s(z,y)$ satisfy the linear differential equation

$$\partial_y c_s(z,y)=2(l,\lambda)^{-1}(\xi_{s+1}^*(z+\lambda,y)-\xi_{s+1}^*(z,y)). \tag{5.29}$$

This equation and the initial conditions $c_s(z)=c_s(z,0)$ uniquely determine the function $c_s(x,y)$. The induction step is completed. We have proved that the ratio of two periodic formal series ϕ_1 and ϕ does not depend on y. This implies the equality (5.24), in which the factor $\rho(z,k)$ is determined by the two sides of the equality at $y=0$.

5.3. Spectral curve. Our next objective is to prove that the λ-periodic wave solutions of the equation (1.41), (1.42) are joint eigenfunctions of commuting operators and to identify X with the Jacobian of the corresponding spectral curve.

We note that the simple shift $z\to z+Z$ enables one to define the λ-periodic wave solutions with meromorphic coefficients along the affine subspaces $Z+\mathbb{C}^d$, $Z\notin\Sigma$. These λ-periodic solutions are connected with each other by ∂_1-invariant factors. Hence, choosing a hyperplane orthogonal to the vector U in a neighbourhood of any point $Z\notin\Sigma$ and fixing the initial conditions on this hyperplane at $y=0$, we determine the coefficients of the series $\phi(z+Z,y,k)$ as *local* meromorphic functions of the variable Z and as *global* meromorphic functions of the variable z.

Lemma 5.4. *Under the assumptions of Theorem 5.1, there is a unique pseudo-differential operator*

$$\mathscr{L}(Z,\partial_x)=\partial_x+\sum_{s=1}^{\infty}w_s(Z)\partial_x^{-s} \tag{5.30}$$

such that

$$\mathscr{L}(Ux + Vy + Z, \partial_x)\,\psi = k\,\psi\,, \tag{5.31}$$

where $\psi = e^{kx+k^2y}\phi(Ux + Z, y, k)$ *is a* λ*-periodic solution of the equation* (1.41). *The coefficients* $w_s(Z)$ *of this operator are meromorphic functions on the Abelian variety* X *that are holomorphic outside the divisor* Θ.

Proof. The construction of $\mathscr{L}$ is standard in the KP theory. We first define $\mathscr{L}$ as a pseudodifferential operator with the coefficients $w_s(Z, y)$ depending on Z and y.

Let us consider a λ-periodic wave solution ψ. The substitution of the formula (5.22) into (5.31) gives a system of equations, uniquely determining the coefficients $w_s(Z, y)$, in the form of differential polynomials in the coefficients $\xi_s(Z, y)$ of the expansion of ψ. The coefficients of ψ are local meromorphic functions of the variable Z. Since λ-periodic wave solutions are connected with each other by ∂_1-invariant factors, the non-uniqueness of ψ does not affect the coefficients of $\mathscr{L}$. Hence, these coefficients are well defined as *global meromorphic functions* on $\mathbb{C}^g \setminus \Sigma$. The codimension of the singular locus is not less than 2. Hence, by the Hartogs theorem, the functions $w_s(Z, y)$ admit continuations to global meromorphic functions on $\mathbb{C}^g$.

It follows from the translation invariance of u that for any constant s the series $\phi(Vs + Z, y - s, k)$ and $\phi(Z, y, k)$ correspond to λ-periodic solutions of the same equation. Thus, they differ by a ∂_1-invariant factor. Hence, $w_s(Z, y) = w_s(Vy+Z)$.

For the same reasons,

$$\partial_1\big(\phi_1(Z + \lambda', y, k)\phi^{-1}(Z, y, k)\big) = 0. \tag{5.32}$$

Thus, the functions w_s are periodic with respect to the lattice Λ, and hence are meromorphic functions on X. This proves Lemma 5.4.

As usual, we denote by $\mathscr{L}^m_+$ the differential part of the operator $\mathscr{L}^m$. By definition, the leading coefficient F_m of the integral part $\mathscr{L}^m_- = \mathscr{L}^m - \mathscr{L}^m_+ = F_m\partial^{-1} + O(\partial^{-2})$ is called the *residue* of $\mathscr{L}^m$,

$$F_m = \operatorname{res}_\partial \mathscr{L}^m. \tag{5.33}$$

It follows from the construction of $\mathscr{L}$ that $[\partial_y - \partial_x^2 + u, \mathscr{L}^n] = 0$. Thus,

$$[\partial_y - \partial_x^2 + u, \mathscr{L}^m_+] = -[\partial_y - \partial_x^2 + u, \mathscr{L}^m_-] = 2\partial_x F_m. \tag{5.34}$$

The functions F_m are differential polynomials in the coefficients w_s of the operator $\mathscr{L}$. Hence, these functions are meromorphic on X.

Lemma 5.5. *The orders of the poles of the Abelian functions* F_m *on* Θ *do not exceed two.*

Proof. We use another standard construction of the KP theory. An arbitrary wave solution determines a unique pseudodifferential operator Φ such that

$$\psi = \Phi e^{kx+k^2y}, \qquad \Phi = 1 + \sum_{s=1}^{\infty} \varphi_s(Ux + Z, y)\partial_x^{-s}. \tag{5.35}$$

The coefficients of Φ are differential polynomials in the coefficients ξ_s of the wave solution. Hence, if ψ is the same wave solution as in the assertion of Lemma 5.3, then the coefficients $\varphi_s(z+Z,y)$ of the corresponding operator Φ are global meromorphic functions of the variable $z\in\mathbb{C}^d$ and local meromorphic functions of the variable $Z\notin\Sigma$. We note that $\mathscr{L}=\Phi(\partial_x)\,\Phi^{-1}$.

We define the dual wave function by the left action of the operator Φ^{-1}, that is, $\psi^+=e^{-kx-k^2y}\Phi^{-1}$. We recall that the left action of a pseudodifferential operator is formally adjoint to the right action, that is, it is defined by the formula $(f\partial_x)=-\partial_x f$. If ψ is a formal wave solution of the equation (1.41), then ψ^+ is a solution of the conjugate equation

$$(-\partial_y-\partial_x^2+u)\psi^+=0. \tag{5.36}$$

As above, one can prove that the coefficients ξ_s^+ of the dual solution have simple poles at the poles of u. Hence, ψ^+ is of the form

$$\psi^+=e^{-kx-k^2y}\phi^+(Ux+Z,y,k),$$

where the coefficients $\xi_s^+(z+Z,y)$ of the formal series

$$\phi^+(z+Z,y,k)=e^{-by}\left(1+\sum_{s=1}^{\infty}\xi_s^+(z+Z,y)\,k^{-s}\right) \tag{5.37}$$

are λ-periodic meromorphic functions of the variable $z\in\mathbb{C}^d$ with simple poles on $\Theta^U(y)$.

The ambiguity in the definition of ψ does not influence the product

$$\psi^+\psi=(e^{-kx-k^2y}\Phi^{-1})(\Phi e^{kx+k^2y}). \tag{5.38}$$

Hence, the coefficients J_s of this product

$$\psi^+\psi=\phi^+(Z,y,k)\phi(Z,y,k)=1+\sum_{s=2}^{\infty}J_s(Z,y)k^{-s} \tag{5.39}$$

are *global meromorphic functions* of the variable Z. Moreover, it follows from the translation invariance of u that these functions are of the form $J_s(Z,y)=J_s(Z+Vy)$. Each of the factors has a simple pole on $\Theta-Vy$. Thus, $J_s(Z)$ is a meromorphic function on X with a second-order pole on Θ.

It follows from the definition of $\mathscr{L}$ that

$$\operatorname{res}_k(\psi^+(\mathscr{L}^n\psi))=\operatorname{res}_k(\psi^+k^n\psi)=J_{n+1}. \tag{5.40}$$

On the other hand, using the equality

$$\operatorname{res}_k(e^{-kx}\mathscr{D}_1)(\mathscr{D}_2e^{kx})=\operatorname{res}_\partial(\mathscr{D}_2\mathscr{D}_1), \tag{5.41}$$

which holds for any pseudodifferential operators, we obtain

$$\operatorname{res}_k(\psi^+\mathscr{L}^n\psi)=\operatorname{res}_k(e^{-kx}\Phi^{-1})(\mathscr{L}^n\Phi e^{kx})=\operatorname{res}_\partial\mathscr{L}^n=F_n. \tag{5.42}$$

This implies that $F_n=J_{n+1}$ and completes the proof of Lemma 5.5.

We denote by $\widehat{\mathbf{F}}$ the linear space generated by the functions $\{F_m,\ m = 0,\ 1,\ 2,\ \ldots\}$, where we formally set $F_0 = 1$. It follows from what was proved above that this is a subspace of the 2^g-dimensional space of Abelian functions with a pole on Θ of order at most two. Hence, for all but $\widehat{g} = \dim \widehat{\mathbf{F}}$ positive numbers n there are constants $c_{i,n}$ such that

$$F_n(Z) + \sum_{i=0}^{n-1} c_{i,n} F_i(Z) = 0. \tag{5.43}$$

The set I of integers for which there are no such constants is called the *gap sequence*.

Lemma 5.6. *Let $\mathscr{L}$ be a pseudodifferential operator constructed from λ-periodic wave functions ψ. In this case the operators*

$$L_n = \mathscr{L}_+^n + \sum_{i=0}^{n-1} c_{i,n} \mathscr{L}_+^{n-i} = 0, \qquad n \notin I, \tag{5.44}$$

satisfy the equalities

$$L_n \psi = a_n(k)\,\psi, \qquad a_n(k) = k^n + \sum_{s=1}^{\infty} a_{s,n} k^{n-s}, \tag{5.45}$$

in which the coefficients $a_{s,n}$ are constant.

Proof. Let us note first that the formula (5.34) implies the equality

$$[\partial_y - \partial_x^2 + u, L_n] = 0. \tag{5.46}$$

Thus, if the function ψ is a λ-periodic wave solution of the equation (1.41), then the function $L_n\psi$ is also a λ-periodic wave solution. This yields $L_n\psi = a_n(Z,k)\psi$, where a is a ∂_1-invariant series. The ambiguity in the definition of ψ does not influence the functions a_n. Hence, the coefficients a_n are well-defined *global* meromorphic functions on $\mathbb{C}^g \setminus \Sigma$. It follows from the ∂_1-invariance of a_n that every function a_n is holomorphic outside Σ as a function of Z. Thus, it admits a holomorphic extension to $\mathbb{C}^g$. It follows from (5.32) that the function a_n is periodic with respect to the period lattice Λ. Thus, a_n does not depend on Z. We note that $a_{s,n} = c_{s,n}$, $s \leqslant n$. This completes the proof of Lemma 5.6.

The operator L_m can be regarded as a $(Z \notin \Sigma)$-parameter family of ordinary differential operators L_m^Z whose coefficients are of the form

$$L_m^Z = \partial_x^n + \sum_{i=1}^{m} u_{i,m}(Ux + Z)\,\partial_x^{m-i}, \qquad m \notin I. \tag{5.47}$$

Corollary 5.1. *The operators L_m^Z commute with each other*:

$$[L_n^Z, L_m^Z] = 0, \qquad Z \notin \Sigma. \tag{5.48}$$

It follows from (5.45) that $[L_n^Z, L_m^Z]\psi = 0$. The commutator is an ordinary differential operator, and therefore the last relation implies (5.48).

Lemma 5.7. *There is an irreducible algebraic curve* Γ *such that for any* $Z \notin \Sigma$ *the commutative ring* $\mathscr{A}^Z$ *generated by the operator* L_n^Z *is isomorphic to the ring* $A(\Gamma, P_0)$ *of meromorphic functions on* Γ *with a unique pole at a smooth point* P_0*. The correspondence* $Z \to \mathscr{A}^Z$ *defines a holomorphic embedding of* $X \setminus \Sigma$ *in the space* $\overline{\mathrm{Pic}}(\Gamma)$ *of torsion-free sheaves* Φ *of rank* 1 *on* Γ:

$$j\colon X \setminus \Sigma \longmapsto \overline{\mathrm{Pic}}(\Gamma). \tag{5.49}$$

Proof. The fundamental assertion of the theory of commuting differential operators ([1], [2], [61], [62], [23]) is that there is a natural correspondence

$$\mathscr{A} \longleftrightarrow \{\Gamma, P_0, [k^{-1}]_1, \Phi\} \tag{5.50}$$

between commutative rings $\mathscr{A}$ of ordinary linear differential operators containing a pair of operators that have coprime orders and are *regular* in a neighbourhood of $x = 0$, on the one hand, and families of algebro-geometric data $\{\Gamma, P_0, [k^{-1}]_1, \Phi\}$, where Γ is an algebraic curve with a fixed first germ $[k^{-1}]_1$ of the local coordinate k^{-1} in a neighbourhood of the smooth point $P_0 \in \Gamma$ and Φ is a torsion-free sheaf of rank 1 on Γ such that

$$H^0(\Gamma, \Phi) = H^1(\Gamma, \Phi) = 0, \tag{5.51}$$

on the other hand. The correspondence becomes one-to-one if one considers the commutative rings $\mathscr{A}$ up to conjugation $\mathscr{A}' = g(x)\mathscr{A}g^{-1}(x)$.

The algebraic curve Γ is called a *spectral curve of the ring* $\mathscr{A}$. The ring $\mathscr{A}$ is isomorphic to the ring $A(\Gamma, P_0)$ of meromorphic functions on Γ with a single pole at the distinguished point P_0. The isomorphism is defined by the equality

$$L_a\psi_0 = a\psi_0, \qquad L_a \in \mathscr{A}, \quad a \in A(\Gamma, P_0). \tag{5.52}$$

Here ψ_0 stands for a common eigenfunction of the commuting operators. For $x = 0$ this function is a section of the sheaf $\Phi \otimes \mathscr{O}(-P_0)$.

Important remark. The correspondence (5.50) depends on the choice of the initial point $x_0 = 0$. Both the spectral curve and the sheaf Φ are determined by the values of the coefficients of the generators of the ring $\mathscr{A}$ and of finitely many derivatives of these coefficients at the initial point. We note that the dependence of the spectral curve on the initial point is trivial, whereas this is not the case for the sheaf, $\Phi = \Phi_{x_0}$.

Using a shift of the initial point, one can show that the correspondence (5.50) can be extended to the case of commutative rings of operators whose coefficients are *meromorphic* functions of the variable x. The rings of operators having a pole at the point $x = 0$ correspond to sheaves for which the condition (5.51) is violated.

Let us consider the spectral curve Γ^Z corresponding to the ring $\mathscr{A}^Z$. By the above remark, this curve is well defined for $Z \notin \Sigma$. The eigenvalues $a_n(k)$ of the operator L_n^Z that are defined in (5.45) coincide with the Laurent expansions of the meromorphic functions $a_n \in A(\Gamma^Z, P_0)$ in a neighbourhood of P_0. These eigenvalues do not depend on Z. Hence, the spectral curve also does not depend on Z, that is, $\Gamma = \Gamma^Z$. This proves the first assertion of the lemma.

It follows from the construction of the correspondence (5.50) that if the coefficients of the operators in the ring $\mathscr{A}$ depend holomorphically on parameters, then the corresponding algebro-geometric spectral data also depend holomorphically on these parameters. This implies that the map j is holomorphic outside the divisor Θ. Using a shift of the initial point and the fact that Φ_{x_0} depends holomorphically on x_0, we see that the map j can be holomorphically continued to $\Theta \setminus \Sigma$. This completes the proof of the lemma.

Our next objective is to prove the existence of a global wave function.

Lemma 5.8. *Under the assumptions of Theorem 5.1, there is a joint eigenfunction of the corresponding commuting operators L_n^Z that has the form $\psi = e^{kx}\phi(Ux+Z,k)$, where the coefficients of the formal series*

$$\phi(Z,k) = 1 + \sum_{s=1}^{\infty} \xi_s(Z)\, k^{-s} \tag{5.53}$$

are global meromorphic functions of the variable Z with simple poles on Θ.

Proof. Let us first consider the case of a smooth spectral curve. In this case, by [1] and [2], the joint eigenfunction of the commuting operators, normalized by the condition $\psi_0\big|_{x=0} = 1$, can be expressed explicitly in terms of the theta function:

$$\widehat{\psi}_0 = \frac{\widehat{\theta}(\widehat{A}(P) + \widehat{U}x + \widehat{Z})\,\widehat{\theta}(\widehat{Z})}{\widehat{\theta}(\widehat{U}x + \widehat{Z})\,\widehat{\theta}(\widehat{A}(P) + \widehat{Z})}\, e^{x\,\Omega(P)}. \tag{5.54}$$

Here $\widehat{\theta}(\widehat{Z})$ is the Riemann theta function constructed from the period matrix of normalized holomorphic differentials on Γ, $\widehat{A}\colon \Gamma \to J(\Gamma)$ is the Abel map, Ω is the Abelian integral corresponding to the normalized Abelian differential $d\Omega$ with a single pole of the form dk at the point P_0, and $2\pi i\widehat{U}$ is the vector of b-periods of $d\Omega$.

Remark. We stress once more that the formula (5.54) was by no means obtained as a result of solving any differential equations. This formula is a consequence of the analytic properties (presented above at the beginning of § 2) of the Baker–Akhiezer function on the spectral curve.

The last factors in the numerator and the denominator of the formula (5.54) do not depend on x. Hence, the non-normalized Baker–Akhiezer function

$$\widehat{\psi}_{BA} = \frac{\widehat{\theta}(\widehat{A}(P) + \widehat{U}x + \widehat{Z})}{\widehat{\theta}(\widehat{U}x + \widehat{Z})}\, e^{x\,\Omega(P)} \tag{5.55}$$

is also an eigenfunction of the commuting operators. In a neighbourhood of P_0 the function $\widehat{\psi}_{BA}$ is of the form

$$\widehat{\psi}_{BA} = e^{kx}\left(1 + \sum_{s=1}^{\infty} \frac{\tau_s(\widehat{Z} + \widehat{U}x)}{\widehat{\theta}(\widehat{U}x + \widehat{Z})}\, k^{-s}\right), \qquad k = \Omega, \tag{5.56}$$

where $\tau_s(\widehat{Z})$ are holomorphic functions.

By the assertion of Lemma 5.7, there is a holomorphic embedding $\widehat{Z} = j(Z)$ of the complement $X \setminus \Sigma$ in $J(\Gamma)$. Let us consider the function $\psi = j^* \widehat{\psi}_{BA}$. It is well defined outside Σ. If $Z \notin \Theta$, then $j(Z) \notin \widehat{\Theta}$. Hence, the coefficients of ψ are regular outside Θ. The codimension of the singular locus is not less than two. Hence, the function ψ can be continued to the whole of X by the Hartogs theorem.

In the case of a singular spectral curve, the proof does not differ in fact from that in the case of smooth curve treated above. It suffices to replace the formula (5.55) by its generalization given by the Sato τ-function theory (for details, see [63]). Namely, the family of algebro-geometric spectral data in (5.50) determines a point of the Sato Grassmannian and, as a consequence, the corresponding τ-function, $\tau(t;\mathscr{F})$. This is a holomorphic function of the variable $t = (t_1, t_2, \dots)$ and a section of the line bundle on $\overline{\operatorname{Pic}}(\Gamma)$. The variable x is identified with the first 'time' variable of the KP hierarchy, $x = t_1$. It follows from the formula for the Baker–Akhiezer function corresponding to a point of the Grassmannian [63] that the function $\widehat{\psi}_{BA}$ given by the formula

$$\widehat{\psi}_{BA} = \frac{\tau(x-k, -\frac{1}{2}k^2, -\frac{1}{3}k^3, \dots; \mathscr{F})}{\tau(x, 0, 0, \dots; \mathscr{F})} e^{kx} \tag{5.57}$$

is a joint eigenfunction of commuting operators. The remaining part of the proof is identical to the smooth case.

Lemma 5.9. *The linear space $\widehat{\mathbf{F}}$ spanned by the Abelian functions $\{F_0 = 1,\ F_m = \operatorname{res}_\partial \mathscr{L}^m\}$ is a subspace of the space $\mathbf{H}$ of Abelian functions generated by F_0 and the functions $H_i = \partial_1 \partial_{z_i} \log \theta(Z)$.*

Proof. We recall that the functions F_n are Abelian functions with a pole of order at most two on Θ. Thus, the *a priori* estimate for the dimension of the space spanned by these functions is $\widehat{g} = \dim \widehat{\mathbf{F}} \leqslant 2^g$. To prove the assertion of the lemma, it suffices to show that $F_n = \partial_1 Q_n$, where Q_n is a meromorphic function with a pole on Θ. Indeed, if Q_n exists, then the equality $Q_n(Z+\lambda) = Q_n(Z) + c_{n,\lambda}$ holds for any vector λ in the period lattice. Hence, one can find a constant q_n and two g-dimensional vectors l_n and l_n' such that $Q_n = q_n + (l_n, Z) + (l_n', h(Z))$, where $h(Z)$ is the vector with the coordinates $h_i = \partial_{z_i} \log \theta$. The last identity means that $F_n = (l_n, U) + (l_n', H(Z))$.

Let us consider the global wave function $\psi(x, Z, k)$ whose existence was proved above. The coefficients $\varphi_s(Z)$ of the corresponding wave operator Φ in (5.35) are global meromorphic functions with poles along Θ.

The left and right actions of the operators are formally conjugate. Thus, for any two pseudodifferential operators we have the equality $(e^{-kx}\mathscr{D}_1)(\mathscr{D}_2 e^{kx}) = e^{-kx}(\mathscr{D}_1\mathscr{D}_2 e^{kx}) + \partial_x(e^{-kx}(\mathscr{D}_3 e^{kx}))$. The coefficients of the operator $\mathscr{D}_3$ are differential polynomials in the coefficients of the operators $\mathscr{D}_1$ and $\mathscr{D}_2$. Thus, it follows from the equalities (5.38)–(5.42) that

$$\psi^+\psi = 1 + \sum_{s=2}^{\infty} F_{s-1} k^{-s} = 1 + \partial_x \biggl(\sum_{s=2}^{\infty} Q_s k^{-s} \biggr). \tag{5.58}$$

The coefficients Q are differential polynomials in the coefficients φ_s of the wave operator. Hence, they are global meromorphic functions with poles along Θ. This proves Lemma 5.9.

To complete the proof of the theorem, we use another fact of the KP theory: the flows of the KP hierarchy determine deformations of the commutative rings $\mathscr{A}$ of ordinary linear differential operators. For a fixed spectral curve Γ the orbits of flows of the KP hierarchy are isomorphic to the generalized Jacobian of the curve $J(\Gamma)=\operatorname{Pic}^0(\Gamma)$, which is by definition the group of equivalence classes of divisors of degree zero (for details, see [1], [2], [8], [63]).

In the Sato form, a KP hierarchy is a system of equations for a pseudodifferential operator $\mathscr{L}$,

$$\partial_{t_n}\mathscr{L}=[\mathscr{L}^n_+,\mathscr{L}]. \tag{5.59}$$

If $\mathscr{L}$ is defined by using a λ-periodic wave solution of the equation (1.41), then the equations (5.59) are equivalent to the equations

$$\partial_{t_n}u=\partial_x F_n. \tag{5.60}$$

The first two 'time' variables of the hierarchy are identified with the variables x and y: $t_1=x$ and $t_2=y$.

The equations (5.60) identify the space $\widehat{\mathbf{F}}_1$ spanned by the functions $\partial_1 F_n$ with the tangent space to the orbit of the KP hierarchy at the point $\mathscr{A}^Z$. By the assertion of the previous lemma, this space is a subspace of the tangent space X. Hence, for any $Z\notin\Sigma$ the deformations of the ring $\mathscr{A}^Z$ caused by the action of the flows of the KP hierarchy belong to X, that is, a holomorphic embedding is defined:

$$i_Z\colon J(\Gamma)\longmapsto X. \tag{5.61}$$

It follows from (5.61) that $J(\Gamma)$ is *compact*.

The generalized Jacobian of an algebraic curve is compact if and only if the curve is *smooth* [64]. Every torsion-free rank-one sheaf on a smooth algebraic curve is a vector bundle, that is, $\overline{\operatorname{Pic}}(\Gamma)=J(\Gamma)$. In this case it follows from (5.49) that i_Z is an isomorphism. We note that the singular locus Σ of the Jacobian of a smooth algebraic curve is empty [8], that is, the embedding j in (5.49) is everywhere defined and is inverse to i_Z. This proves Theorem 5.1 and thus completes the proof of Theorem 1.4.

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V. M. Buchstaber
Steklov Institute of Mathematics,
Russian Academy of Sciences;
School of Mathematics, University of Manchester
E-mail: `buchstab@mi.ras.ru`

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I. M. Krichever
Columbia University;
Landau Institute of Theoretical Physics
E-mail: `krichev@math.columbia.edu`