SELF-SIMILAR SOLUTIONS OF EQUATIONS OF KORTEWEG–DE VRIES TYPE

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The present paper was stimulated by the recent results [1] on the construction of self-similar solutions of the modified Korteweg–de Vries equations and the sin-gordon equations. Such solutions can be described by ordinary equations of Painleve type [2], for which a commutation representation is found [1]. The problem of construction of simultaneous eigenfunctions of auxiliary operators reduces to the Riemann problem. According to [1], the analogues of the dispersion data are the transition matrices at the jumps of this function on "anti-Stokes."

The principal distinction of the proposed scheme is the use of Lax pairs for the initial partial differential equations. This allows one to unite in one construction the construction of self-similar solutions and other well-known classes of exact solutions: multisoliton and rational. The general solutions obtained in the realms of our construction correspond to "solitons against the background of self-similar solutions." There is no difficulty in carrying over all the constructions to nontrivial algebraic curves (multisoliton solutions correspond in the algebrogeometric constructions to degenerations of these curves to rational ones). This will be done in the detailed publication of the results of this paper.

1. For definiteness we restrict ourselves to Lax' equations for operators with scalar coefficients. We note that here, as also in the construction of finite-zone solutions of rank $l > 1$ (cf. [3, 4]), the construction of simultaneous eigenfunctions of scalar differential operators uses the vectorial Riemann problem.

Let φ_s be the angle giving on the λ plane the s-th ray defined by the equation

$$\operatorname{Re}\left[\lambda^m\left(\exp\left(\frac{2\pi i}{n}k\right)-\exp\left(\frac{2\pi i}{n}l\right)\right)\right]=0, \quad 1\leqslant k,\ l\leqslant n. \tag{1}$$

Let us associate with each ray a constant matrix G_s such that $G_s^{kk} = 1$, and the element G_s^{kl} ($k \neq l$) can be nonzero only for those indices for which on the corresponding ray (1) holds and for which in a neighborhood of this ray the left side of (1) decreases upon counterclockwise rotation. We note that if $G_s^{kl} \neq 0$, then $G_s^{lk} = 0$. The indicated collection of rays is invariant with respect to rotation by $2\pi/n$. Let $\sigma(s)$ be the corresponding permutation of indices. We require that the relation

$$G_{\sigma(s)} = M^{-1}G_sM, \qquad M=\begin{pmatrix}0 & 0 & \dots & 0 & 1\\ 1 & 0 & \dots & 0 & 0\\ 0 & 1 & \dots & 0 & 0\\ \cdot & \cdot & \cdot & \cdot & \cdot\\ 0 & & \dots & 1 & 0\end{pmatrix} \tag{2}$$

be satisfied.

Let $\nu_{s'}$ and $\mu_{s'}$, $1 \leq s' \leq Nn$ be two collections of complex numbers, invariant with respect to rotation by $2\pi/n$, and $\tau(s')$ be the corresponding permutation of indices. To them we associate constant vectors $\alpha_{s'} = (\alpha_{s'1}, \dots, \alpha_{s'n})$ and $\beta_{s'} = (\beta_{s'1}, \dots, \beta_{s'n})$ such that $\alpha_{\tau(s')} = \alpha_{s'}M$, $\beta_{\tau(s')} = \beta_{s'}M$. The pairs $(\nu_{s'}, \alpha_{s'})$, $(\mu_{s'}, \beta_{s'})$ are called "Tyurin parameters" as in [3, 4].

To each collection of enumerated "data of the inverse problem" corresponds a unique vector-function $\psi(x, t, \lambda) = (\psi_1, \dots, \psi_n)$ such that

1°. The function ψ has the form $v(x, t, \lambda)\Psi_0$, where $v(x, t, \lambda)$ is a piecewise meromorphic vector-function on the extended λ plane, and the matrix $\Psi_0 = \exp(\lambda Qx + \lambda^m Q^m t)$, $Q^{kl} = \exp[(2\pi i/n)k]\delta_{kl}$.

2°. Outside ∞, ψ is piecewise meromorphic, has jumps on the rays $re^{i\varphi_s}$, $r > 0$, while

$$\psi^{+s}(r) = \psi^{-s}(r)G_s, \tag{3}$$

where $\psi^{\pm s}(r) = \psi(re^{i\varphi_s \pm i0})$.

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3°. The poles of ψ lie at the points $\nu_{s'}$, and for the residues of its components one can use the relation

$$\alpha_{s'l}\operatorname{res}_{\nu_{s'}}\psi_l = \alpha_{s'k}\operatorname{res}_{\nu_{s'}}\psi_k, \qquad 1 \leqslant k,\ l \leqslant n. \tag{4}$$

Moreover, let

$$\sum_{l=1}^{n} \beta_{s'l}\psi_l(x, t, \mu_{s'}) = 0. \tag{5}$$

4°.

$$\psi\left(x, t, \lambda \exp\frac{2\pi i}{n}\right) = \psi(x, t, \lambda) M. \tag{6}$$

LEMMA 1. There exists a unique vector-function ψ satisfying the enumerated requirements and normalized by the condition $v(x, t, \lambda) \to (1, \ldots, 1)$, $\lambda \to \infty$.

The problem of constructing ψ is equivalent with the Riemann problem on constructing $v(x, t, \lambda)$. Condition 2° is transformed here into

$$v^{+s}(r) = v^{-s}(r)\widetilde{G}_s(r), \tag{3'}$$

where $\widetilde{G}_s(r) = \Psi_0(re^{i\varphi_s})G_s\Psi_0^{-1}(re^{i\varphi_s})$. The restrictions on the form of G_s are sufficient for the existence of a solution of the Riemann problem. In fact, we consider the piecewise constant function G, equal to G_s in the sectors $\varphi_s \leq \varphi \leq \varphi_s + \delta\varphi$, 1 outside them. Then $\psi_1 = \psi G^{-1}$ satisfies all the requirements imposed on ψ, except for one: it has jumps on the rays $\varphi_s + \delta\varphi$. For small $\delta\varphi > 0$, it follows from the restrictions on the form G_s that the jump function $G_s(r)$ for $\varphi_s + \delta\varphi$ decreases exponentially as $r \to \infty$. The number of linearly independent solutions of the problem (3), (4), (5) is equal to n. Condition (6) singles out a vector-function ψ which is unique up to proportionality.

The operators ∂_x, ∂_t and the operator of multiplication by λ^n transform ψ into a function subject to the same restrictions as ψ, but $v(x, t, \lambda)$, which figures in condition 1°, can have a pole at $\lambda = \infty$. In the standard way, comparing singular parts one gets

THEOREM 1. There exist unique operators

$$L_n = \sum_{i=0}^{n} u_i(x, t)\partial_x^i, \qquad L_m = \sum_{i=0}^{m} v_i(x, t)\partial_x^i$$

with scalar coefficients, such that

$$L_n\psi = \lambda^n\psi, \qquad (\partial_t - L_m)\psi = 0.$$

COROLLARY 1. The operators L_n and L_m satisfy the equation

$$[L_n, \partial_t - L_m] = 0.$$

COROLLARY 2. Suppose in the given inverse problem the Tyurin parameters are absent, i.e., N = 0. Then the construction indicated leads to a self-similar solution of Lax' equation.

The assertion of the corollary follows from the fact that for N = 0 the data are invariant with respect to the scale group, which means $\psi(x, t, \lambda) = \psi(\beta x, \beta^m t, \beta^{-1}\lambda)$. Simple calculation shows that the number of parameters is equal to $m(n-1)$, i.e., for mutually prime n and m it is equal to the dimension of the space of self-similar solutions.

If all $G_s = 1$, then the construction indicated gives a soliton solution.

Remark. Change of (5) to a system of conditions analogous to [5] includes in this scheme also the construction of rational solutions.

II. Two-Dimensional Systems of KdeV Type

Let $\psi(x, y, t, \lambda)$ be defined by 1°–4° with the one change in 1°:

$$\Psi_0(x, y, t, \lambda) = \exp(\lambda Qx + \lambda^{m_1}Q^{m_1}y + \lambda^m Q^m t).$$

Without loss of generality one can assume that $m_1 < m$. Here $\psi(x, y, t, \lambda)$ exists and is unique.

THEOREM 2. There also exist unique scalar operators L_m and L_{m_1}, whose coefficients depend on x, y, t, such that

$$(\partial_y - L_{m_1})\psi = (\partial_t - L_m)\psi = 0.$$

COROLLARY 1. $[\partial_y - L_{m_1}, \partial_t - L_m] = 0$.

COROLLARY 2. For $N = 0$, the solutions of the equations obtained are self-similar.

For $m_1 = 2$, $m = 3$ we get a self-similar solution of the Kadomtsev–Petviashvili equation.

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ASYMPTOTICS CLOSE TO THE BOUNDARY OF A SPECTRAL FUNCTION OF AN ELLIPTIC SECOND-ORDER DIFFERENTIAL OPERATOR

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1. Let A be a positive-definite elliptic differential operator in $L_2(M)$; M be a bounded domain in $\mathbf{R}^m$, $m \geq 2$. The operator A is defined by a differential expression of the form

$$Au = -\sum_{i,j=1}^{m} a_0^{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{m} a_1^i(x)\frac{\partial u}{\partial x_i} + a_2(x)u \tag{1}$$

on functions satisfying some self-adjoint boundary condition (Dirichlet or of the third kind). It will be assumed that $a_0^{ij}(x) = a_0^{ji}(x), a_1^i(x), a_2(x) \in C^\infty(\overline{M})$, the boundary ∂M is infinitely differentiable, and the operator A is uniformly elliptic, i.e., $\sum_{i,j=1}^{m} a_0^{ij}(x)\xi_i\xi_j \geqslant C|\xi|^2,\ C > 0,\ x \in \overline{M}$.

2. The fundamental result of the present note is the

THEOREM. Let A be a differential operator of the form (1), $e(x, y, \lambda)$ be the spectral function of the operator A. Then for any $\varepsilon > 0$ there exist constants $C_\varepsilon^{1,2}$ such that

$$\left| e(x, y, \lambda) - (2\pi)^{-m/2}\rho(x)\left\{\left(\frac{\sqrt{\lambda}}{\tau}\right)^{m/2} J_{m/2}(\sqrt{\lambda}\tau) \pm \left(\frac{\sqrt{\lambda}}{\tau_{\text{neq}}}\right)^{m/2} J_{m/2}(\sqrt{\lambda}\tau_{\text{neg}})\right\}\right| \leqslant C_\varepsilon^1 \lambda^{(m-(3/4)+\varepsilon)/2} \tag{2}$$

for $\tau = \tau(x, y) \leqslant C_\varepsilon^2 \lambda^{-(1/4)-\varepsilon}$, $\lambda \to \infty$.

We explain the use of the notation. $\tau = \tau(x, y)$ is the geodesic distance between x and y, $\tau_{\text{neg}} = \tau_{\text{neg}}(x, y)$ is the "negative eikonal" (cf. [4] for more detail). For $n(x) \geq C\lambda^{-1/4}$, τ_{neg} is defined as the geodesic length of a minimal two-component curve joining x and y and having vertex on ∂M, for $n(x) \leq C\lambda^{-1/4}$, $\tau_{\text{neg}} = \{s^2(x, y) + (n(x) + n(y))^2\}^{1/2}$, where $s(x, y)$ is the geodesic distance along ∂M between the projections on ∂M of x and y. Here $n(x)$ [respectively, $n(y)$] is the geodesic distance in the metric generated by A, from x to ∂M (respectively, from y to ∂M). $\rho(x) = \det^{1/2}\| a_{ij}^0 \|$, where $\| a_{ij}^0(x) \| = \| a_0^{ij}(x) \|^{-1}$. Finally, $J_{m/2}(z)$ is the Bessel function of index $m/2$. The sign + in (2) corresponds to a boundary condition of the third kind, the sign − to a Dirichlet boundary condition.

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