Let $\mathfrak{M}(r, d, \varepsilon)$, where $r > 0$, $d > 0$, $0 < \varepsilon < \pi$, denote the set of all convex open bounded polygons $M\ (\subset R^2)$ such that:

1) the vertices of M belong to Z^2,

2) the coordinate origin 0 belongs to M,

3) the distance of 0 from the boundary of M is greater than r,

4) the length of each side of M is greater than d,

5) the size of each angle of M is greater than $\pi - \varepsilon$.

THEOREM. Let $A\ (\in \sigma)$ be an invertible operator. Then $\exists r\ (>0)\ \exists d\ (>0)\ \exists \varepsilon (\in (0, \pi))\ \exists C\ (>0)\ \forall M\ (\in \mathfrak{M}(r, d, \varepsilon))$, the operator $P_{M+\varkappa} A P_M$ is invertible as an operator in $\mathrm{Hom}\,(P_M L_2(T), P_{M+\varkappa} L_2(T))$, and

$$|(P_{M+\varkappa} A P_M)^{-1}| \leqslant C,$$

where $\varkappa = \varkappa(A)$, $M + \varkappa$ is the set of all points of the plane of the form $m + \varkappa$, where $m \in M$.

COROLLARY. When the hypotheses of the theorem hold, it is always possible to choose a sequence of polygons $M_n \in \mathfrak{M}(r, d, \varepsilon)$ such that the sequences P_{M_n}, $P_{M_n+\varkappa}$ of projections converge strongly to the identity operator, and hence the projection method using the system of projections $\{P_{M_n}, P_{M_n+\varkappa}\}$ is applicable to the operator A.

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RATIONAL SOLUTIONS OF THE KADOMTSEV – PETVIASHVILI EQUATION AND INTEGRABLE SYSTEMS OF N PARTICLES ON A LINE

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Recently, interesting rational solutions have been found for the Korteweg–de Vries (KdV), Kadomtsev–Petviashvili (KP), Bürgers, and certain other equations (cf. [1–4]). For the KdV equation it was shown in [1] that the set of all rational solutions is a degenerate family of finite-zone solutions obtained by deformation of the Lamé potentials $n(n + 1)\,\wp(x)$ with respect to t using the KdV equation and its higher analogs. The first solution of this type was obtained in [5]. In [1], the connection was first remarked between the dynamics of the poles of these solutions and the motion of a Hamiltonian system of particles on a line.

In this note we set forth an algorithm for constructing certain rational solutions of the Zakharov–Shabat equations. In the particular case of the Kadomtsev–Petviashvili equation, a more complete description of all the rational solutions is given. The method developed allows us to completely identify the motion of the poles of the rational solutions of the KP equation with the Hamiltonian flows arising in Moser's theory [6] of a system of N particles on a line with Hamiltonian $H = \frac{1}{2}\sum_{j=1}^{N} p_j^2 + \sum_{i<j} 2\,(x_j - x_i)^{-2}$.

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1. Assume given a set of polynomials $Q(k)=\sum_{i=0}^{n} c_i k^i$, $R(k)=\sum_{i=0}^{m} r_i k^i$, $q_1(k)=\sum_{i=0}^{N} h_i k^i$, $h_N=1$, complex numbers $\varkappa_s$, and rectangular matrices $A_s = a_{ij}^s$, $1 \le i \le l_s$ of rank l_s, $\sum_s l_s = N$.

There exists a unique function $\psi(x, y, t, k)$ of the form

$$\psi(x, y, t, k) = \left(1 + \frac{q(x, y, t, k)}{q_1(k)}\right) e^{kx+Q(k)y+R(k)t}$$

such that the coefficients of its expansion at the points $\varkappa_s$ of the form

$$\psi(x, y, t, k) = \sum_{j=0}^{\infty} \zeta_{j,s}(x, y, t)(k-\varkappa_s)^j$$

satisfy the system of equations $\sum_j a_{ij}^s \zeta_{j,s}(x, y, t) = 0$. The coefficients of the polynomial $q(x, y, t, k) = \sum_{i=0}^{N-1} b_i(x, y, t) k^i$ moreover turn out to be rational functions of the arguments.

THEOREM 1. There exist unique operators $L_1 = \sum_{i=0}^{n} u_i(x, y, t)\frac{d^i}{dx^i}$, $L_2 = \sum_{i=0}^{m} v_i(x, y, t)\frac{d^i}{dx^i}$ such that $\left(L_1 - \frac{\partial}{\partial y}\right)\psi = \left(L_2 - \frac{\partial}{\partial t}\right)\psi = 0$. The coefficients of these operators are differential polynomials in the functions $b_i(x, y, t)$ and hence are rational functions of their arguments.

COROLLARY. For the operators constructed above, the equality $\left[L_1 - \frac{\partial}{\partial y}, L_2 - \frac{\partial}{\partial t}\right] = 0$ holds.

The systems of equations for the coefficients of the operators L_1 and L_2 which are equivalent to the last equality are called the Zakharov–Shabat equations.

2. Let $Q(k) = 0$, $R(k) = k^2$, $l_s = 1$. The function $\psi(x, t, k)$ defined by these data and the polynomial $q_1(k)$ satisfies the nonstationary Schrödinger equation

$$\left(\frac{\partial}{\partial t} - \frac{d^2}{dx^2} - u(x, t)\right)\psi(x, t, k) = 0. \quad (1)$$

The potential $u(x, t)$ is equal to $-2\sum_{j=1}^{N}(x - x_j(t))^{-2}$. The function $\psi(x, t, k)$ can be written in the form

$$\psi(x, t, k) = \left(1 + \sum_{j=1}^{N} a_j(t, k)(x - x_j(t))^{-1}\right) e^{kx+k^2t}, \quad (2)$$

where the $a_j(t, k)$ depend rationally on the variable k.

THEOREM 2. Equation (1) has a solution of the form (2) if and only if the matrices Λ and T with matrix elements $\Lambda_{jk} = \dot{x}_j\delta_{jk} + \frac{2(1-\delta_{jk})}{x_j - x_k}$, $T_{jk} = \frac{2(1-\delta_{jk})}{(x_j - x_k)^2} - \sum_{s\ne j}\frac{2\delta_{jk}}{(x_j - x_s)^2}$ satisfy the equation $\left[\frac{\partial}{\partial t} - T, \Lambda\right] = 0$.

This matrix representation for the equations of motion of the Hamiltonian system of N particles on a line with Hamiltonian $H = \frac{1}{2}\sum_{j=1}^{N} p_j^2 + \sum_{i<j} 2(x_i - x_j)^{-2}$ was found in [6] and used to prove the complete integrability of the system.

There are thus 2N parameters: the coefficients of the polynomial $q_1(k)$ and the points $\varkappa_s$, $1 \le s \le N$ determining the function $\psi(x, t, k)$, which give the generic solution of the equations of motion for a Moser system of particles. This result allows us to find explicit formulas for the solutions of the last equations. Formulas of this type were first obtained in [7]. The method developed makes it easy to obtain analogous formulas for all the higher Hamiltonian flows (cf. the formulas of the following paragraph).

Introduce the matrix Θ with matrix elements

$$\Theta_{si} = (x + 2\varkappa_s t)\varkappa_s^i + i\varkappa_s^{i-1} - \varkappa_s^i\left(\sum_{j=1}^{N} j h_j \varkappa_s^{j-1}\right)\left(\sum_{j=0}^{N} h_j \varkappa_s^j\right)^{-1}.$$

COROLLARY. If $x_j(t)$ are the coordinates of the particles of a Moser system, the equality $\prod_{j=1}^{N}(x-x_j(t)) = \det\Theta$ holds.

3. The Zakharov–Shabat equations in the case of the operators $L_1 = d^2/dx^2 + u(x, y, t)$, $L_2 = d^3/dx^3 + {}^3/_2 u\,d/dx + w(x, y, t)$ imply that the function $u(x, y, t)$ satisfies the KP equation

$$\frac{3}{4}\frac{\partial^2 u}{\partial y^2} + \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial t} + \frac{1}{4}\left(6u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3}\right)\right) = 0.$$

THEOREM 3. The function $u(x, y, t)$ is a solution of KP equations depending rationally on the variable x and decreasing to zero as $x \to \infty$, if and only if $u(x, y, t) = -2\sum_{j=1}^{N}(x - x_j(y, t))^{-2}$, and there exists a function

$$\psi(x, y, t, k) = \left(1 + \sum_{j=1}^{N} a_j(y, t, k)(x - x_j(y, t))^{-1}\right) e^{kx+k^2y+k^3t},$$

satisfying the equation $\left(\frac{\partial}{\partial y} - L_1\right)\psi = \left(\frac{\partial}{\partial t} - L_2\right)\psi = 0$.

COROLLARY 1. The dynamics of the poles of the rational solutions of the KP equations with respect to the variable y coincides with the motion of the Moser system, and is given with respect to the variable t by the higher Hamiltonian $F_3 = {}^1/_3 \operatorname{Sp}\Lambda^3$ or, what is the same, by the matrix equation $\left[\frac{\partial}{\partial t} - T_2, \Lambda\right] = 0$, where $T_2 = \tilde{T} - {}^3/_2\Lambda T$, $\tilde{T}_{jk} = \sum_{s\neq j}\frac{3\delta_{jk}}{(x_j - x_s)^3} - \frac{3(1-\delta_{jk})}{(x_j - x_k)^3}$.

COROLLARY 2. The general rational solutions of the KP equation which decay as $x \to \infty$ are given by the equation

$$u(x, y, t) = 2\frac{d^2}{dx^2}\ln\det\tilde{\Theta}, \tag{3}$$

where $\tilde{\Theta}_{si}(x, y, t) = \Theta_{si}(x, y) + 3\varkappa_s^{2+i}t$.

The solutions which are nonsingular for all real x and y were found in [4]. We remark that the decay condition in the variable y is a consequence of the decay condition in the variable x.

COROLLARY 3. The dynamics of the poles of the rational solutions of the KdV equation, i.e., the rational solutions of the KP equation not depending on the variable y, coincides with the Hamiltonian flow determined by F_3 restricted to the fixed points of the initial Hamiltonian flow. The number of these poles is equal to $n(n + 1)/2$, and the solutions themselves are given by Eq. (3) if we put

$$\tilde{\Theta}_{si} = \frac{\partial^{3n-2i-s}}{\partial x^{3n-2i-s}}\frac{\partial^{2n+1}}{\partial k^{2n+1}}\frac{e^{kx+k^3t}}{\varphi_1(k)}\bigg|_{k=0}.$$

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