**Topology, fall 2022.**

**NAME:**

**Topology (fall 2022). Midterm exam I, October 6**

You can solve problems in any order. Justify your answers.

I. (15 points) Let $A \subset X$ be a subspace of the topological space $X$.

(a) State the definition of the closure $\overline{A}$ of $A$ in $X$.

(b) Prove that $\overline{A \cup B} = \overline{A} \cup \overline{B}$ for any subspaces $A, B \subset X$.

II. (30 points) (a) Let $\mathcal{T}$ be the topology on $X = \{a, b, c, d\}$ with the subbasic $\mathcal{S} = \{\{a, b\}, \{b, c\}, \{b, c, d\}\}$. Give an example of a basis $\mathcal{B}$ for this topology. (It may be useful to draw a picture of elements of $X$ and various open sets you can build from the subbasis.)

(b) Compute the closure $\overline{A}$ of the subspace $A = \{a, c\}$ of $X$ in this topology.

(c) Consider the map $f : X \longrightarrow Y$ to the two-element discrete topological space $Y = \{0, 1\}$ given by $f(a) = f(d) = 0$, $f(b) = f(c) = 1$. Is $f$ continuous?

(d) Let $A = \{a, c\} \subset X$. Determine the subspace topology on $A$. Is $A$ discrete? Is $A$ connected?

(e) Consider the infinite sequence $b, b, b, \ldots$ Find all limit points of this sequence in $X$.

III. (20 points) Which of the following maps are continuous? Briefly justify your answer.

1) The map $f : \mathbb{R}_\ell \longrightarrow \mathbb{R}$ from $\mathbb{R}$ with the lower limit topology to $\mathbb{R}$ with the standard topology given by $f(x) = 3x$.

2) The identity map from $X$ in problem II above to $X$ with the indiscrete topology.

3) The map from the Cantor set $C = \prod_{\mathbb{N}}\{0, 1\}$ to itself taking a sequence $b_1 \times b_2 \times \dots$ to $b_2 \times b_3 \times \dots$ (the map which forgets the first coordinate).

4) The map from the Cantor set $C = \prod_{\mathbb{N}}\{0, 1\}$ to itself taking a sequence $b_1 \times b_2 \times b_3 \dots$ to $b_1 \times b_1 \times b_2 \times b_2 \times b_3 \times b_3 \dots$ (repeat each entry twice).

IV. (15 points) Mark the square in the *first* column, respectively **second** column, if the corresponding subset of $\mathbb{R}^2$ with the standard topology is *open*, respectively **connected**.

☐ ☐ $\{(x, y)|x \in \mathbb{Q}, y \in \mathbb{R}.\}$
☐ ☐ $\{(x, y)| -1 < x < 1 \text{ and } 1 < y < 3\}$
☐ ☐ $\{(x, y)|x = 1 \text{ or } y = 1\}$
☐ ☐ $\{(x, y)|xy > 1\}$
☐ ☐ $\{(x, y)|x \neq y\}$

V. (20 points) (a) Given topological spaces $X, Y$, state the definition of the product topology on $X \times Y$.

(b) Prove that $X \times Y$ is Hausdorff if both $X, Y$ are Hausdorff.

VI. *(Optional problem, extra credit, 10 points).* Suppose that $X_i$, $i \in \mathbb{N}$ are discrete topological spaces. Prove that the product

$$X = X_1 \times X_2 \times X_3 \times ...$$

with the box topology is discrete.