Correction from last lecture.

$\mathbb{R}^{\mathbb{N}}$ or $\prod_{i\in\mathbb{N}} X_i$, $\prod_{\alpha\in J} X_\alpha$ metric spaces. To define uniform metric, set

$D(x,y) = \sup_{i\in\mathbb{N}} (\bar{d}_i(x_i,y_i))$ $\qquad \bar{d}_i(x_i,y_i) = \min(d_i(x_i,y_i), 1)$

first cut to 1, then take sup.

In class we used $\tilde{D}(x,y) = \sup(d_i(x_i,y_i))$ which may give ∞.

Same balls $B_D(x,\epsilon)$ and $B_{\tilde{D}}(x,\epsilon)$, $\epsilon < 1$ $\Rightarrow$ same topology

(same basis $\{B_D(x,\epsilon) \mid \epsilon<1,\ x\in\prod X_i\}$)

Include Thm 21.1 (lect 7, page 2)

$\mathbb{R}$ have addition, multiplication, $x \mapsto -x$, division

Prop 1) $+, \cdot$ are continuous maps $\mathbb{R}\times\mathbb{R} \to \mathbb{R}$

2) $x \mapsto -x$ is a continuous map $\mathbb{R} \to \mathbb{R}$

3) x/y is a continuous map $\mathbb{R}\times\mathbb{R}^* \to \mathbb{R}$ $\qquad \mathbb{R}^* = \mathbb{R}\setminus\{0\}$

Pf Exercise

Corollary 1) if $f,g: X \to \mathbb{R}$ are continuous, then $f+g$, $f-g$, fg are continuous $X \to \mathbb{R}$

2) if $g(x) \neq 0 \ \forall x \Rightarrow f/g$ is continuous

if $g(x)=0$ for some x, let $Z \subset X$, $Z = \{x \mid g(x)=0\}$

$X\setminus Z \xrightarrow{g} \mathbb{R}^*$. Then f/g is continuous on $X\setminus Z$.

For now, skip uniform continuity (end of §21) and §22 (Quotient topology)

§ 23 Connectedness

Def A separation of a top. space X is a pair U, V:

- $U \cap V = \emptyset$ (disjoint), $U \cup V = X$ (all of X)
- U, V are both open. (or both closed)
- $U, V \neq \emptyset$

$\iff U \subset X$, $U \neq \emptyset, X$, U is clopen (both closed and open).

Equivalently, $X = U \cup V$ disjoint union, $\emptyset \neq U$, $\emptyset \neq V$ and neither of U, V contains a limit point of the other. $\bar{U} = U$, $\bar{V} = V$

X connected if it has no separation

Cantor set - lots of separations

connected, indiscrete top space is connected, $[a,b]$, $\mathbb{R}$ is connected.

 connected, X has an isolated point $\Rightarrow$ not connected

$\mathbb{R} \setminus \{a\}$, $\mathbb{Q}$ not connected.

$(0,1) \cup [2,3)$ is not connected,

$[-1,1]$: $[-1,0)$ and $[0,1]$ is not a separation

$X = \{x \times y \mid y = 0\} \cup \{x \times y \mid x > 0 \text{ and } y = 1/x\}$

not connected. neither contains a limit pt. of the other.

take $y = \frac{1}{2x}$

Prop (23.2) If $C \cup D$ is a separation of X, $Y \subset X$ is connected $\Rightarrow$

$Y \subset C$ or $Y \subset D$.

Pf: separation of X induces a separation of Y if Y contains points of both subspaces.

Thm (23.3) if $A_\alpha \subset X$ is connected $\forall \alpha \in J$ and $\bigcap_{\alpha \in J} A_\alpha \neq \emptyset \Rightarrow$

$Y = \bigcup_{\alpha \in J} A_\alpha$ is connected.

Proof let $p \in \bigcap_{\alpha \in J} A_\alpha$. Suppose $C \cup D$ is a separation of Y. let $p \in C$.

A_α -connected $\Rightarrow A_\alpha \subset C$ or $A_\alpha \subset D \Rightarrow A_\alpha \subset C \Rightarrow Y \subset C$. □

(23.4)

Thm let $A \subset X$ be connected. If $A \subset B \subset \bar{A}$ then B is connected.
(including the case $B = \bar{A}$).

closure in X

Proof If $B = C \cup D$ separation $\Rightarrow A \subset C$ (for instance). $\Rightarrow \bar{A} \subset \bar{C}$

but $C \subset B \subset A \Rightarrow \bar{C} \in \bar{A} \Rightarrow \bar{C} = \bar{A}$

$\bar{C}$ and D are disjoint in B □.

Thm (23.5) if $f: X \to Y$ continuous, X-connected $\Rightarrow f(X)$ connected.

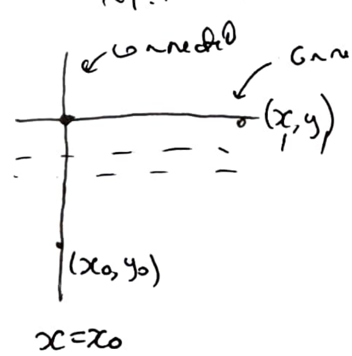
Proof if $C \cup D = f(X)$ separation of $f(X)$ then $f^{-1}(C) \cup f^{-1}(D)$ is a separation of X. □

Thm (23.6) X, Y connected $\Rightarrow X \times Y$ connected

$X_1 \ldots X_n$ connected $\Rightarrow \prod_{i=1}^{n} X_i$ connected.

Proof

$X \times Y$



$A_{y_1} = \{x = x_0\} \cup \{y = y_1\}$ is connected

$\bigcup_{y_1 \in Y} A_{y_1}$ connected.

L - simply ordered, $|L| > 1$.

Def L is a linear continuum if

(1) L has the least upper bound property → any nonempty subset bounded from above has the least upper bound

(2) if $x < y$ $\exists z$ s.t. $x < z < y$.

Examples
$\mathbb{R}$, $[a,b]$, $[a,b)$, $(a,b]$

$\mathbb{Q}$ is not a linear continuum. take set $\{a \in \mathbb{Q} \mid a < \sqrt{2}\}$

Thm If L is a linear continuum in the order topology, then L is connected and so are intervals & rays in L.

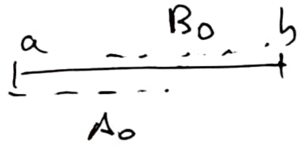
Proof $Y \subset L$ is convex if $\forall a,b \in Y, a < b \Rightarrow [a,b] \subset Y$. Prove that any convex subset is connected.

if $Y = A \cup B$ ← disjoint, nonempty, open in Y. (separation)

$a \in A, b \in B$. Let $a < b$. $[a,b] \subset Y$

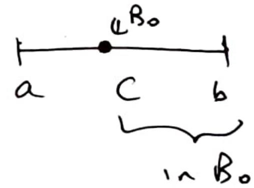
Let $A_0 = A \cap [a,b]$, $B_0 = B \cap [a,b]$ $\quad$ A_0, B_0 open in $[a,b]$ in subspace top

$\Rightarrow A_0, B_0$ is a separation of $[a,b]$



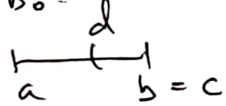
let $c = \sup A_0$.

Case 1: If $c \in B_0 \Rightarrow c \neq a$, $c = b$ or $a < c < b$

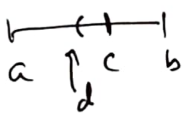


$\Rightarrow \exists$ interval $(d,c] \subset B_0$ since B_0 is open

if $c = b$ contradiction (d is a smaller u.b. than c)

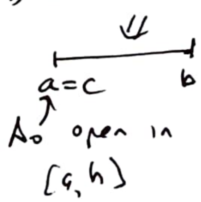


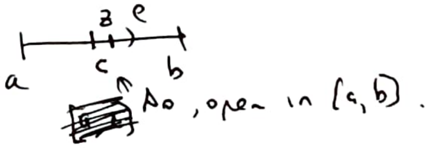
if $c < b$



$(d,b] \cap A_0 = \emptyset \Rightarrow d$ smaller u.b. than c

Case 2 if $c \in A_0$, $c \neq b \Rightarrow c = a$ or $a < c < b$





Corollary $\mathbb{R}$, intervals/rays in $\mathbb{R}$ are connected

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Example $\mathbb{R}^{\mathbb{N}}$, product top $\Rightarrow$ connected $\quad \prod_{\alpha \in J} X_\alpha$, X_α - each connected $\Rightarrow$ connected

$\mathbb{R}^{\mathbb{N}}$, box topology $\Rightarrow \exists$ a separation.