Continue with § 19-20.

Box topology on $\prod_{i\in\mathbb{N}} X_i$. Basis $\mathcal{B}$ $\prod_{i\in\mathbb{N}} U_i$, $U_i \in X_i$ open.

Exercise this is a basis of a topology → "much" finer than product top.

if each X_i is a metric space, d_i metric, also have uniform topology –
$D(x,y) = \sup_i (d_i(x_i,y_i))$ topology for the metric D.
↪ better take $\sup(\bar{d}_i(x_i,y_i))$ $\bar{d}_i = \min(d_i, 1)$ to avoid possible $D(x,y)=\infty$

$\mathbb{R}^{\mathbb{N}}$ points are $x = (x_1, x_2 \ldots)$ $x_i \in \mathbb{R}$

Box topology – basis neigh. of $0 = (0 \ldots 0)$ are $(-\epsilon_1,\epsilon_1)\times(-\epsilon_2,\epsilon_2)\times\ldots$ can be made small

Product top – large basis neigh

Uniform topology
$$B(x,\epsilon) = \{(y_1, y_2 \ldots) \mid |y_i - x_i| < \epsilon \ \forall i\}$$

Prop On $\mathbb{R}^{\mathbb{N}}$, Box top. is strictly finer than uniform top, uniform top. is strictly finer than product top.

Proof: need to look at nestings of basis sets for various topologies

For box top, pick neighbourhood $\prod_{n\geq 0}(-\frac{1}{n},\frac{1}{n})$ of 0. Cannot fit any basis neighbourhood of 0 from the other two topologies there.

For product top, can use $U = \prod_{i=1}^{n}(-\epsilon,\epsilon)\times\mathbb{R}^{\mathbb{N}}$ around 0.

for uniform top, $U = \{(y_1 \ldots) \mid |y_i - x_i| < \epsilon \ \forall i\}$

box s. finer than uniform s. finer than product D.

Munkres writes $\mathbb{R}^{\omega}$ instead of $\mathbb{R}^{\mathbb{N}}$.

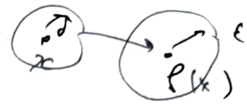
Example $\mathbb{R} \xrightarrow{f} \mathbb{R}^{\mathbb{N}}$ $f(t) = (t, 2t, 3t \ldots)$ continuous in prod. top.
not continuous in uniform topology
not continuous in box topology

$x_n = (\frac{1}{n}, \frac{1}{n}, \ldots \frac{1}{n} \ldots)$
$x_n \to (0, \ldots 0 \ldots) = 0$ in product top
converges to 0 in uniform topology
does not converge in box topology → use $\prod_{n\geq 1}(-\frac{1}{n^2},\frac{1}{n^2})$

Thm (21.1) Let $f: X \to Y$ and X, Y be metrizable, via d_X, d_Y.
Then f continuous $\iff$ for $x \in X, \varepsilon > 0$ $\exists \delta$ s.t.

$$d_X(x,y) < \delta \Rightarrow d_Y(f(x), f(y)) < \varepsilon.$$

Pf: fix x, ε, consider $f^{-1}(B(f(x), \varepsilon))$. Open, contains δ-ball $B(x, \delta)$.



Converse: see Munkres. □

lemma (21.2) (sequence lemma) Let X - top space, $A \subset X$.
If $\exists$ a sequence of points of A converging to x, then $x \in \bar{A}$.
The converse holds if X is metrizable (more generally, if X is first-countable)

Proof: $\Rightarrow$ Suppose $x_n \to x$, $x_n \in A$. Then $\forall U \ni x$ neighbourhood contains a point of A, so $x \in \bar{A}$. $\Leftarrow$ Assume X metrizable, $x \in \bar{A}$. d - metric, $\mathcal{T} = \mathcal{T}_d$
$\forall n > 0$ take $B_d(x, \frac{1}{n})$ choose $x_n \in A \cap B_d(x, \frac{1}{n})$.
Then $x_n \to x$. $\forall U \ni x$ open, $\exists n$ $B_d(x, \frac{1}{n}) \subset U$. for $N > \frac{1}{n}$ $\forall i \geq N$ $x_i \in U$. □

X has a countable basis at pt x if $\exists$ $\{U_n\}_{n \in \mathbb{N}}$ of neighbourhoods of x.
s.t $\forall U \ni x$ open, $U \supset U_n$ some n.

X is first countable if it has a countable basis at each pt.

Prop Any metric space is first countable. □

lemma 21.2 holds for first-countable spaces.

Example $\mathbb{R}^{\mathbb{N}}$ with box topology is not first-countable.
Consider $A = \{x \mid x_i > 0 \ \forall i\}$ $\bar{A} \ni \underline{0}$. Assume $\exists$ $x_n \in A$ $\xrightarrow[n \to \infty]{} 0$

$A \ni x_1 = (x_{11}, x_{12}, \ldots)$
$A \ni x_2 = (x_{21}, x_{22}, \ldots)$

Pick $\varepsilon_i < x_{ii}$ $\underline{\varepsilon} = (\varepsilon_1, \varepsilon_2 \ldots)$
Then $U = \prod_i (-\varepsilon_i, \varepsilon_i)$, U does not contain any x_i,

$\mathbb{R}$: have addition, multiplication, $x \mapsto -x$, division

Prop 1) $+, -, \cdot$ are continuous maps $\mathbb{R}\times\mathbb{R} \to \mathbb{R}$
2) division is a continuous map $\mathbb{R}\times(\mathbb{R}\setminus\{0\}) \to \mathbb{R}$

Pf: Exercise.

Corollary: if $X \xrightarrow{f,g} \mathbb{R}$ are continuous, then $f+g$, fg, $f-g$ are continuous

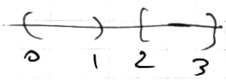
if $g(x)\neq 0\ \forall x \Rightarrow f/g$ is continuous

if $g(x)=0$ for some x, let $Z\subset X$, $Z=\{x \mid g(x)=0\}$ $\quad X \xrightarrow{(f,g)} \mathbb{R}\times\mathbb{R} \longrightarrow \mathbb{R}$

$X\setminus Z \xrightarrow{g} \mathbb{R}^*$. Now f/g is continuous on ~~$X\setminus P$~~ $X\setminus Z$.

A separation of X: pair of $\underbrace{\text{open } U, V}_{\text{nonempty}} \subset X$, $U\cap V=\emptyset$, $U\cup V = X$.

X is connected if there is no separation

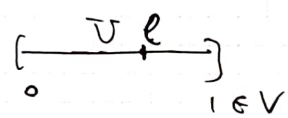


$(0,1)\cup[2,3]$ not connected

Prop $[a,b]$, $\mathbb{R}$, $[a,b)$ are connected.

if $[0,1]$ not connected, $[0,1] = U \cup V$ (clopen). $\exists$ U - bounded. take $1\in V$ the least upper bound of U ($\exists$ by properties of real line.)
$l\in[0,1]$

$l\in \bar U$ or $l\in \bar V$. if $l\in \bar U$, $\bar U$ is not open
if $l\in \bar V$, $\bar V$ is not open



if $X = Y\cup Z$ (both connected), $Y\cap Z\neq\emptyset \Rightarrow X$ connected.

$\mathbb{Q}$, $\mathbb{C}$, not connected

Get numerical invariant of some top. spaces
(# of connected components)