SRS
SwissMAP Research Station in Les Diablerets

lecture 6 Topology fall 2022

Last time: defined metric spaces (X, d) ← distance f-n

$$B_d(x,\epsilon) = \{y \mid d(x,y) < \epsilon\}$$

Basis for metric topology $\mathcal{B}_d = \{ B_d(x,\epsilon) \mid x \in X, \epsilon > 0 \}$. $\mathcal{T}_d$

A topological space $(X,\mathcal{T})$ is <u>metrizable</u> if $\exists$ a metric d on X, s.t. $\mathcal{T} = \mathcal{T}_d$

<u>Prop</u> X is metrizable $\Rightarrow$ X is Hausdorff.

<u>Def</u> (X,d) metric space, $A \subset X$ is called <u>bounded</u> if $\exists M > 0$ s.t. $d(a_1,a_2) \leq M$ $\forall a_1, a_2 \in A$. If A bounded, $A \neq \emptyset$ the diameter of A is

$$\operatorname{diam}(A) = \sup(d(a_1,a_2) \mid a_1,a_2 \in A).$$

<u>Boundedness</u> is not a topological property, may depend on choice of metric.

$(0,1)$ vs $\mathbb{R}$ (homeomorphic, 2 metrics).

A way to make a metric bounded:

Thm (20.1) Let (X,d) be a metric space. Define $\bar{d}: X \times X \to \mathbb{R}$ via

$$\bar{d}(x,y) = \min(d(x,y), 1).$$

Then $\bar{d}$ is a metric that induces the same topology as d. (standard bounded metric for d)

<u>Proof</u>: Conditions (1), (2) hold. For (3) need to check

$\bar{d}(x,z) \leq \bar{d}(x,y) + \bar{d}(y,z)$.

Metric:

(1) $d(x,y) \geq 0$, $d(x,y) = 0 \Leftrightarrow x = y$

(2) $d(x,y) = d(y,x)$

(3) $d(x,z) \leq d(x,y) + d(y,z)$

If $d(x,y) \geq 1$ or $d(y,z) \geq 1 \Rightarrow$ RHS ≥ 1 while LHS ≤ 1, holds.

Otherwise, $d(x,y), d(y,z) < 1$.

$\bar{d}(x,z) \leq d(x,z) \leq d(x,y) + d(y,z) = \bar{d}(x,y) + \bar{d}(y,z)$.

$\Rightarrow \bar{d}$ is a metric. For bases in both topologies can take open balls of radii $\epsilon < 1$

$B_d(x,\epsilon) = B_{\bar{d}}(x,\epsilon)$ for $\epsilon < 1$. $\Rightarrow$ Same topology. $\square$

Example $(\mathbb{R}, d) \to (\mathbb{R}, \bar{d})$ weird.

Distances in $\mathbb{R}^n$

$\underline{x} = (x_1, \dots x_n) \in \mathbb{R}^n$ norm $\|x\| = (x_1^2 + \dots + x_n^2)^{1/2}$

Euclidean metric d $d(x,y) = \|x-y\| = ((x_1-y_1)^2 + \dots + (x_n-y_n)^2)^{1/2}$

(comes from inner product on $\mathbb{R}^n$, has good symm properties $O(n)$)

Square metric $\rho(x,y) = \max(|x_1-y_1|, \dots |x_n-y_n|)$. (works in general $(X, d_1), (Y, d_2) \to (X \times Y, \rho)$)

Prop ρ is a metric. Pf: $\forall i$ $|x_i - z_i| \le |x_i - y_i| + |y_i - z_i| \le \rho(x,y) + \rho(y,z)$
$\Rightarrow \rho(x,z) = \max(|x_i - z_i|) \le \rho(x,y) + \rho(y,z)$.

✓ Basis el't

(20.2)
Lemma ✓ $(X,d), (X,d')$ two metrics on X, $\mathcal{T}, \mathcal{T}'$ assoc. topologies.
$\mathcal{T}'$ is finer than $\mathcal{T}$ iff $\forall x \in X, \forall \varepsilon > 0$ $\exists \delta > 0$ s.t. $B_{d'}(x,\delta) \subset B_d(x,\varepsilon)$.

Thm (20.3) d and ρ metrics on $\mathbb{R}^n$ induce the same topology.

1, $\sqrt{n} \to \lambda_1, \lambda_2 > 0$ may depend on x.

Pf: $\rho(x,y) \le d(x,y) \le \sqrt{n}\, \rho(x,y)$

$B_d(x,\varepsilon) \subset B_\rho(x,\varepsilon)$, $B_\rho(x, \varepsilon/\sqrt{n}) \subset B_d(x,\varepsilon)$ □.

use lemma

Thm Let $(X,d_1), (Y,d_2)$ metric spaces. $X \times Y$ is a metric space with ρ metric (above)
Then $\mathcal{T}_\rho$ on $X \times Y$ coincides with the product topology for $(X, \mathcal{T}_{d_1})$ and $(Y, \mathcal{T}_{d_2})$.
Same for $\mathcal{T}_d$.

Pf: Exercise.

Have a topology on $X_1 \times \dots \times X_n$ given topologies on X_i, $i \geq 1$.
How to define a topology on an infinite product?
Munkres §19.

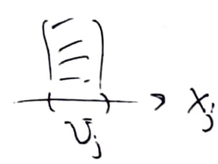
Consider countable product $\prod_{i \geq 1} X_i = X_1 \times X_2 \times \dots$

uncountable even if each X_i is finite, $X_i \neq \emptyset$, some have more than 1 el't.

$\prod_{i \geq 1} X_i \xrightarrow{\pi_j} X_j$ $j \geq 1$. Introduce weakest topology so that each p_j continuous.

Subbasis $\mathcal{S} = \{ \underline{x} \mid \forall i \in \mathbb{N},\ x_i \in U_i,\ U_i \text{ open in } X_i \}$ $\underline{x} = (x_i)_i$ $x_i \in X_i$

$\mathcal{S}_j = \{ \pi_j^{-1}(U_j) \mid U_j \text{ open in } X_j \}$
$\mathcal{S} = \bigcup \mathcal{S}_j$
Likewise for more general products $\prod_{\alpha \in J} X_\alpha$



take finite intersections. Get a basis $\mathcal{B}$ with elements

$$B = \pi_{\beta_1}^{-1}(U_{\beta_1}) \cap \pi_{\beta_2}^{-1}(U_{\beta_2}) \cap \dots \cap \pi_{\beta_n}^{-1}(U_{\beta_n}).$$

$$\pi_\beta : \prod_{\alpha \in J} X_\alpha \to X_\beta$$

$$S_\beta = \{ \pi_\beta^{-1}(U_\beta) \mid U_\beta \text{ open in } X_\beta \}$$

$$S = \bigcup_{\beta \in J} S_\beta$$

$x = (x_\alpha) \in B$ iff β_i coordinate is in U_{β_i}

$$B = \prod_{\alpha \in J} U_\alpha \qquad U_\alpha = X_\alpha \text{ if } \alpha \neq \beta_1 \dots \beta_n$$

This is the product topology on $\prod_{\alpha \in J} X_\alpha$.

weakest topology s.t. $\pi_\beta : \prod_{\alpha \in J} X_\alpha \to X_\beta$ is continuous $\forall \beta \in J$.

These products come up in studying spaces of functions $f : [0,1] \to \mathbb{R}$. $f(x)$ determined by its values $f(x) \in \mathbb{R}^I$ product of $\mathbb{R}$'s parametrized by I

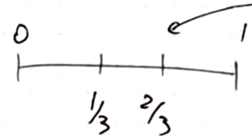
Box topology: pick $U_\alpha \subset X_\alpha$ each $\alpha \in J$. Basis $\mathcal{B}$ consists of products $\prod_{\alpha \in J} U_\alpha$

Exercise both are topologies on $\prod X_\alpha$. Box topology is finer than product top.

Example $J = \mathbb{N}$ $X_i = \{0,1\}$. Then 1) box topology makes $\prod_{i=1}^{\infty} \{0,1\}$ a discrete top space

2) $\prod_{\mathbb{N}} \{0,1\}$ with prod. top. is homeomorphic to the Cantor set.

Cantor set C:



remove open interval $(\frac{1}{3}, \frac{2}{3})$

repeat with each remaining closed interval

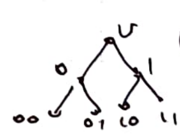
From $[a,b]$, inductively remove $(a + \frac{b-a}{3}, a + \frac{2}{3}(b-a))$, repeat.

C is closed

Prop Points of C are in a bijection with infinite sequences of 0, 1.

Nested sequence of closed intervals.

$[\ [\ [\ ]\ ]\ ]$
lengths $\to 0$
$\Rightarrow \exists!$ common point.



paths in full infinite binary tree

Open sets in C?

$x = a_1 a_2 \dots a_1 \quad a_n a_{n+1} \dots$

all $y = a_1 \dots a_n b_{n+1} \dots$ are in open neighborhood

These sets are both open and closed (clopen) Recognize product topology.

$U_{a_1 \dots a_n **}$ $\quad U_{*a_2*}$

(later appears as topology on p-adic numbers) Thm C is homeomorphic to $\prod_{i=1}^{\infty} \{0,1\}$ prod. top.

what if sets have more than 2 elements

X_i - finite discrete top spaces, $|X_i| \geq 2$ $\forall i$ to exclude trivial case

<u>Exercise</u> $C \overset{\text{homeo}}{\simeq} \prod_{i=1}^{\infty} X_i$, product topology.

Note that Cantor set topology is metrizable, $C \subset [0,1]$.

Suppose (X_i, d_i) metric, so $\mathcal{T}_i$ comes from a metric.

Is $\underbrace{\prod_{i \in \mathbb{N}} X_i}_{\tilde{X}}$ metrizable? Can we squeeze basis open sets in $\tilde{X}$ into balls of small radii?

Idea: make d_i smaller as $i \to \infty$; change d_i to $\overline{d_i}$ distance at most 1; then divide by i.

metric $\frac{\overline{d_i}}{i}$ on X_i diameter $\operatorname{diam}(X_i, \frac{\overline{d_i}}{i}) \leq \frac{1}{i}$

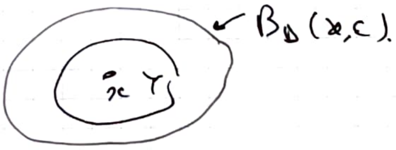
take $D(x,y) = \sup\left(\frac{\overline{d_i}(x_i,y_i)}{i}\right)$ sup metric.

take x, ε consider $B_D(x,\varepsilon)$. if y differs from x in coordinates n with $n > \frac{1}{\varepsilon} \Leftrightarrow \frac{1}{n} < \varepsilon$ $y_n \neq x_n$ then $D(x,y) < \varepsilon$

for $m < \frac{1}{\varepsilon}$ take ε-balls $B_{\overline{d_m}}(x_m, \varepsilon)$. Product of these balls for $m < \frac{1}{\varepsilon}$ and everything X_n for $n > \frac{1}{\varepsilon}$

$$Y = \prod_{m=1}^{N} B_{\overline{d_m}}(x_m, \varepsilon) \times \prod_{n > N} X_n \text{ - open in product topology}$$

$N > \frac{1}{\varepsilon}$



<u>Exercise</u>: Complete the proof or read in Munkres. $\square$.