lecture 4 — Topology, fall 2022
Sept. 15

1) Complete Hausdorff spaces (end of lect. 3)

2) Complete discussion of order topology (end of lect. 2)

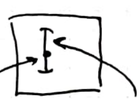
Prop An order topology is Hausdorff.

Let $x, y \in X$ $(X, <)$, $x < y$ → if $\exists z$ $x < z < y$ (1)
→ if $\nexists z$ $x < z < y$ (2)

Case (1): Consider $(-\infty, z)$, $(z, +\infty)$ – disjoint open neighbourhoods of x, y
$x \in (-\infty, z)$, $y \in (z, +\infty)$ $(-\infty, z) = \bigcup \{ (a, z) \mid a < z \} \cup \{ [a_0, z) \text{ if } \exists \text{ smallest } a_0 \}$ open

Case (2): then $(-\infty, y)$ (open) neigh of x, $(x, +\infty)$ neighbourhood of y. ; likewise for $(z, +\infty)$ – open

Consider I_0^2 ordered square $I = [0, 1]$ I^2, order topology
$(x_1, y_1) < (x_2, y_2)$ if $x_1 < x_2$ or $x_1 = x_2$, $y_1 < y_2$



(x, y) if $y \neq 0, 1$ "small" open neighbourhood

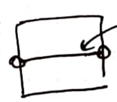
$(x+\varepsilon, y)$

$(x, 0)$

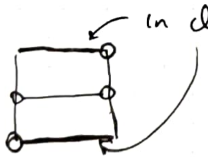
neighbourhoods of $(x, 0)$ are "large" in ~~order~~ I_0^2.

likewise for $(x, 1)$

Let $A = (0, 1) \times \{\frac{1}{2}\}$



then $[0, 1) \times \{1\} \in \bar{A}$
$(0, 1] \in \{0\} \in \bar{A}$



in closure

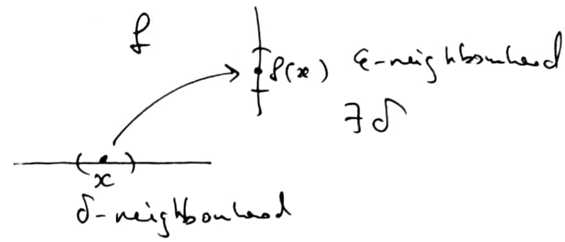
Exercise: Completes this discussion to determine $\bar{A}$ in I_0^2.

(Munkres §18)

Def $f: X \to Y$ a map of top. spaces is continuous if $\forall V \subset Y$ open $\Rightarrow f^{-1}(V) \subset X$ open.

This generalizes a familiar notion of continuous maps $f: \mathbb{R} \to \mathbb{R}$ (or maps of subspaces of $\mathbb{R}$, of $\mathbb{R}^n$).

$f^{-1}(\text{open})$ is open. $\Leftarrow$ $f^{-1}((f(x)-\epsilon, f(x)+\epsilon))$ contains $(x-\delta, x+\delta)$



if $\mathcal{B}$ is a basis for $\mathcal{T}_Y$, continuity of f follows from $f^{-1}(B)$, $B \in \mathcal{B}$ being open (for all B), since $\forall$ open $V \subset Y$ has $V = \bigcup_{\alpha \in J} B_\alpha \Rightarrow$

$$f^{-1}(V) = \bigcup_{\alpha \in J} f^{-1}(B_\alpha).$$

if $\mathcal{S}$ is a subbasis for top. of Y, enough to check $f^{-1}(S)$ - open $\forall S \in \mathcal{S}$.

$f^{-1}(B) = f^{-1}(S_1) \cap \ldots \cap f^{-1}(S_n)$ $\quad B = S_1 \cap \ldots \cap S_n$ basis.

Prop 1) identity map $id_X: X \to X$ is continuous. 2) composition of continuous maps is continuous $X \xrightarrow{f} Y \xrightarrow{g} Z$ f, g - continuous $\Rightarrow gf$ continuous.

3) $(X, \mathcal{T}) \xrightarrow{id} (X, \mathcal{T}')$ is continuous if $\mathcal{T}$ is finer than $\mathcal{T}'$

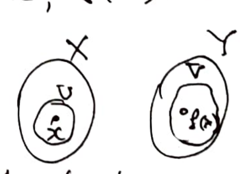
Thm (18.1) Let $f: X \to Y$ map of top. spaces. TFAE

(1) f is continuous

(2) $\forall A \subset X$: $f(\bar{A}) \subset \overline{f(A)}$

(3) $f^{-1}(C)$ closed in X $\forall$ closed $C \subset Y$.

(4) $\forall x \in X$, $\forall$ neighbourhood V of $f(x)$ in Y $\exists$ neigh U of x, $f(U) \subset V$.



Pf: (1) $\Leftrightarrow$ (3) pass to complements.

(1) $\Rightarrow$ (2) let $x \in \bar{A}$, want to show $f(x) \in \overline{f(A)}$

let V - neigh. of $f(x)$. $f^{-1}(V)$ - open in X, contains $x \Rightarrow$ intersects A at pt. y.

$\Rightarrow V$ intersects $f(A)$ at pt. $f(y) \Rightarrow f(x) \in \overline{f(A)}$

(2) $\Rightarrow$ (3). Let $C \subset Y$ closed, $A = f^{-1}(C)$. To show A closed in X check if $\bar{A} = A$.

$f(A) = f(f^{-1}(C)) \subseteq C$. $\Rightarrow$ if $x \in \bar{A}$ then $f(x) \in f(\bar{A}) \subset \overline{f(A)} \subset \bar{C} = C \Rightarrow x \in f^{-1}(C) = A$.

(1) $\Rightarrow$ (4) easy, (4) $\Rightarrow$ (1) easy.

Homeomorphism (isomorphism of top. spaces)

<u>Def</u> $f: X \to Y$ is a homeomorphism if it's bijective and f^{-1} is a continuous map.

<u>Equivalent def</u>: $\exists\, g: Y \to X$ continuous s.t. $gf = id_X$, $fg = id_Y$

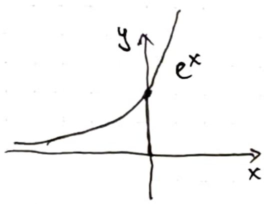
$$\begin{tikzcd} X \arrow[r, bend left, "f"] & Y \arrow[l, bend left, "g"] \end{tikzcd}$$

<u>Examples</u> 1) discrete top spaces are homeomorphic iff they have the same cardinality.

2) same for indiscrete; same for finite complement topology spaces

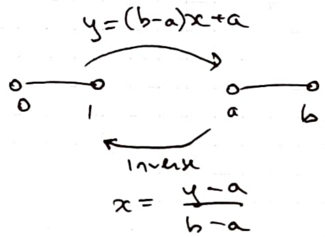
3) $(0,1) \to (a,b)$ $\quad (0,1) \cong (a,b)$ likewise for semiopen & closed intervals

$[0,1) \simeq [a,b) \simeq (a,b]$



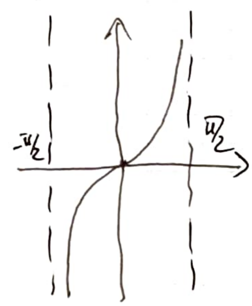
$(-\infty, 0) \underset{\ln x}{\overset{e^x}{\rightleftarrows}} (0,1)$ $\qquad (0,1) = (-\infty, 0) \simeq (0, +\infty)$

likewise $[0,1) \simeq [a, +\infty)$



$\tan: (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}$ continuous, bijective

arctan - continuous



many other functions work

<u>Exercise</u>: Each top. space X has the group of homeomorphisms $Homeo(X)$, often very large

bijective, continuous (monotonic) f'n $[0,1] \to [0,1]$.

$x \mapsto x^2$

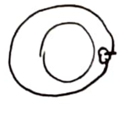
$x \mapsto \sqrt{x}$

$(X, \mathcal{T}) \xrightarrow{id} (X, \mathcal{T}')$ for a homeomorphism, need $\mathcal{T} = \mathcal{T}'$.

If $X \to Y$ injective, $f(X) = Z$ and $f_Z : X \to Z$ is a homeomorphism, say that f is a <u>topological imbedding</u> or <u>imbedding</u> of X in Y.

$[0,1) \longrightarrow S^1$-circle, $S^1 \subset \mathbb{R}^2$, $S^1 = \{(x,y) \mid x^2+y^2=1\}$ bijective, <u>not</u> a homeomorphism.

$t \longmapsto f(t) = (\cos 2\pi t, \sin 2\pi t)$



Thm (18.2)

a) Constant f'n $f: X \to Y$, $f(X) = y_0 \in Y$ continuous.

b) $A \subset X$ subspace, inclusion $j: A \hookrightarrow X$ is continuous

c) Composition (see earlier)

d) $f: X \to Y$, $A \subset X$ subspace $\Rightarrow f|_A : A \to Y$ is continuous

e) $f: X \to Y$, let $Z \supset f(X)$ then restriction $f: X \to Z$ is continuous.
if $Z \supset Y$ as a subspace, $f: X \to Z$ is continuous

f) (local) $f: X \to Y$ is continuous if $X = \bigcup_\alpha U_\alpha$ s.t. $f|_{U_\alpha}$ continuous $\forall \alpha$.

Proof a), b), c), d), e) straightforward

f) Let $V \subset Y$ open $\Rightarrow f^{-1}(V) \cap U_\alpha = (f|_{U_\alpha})^{-1}(V)$ pts x in U_α s.t. $f(x) \in V$

open in $U_\alpha \Rightarrow$ open in X

$f^{-1}(V) = \bigcup_\alpha (f^{-1}(V) \cap U_\alpha)$ also open.

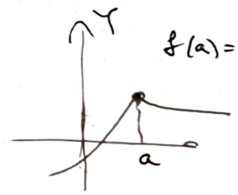
Thm (18.3, pasting lemma). Let $X = A \cup B$, A, B closed in X,
$f: A \to Y$, $g: B \to Y$ continuous, $f|_{A\cap B} = g|_{A \cap B} \Rightarrow f, g$ extend to a continuous $h: X \to Y$, $h(x) = f(x)$, $x \in A$, $h(x) = g(x)$, $x \in B$.

Pf Let $C \subset Y$ closed. $h^{-1}(C) = f^{-1}(C) \cup g^{-1}(C)$ closed in B, B closed in $X \Rightarrow$ closed in X.

closed in $A \to$ closed in X

Thm also holds if U, V open in X, then use Thm 18.2

Example:



$A = (-\infty, a]$, $B = [a, +\infty)$

Thm (18.4) $f: A \to X \times Y$ given by
$f(a) = (f_1(a), f_2(a))$. f is continuous iff $f_1 : A \to X$, $f_2 : A \to Y$ are continuous.

Pf Exercise or Munkres p. 110.