X - top. space

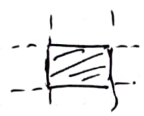
<u>Def</u> $A \subset X$ is <u>closed</u> if $X \setminus A$ is open

<u>Examples</u>: 1) Discrete topology - any set is closed

2) Indiscrete topology - only $\emptyset$, X are closed

3) $\mathbb{R}$, standard topology: for example, $[a,b]$ is closed since its complement is open $(-\infty, a) \cup (b, +\infty)$

4) $\mathbb{R}^2$ closed rectangle $[a,b]\times[c,d]$ (will see more general example soon)



complement is a union of 4 open sets.

<u>Prop</u>: $A \subset X$ closed, $B \subset Y$ closed $\Rightarrow$ $A \times B \subset X \times Y$ closed (product topology).

5) In a finite complement topology on X only finite sets and X are closed

6) Let $Y = [0,1] \cup (2,3) \subset \mathbb{R}$, Y-induced topology. Then $[0,1]$ and $(2,3)$ are closed in Y.

Theorem (17.1 in Munkres) In a topological space X

(1) $\emptyset$, X are closed.

(2) Arbitrary intersections of closed sets are closed

(3) Finite unions of closed sets are closed.

<u>Proof</u>: Use DeMorgan's laws that complement swaps unions & intersections

$\{A_\alpha\}_{\alpha \in J}$, $A_\alpha \subset X$ $\qquad X \setminus \bigcap_{\alpha \in J} A_\alpha = \bigcup_{\alpha \in J} (X \setminus A_\alpha)$

$$X \setminus \bigcup_{i=1}^{n} A_i = \bigcap_{i=1}^{n} (X \setminus A_i)$$

<u>Example</u> In the disjoint union $X \sqcup Y$ of top. spaces X, Y closed sets have the form $A \sqcup B$, $A \subset X$, $B \subset Y$ closed.

Suppose $A \subset Y \subset X$ X-top space, Y has subspace topology. Then A may be closed in Y but not in X (take $A = Y$ and $Y \subset X$ not closed).

For $A \subset Y \subset X$ we say that A is <u>closed</u> in Y if $A \subset Y$ & A closed in the subspace topology of Y.

<u>Thm</u> (17.2) Let $Y \subset X$. Then $A \subset Y$ closed iff $A = Y \cap C$, for some closed set $C \subset X$.

U should $X \setminus U$

<u>Proof</u> A closed in $Y \Leftrightarrow Y \setminus A$ open in $Y \Leftrightarrow Y \setminus A = U \cap Y$, $U \subset X$ open $\Leftrightarrow A = (X \setminus U) \cap Y$

closed in X

<u>Thm</u> (17.3) If $A \subset Y$ closed, $Y \subset X$ closed $\Rightarrow$ $A \subset X$ closed

<u>Pf:</u> $Y \setminus A = \underset{\text{open in } X}{U} \cap Y$, $X \setminus Y$ open

$\Rightarrow X \setminus A = \underbrace{(X \setminus Y) \cup U}_{\text{open}}$ □

<u>Closure, Interior of a set</u>

$A \subset X$ subset (or subspace).

Interior of A, $\mathrm{Int}(A)$ - union of all open sets contained in A. = largest open (in X) subset of A

Closure of A in X: $Cl(A)$ or $\overline{A}$ or $Cl_X(A)$

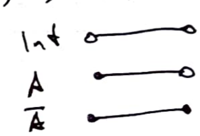
Intersection of all closed sets containing A = smallest closed subset of X that contains A.

$$\boxed{\mathrm{Int}\, A \subset A \subset \overline{A}}$$

<u>Example:</u> $A = [0,1) \subset \mathbb{R}$

$\mathrm{Int}\, A = (0,1)$

$\overline{A} = [0,1]$



$(0,1) \subset [0,1) \subset [0,1]$

A open $\Leftrightarrow$ $\mathrm{Int}\, A = A$

A closed $\Leftrightarrow$ $\overline{A} = A$

If $A \subset Y \subset X$ then $\overline{A} = Cl_X(A)$ may be different from $Cl_Y(A)$

<u>Example:</u> take $A = Y = [0,1)$, $X = \mathbb{R}$ $Cl_Y(A) = [0,1)$, $Cl_X(A) = [0,1]$.

Theorem (17.4). For $A \subset Y \subset X$ we have $Cl_Y(A) = \overline{A} \cap Y$.

<u>Pf:</u> $\overline{A}$ closed in $X \Rightarrow \overline{A} \cap Y$ closed in Y. Let $B := Cl_Y(A)$. Then $B \subset \overline{A} \cap Y$ (*)

intersection of closed sets in Y that contain A.

B closed in $Y \Rightarrow B = \underset{\text{closed in } X}{C} \cap Y$. $\Rightarrow \overline{A} \subset C$ since $C \supset A$

$\Rightarrow \underbrace{\overline{A} \cap Y \subset C \cap Y = B}_{(**)}$. (*), (**) imply $B = \overline{A} \cap Y$. □.

Thm (17.5) Let $A \subset X$. Then (1) $x \in \overline{A}$ iff any open U that contains x intersects A. (2) for a basis $\mathcal{B}$ of $(X, \mathcal{T})$, $x \in \overline{A}$ iff $\forall B \subset \mathcal{B}$, $x \in B$ we have $B \cap A \neq \emptyset$. (any basis element B containing x intersects A).

Pf: (1) $\Leftrightarrow$ $x \notin \overline{A}$ iff some open U contains x. (2) follows as well.

Terminology: U is an open set containing x $\Leftrightarrow$
U is an (open) neighbourhood of x

Limit points

Def: Let $x \in A \subset X$. x is a limit point of A if $\forall$ neighbourhood of x intersects A in some point other than x itself.

x limit point of $A \iff x \in \overline{A \setminus \{x\}}$

x may or may not be in A $\qquad$ A' - set of limit points of A.

Example 1) $A = (0,1) \subset \mathbb{R}$. limit points are $[0,1]$.

2) $A = \mathbb{Z} \subset \mathbb{R}$. There are no limit points $\mathbb{Z}' = \emptyset$

3) $A = \{\frac{1}{n}\}_{n \geq 1}$ 0 is the only limit point of A.

Thm (17.6) For $A \subset X$ we have

$$\overline{A} = A \cup \underbrace{A'}_{\text{limit points}}$$

Proof: Straightforward. $X \setminus \overline{A} = \{x \in X \mid \exists \text{ open } U, x \in U, U \cap A = \emptyset\}$ $\square$.

Corollary (17.7) $A \subset X$ is closed iff it contains all its limit points.
Indeed $A = \overline{A} \iff A' \subset A$. $\square$

Closed points.
In $\mathbb{R}$, each point is a closed set. Same in $\mathbb{R}^n$ & subspaces of $\mathbb{R}^n$

If each point of X is closed $\Leftrightarrow$ finite subsets of X are closed.

This is called T_1 axiom or T_1 property.

1) $\mathbb{R}^n$ is T_1

2) X is T_1, $Y \subset X \Rightarrow Y$ is T_1

3) X, Y are $T_1 \Rightarrow X \times Y$ are T_1.

4) finite complement topology is T_1

$X = \{a, b\}$ $\quad \mathcal{T} = \{\emptyset, \{b\}, \{a, b\}\}$

(*) b is not closed (its complement is not open)
not a T_1 space. Ex: finite space is $T_1 \iff$ its discrete.

In X, say that $x_1, x_2, \ldots$ converges to x if $\forall U \ni x$ neighbourhood $\exists N$ s.t. $x_i \in U$ $\forall i \geq N$.

$x_n = b \ \forall n$ $\quad b, b, \ldots$ converges to both a and b in (*).

Hausdorff spaces

Def X is Hausdorff if $\forall x_1, x_2 \in X$, $x_1 \neq x_2$ $\exists$ open $U_1 \ni x_1$, $U_2 \ni x_2$, $U_1 \cap U_2 = \emptyset$ (disjoint)

Thm (17.8) Points and finite sets in Hausdorff X are closed.

Proof why is $\{x_0\}$ closed? $\forall x \neq x_0$ has $\underset{\text{open}}{U_x}$ disjoint from x_0. $\bigcup_{x \neq x_0} U_x = X \setminus \{x_0\}$ is open

Example: 1) $\mathbb{R}$ is Hausdorff,

2) finite complement topology on an infinite set is not Hausdorff

3) Exercise: X, Y Hausdorff $\Rightarrow$ $X \times Y$ is Hausdorff.

4) X-Hausdorff, $Y \subset X$ $\Rightarrow$ Y is Hausdorff.

5) Any order topology is Hausdorff.

6) Any Hausdorff top. on a finite set is discrete.

Thm (17.9). Let X be Hausdorff (or even T_1). $A \subset X$. Then x is a limit point of A iff $\forall$ neighbourhood of x contains infinitely many points of A

Pf: See Munkres or exercise. $\square$.

Thm (17.10) X is Hausdorff $\Rightarrow$ a sequence of points of X converges to at most one point of X.

Pf: if $\{x_n\}_{n \geq 1} \longrightarrow x \in X$ (converges to), let $y \in X$, $y \neq x$. Pick disjoint neighbourhoods of ~~the~~ x and y, $x \in U$, $y \in V$, $U \cap V = \emptyset$. Then U contains all x_n, $n \geq N$ (some N). $\Rightarrow V$ contains at most fin many x_n . $\square$.

Exercise: X is Hausdorff iff the diagonal $\Delta = \{x \times x \mid x \in X\}$ is a closed subset of $X \times X$.