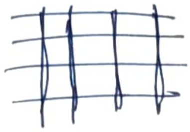
Covering $\begin{array}{c} E \\ \downarrow p \\ B \end{array}$ loc. triv. $b \in U$ $p^{-1}(U) \cong U \times Y$ $p^{-1}(b)$ has discrete topology

surjective

$\mathbb{R} \to S^1$ $\quad \mathbb{C}^* \to \mathbb{C}^*$ $\quad S^1 \to S^1$ product coverings

$t \mapsto (\cos 2\pi t, \sin 2\pi t)$ $\quad z \to z^n$ $\quad z \to z^n$

$\mathbb{R} \times \mathbb{R} \xrightarrow[p \times p]{} S^1 \times S^1$



$b_0 \times b_0$ $\quad b_0 \in S^1$

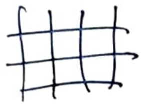
 $\longrightarrow$ 

figure 8

$\begin{array}{ccc} E & \to & B \\ \cup & & \cup \\ E_0 & \to & B_0 \end{array}$ figure 8.

$B_0 = b_0 \times S^1 \cup S^1 \times b_0$

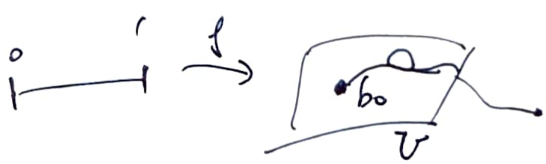
$b_0 \times b_0$

$\mathbb{R} \times \mathbb{Z} \cup \mathbb{Z} \times \mathbb{R}$

Given
$$\begin{tikzcd} & E \arrow[d, "p"] \\ X \arrow[r, "f"] \arrow[ur, dashed, "\tilde{f}"] & B \end{tikzcd}$$
lifting of f is a map $\tilde{f}: X \to E$ s.t. $p \circ \tilde{f} = f$

__Principle__ if $\begin{array}{c} E \\ \downarrow p \\ B \end{array}$ is a covering map, then paths & path homotopies can be lifted

__Prop__ (lemma 54.1). Let $p: E \to B$ be a covering map, $p(e_0) = b_0$. Any path $f: [0,1] \to B$ beginning at b_0 has a unique lifting to a path $\tilde{f}$ in E beginning at e_0



$f^{-1}(U)$ is open in $[0,1]$, contains 0.

$\Downarrow$

$f([0,\varepsilon]) \subset U$ some ε.

$\Downarrow$

$f([0,\varepsilon])$ has a unique lifting. $\tilde{f}(0) = e_0$

(use that $[0,\varepsilon]$ is connected) generalize. B has a covering by $\forall\, b \mapsto U_b$ - trivial

apply f^{-1} get induced covering of $[0,1]$ by open sets.

has a Lebesgue #. (metric, compact). δ. choose n. $\frac{1}{n} < \delta$.

$[0, \frac{1}{n}$ $\quad s_0 = 0, s_1 = \frac{1}{n}..$ $\quad s_i = \frac{i}{n}$ $\quad [\frac{i}{n}, \frac{i+1}{n}] \subset U_{b_i}$ $\quad i = 0, .., n-1$

assume $\exists!$ lifting of $[0, \frac{i}{n}]$ to $\tilde{f}$

$\exists!\, V_{b_i}$ contains $\tilde{f}(\frac{i}{n})$. $\quad p^{-1}(U_{b_i}) = U_{b_i} \times Y$.

$[\frac{i}{n}, \frac{i+1}{n}] \longrightarrow U_{b_i}$

get a unique lift - A continuous map

□.

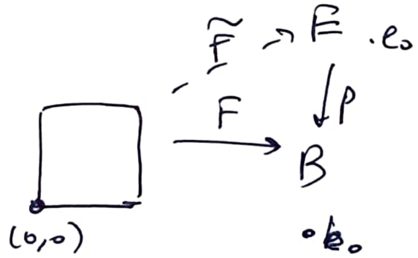
Prop (lemma 54.2). Let $E \xrightarrow{p} B$ be a covering map, let

$F: I \times I \longrightarrow B$ be continuous, $F(0,0) = b_0$. $\Rightarrow \exists!$ lifting of F

to a continuous map $\tilde{F}: I \times I \longrightarrow E$ s.t $\tilde{F}(0,0) = e_0$.

s.t $\tilde{F}(0,0) = e_0$. If F is a path homotopy, then $\tilde{F}$ is a path homotopy.

Proof: Same as before



Can extend to a small square $[0, \frac{1}{n}] \times [0, \frac{1}{n}]$. .

Covering of B by open sets over which p is trivial.

Get covering of $I \times I$ Lebesgue #. δ. choose n s.t $\frac{2}{n} < \delta$

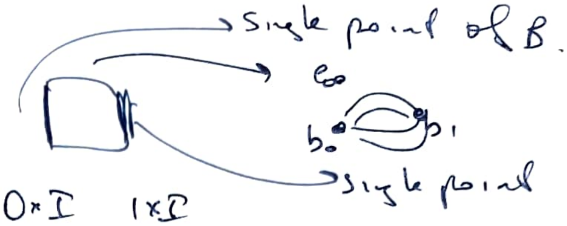
$\forall$ square $\frac{1}{n} \times \frac{1}{n}$ is in some open set of the covering.

lifting already constructed

continuous.

Unique lifting

If F is a path-homotopy,



have a lifting $\tilde{F}(0\times I) \subset p^{-1}(b_0)$ - discrete. $\Rightarrow$ $\tilde{F}(0\times I) = \{e_0\}$.

$\Rightarrow \tilde{F}$ is a path-homotopy.

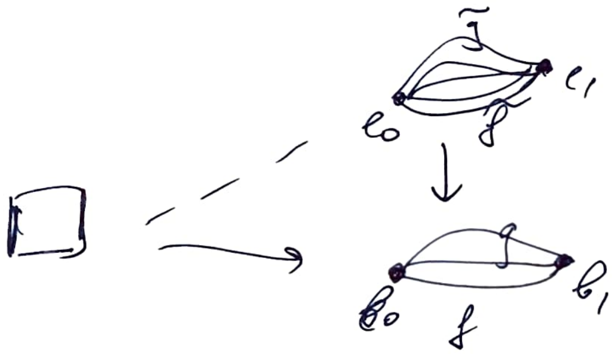
<u>Thm</u> (54.3) $p: E \to B$ covering, $p(e_0) = b_0$. Let f, g be paths in B from b_0 to b_1; $\tilde{f}, \tilde{g}$ their liftings to paths in E at e_0.

If f, g are path-homotopic then $\tilde{f}, \tilde{g}$ end at the same point of E and are path-homotopic.

<u>Proof</u> $F: I\times I \to B$ a path homotopy between f, g.

$F(0,0) = b_0$. Let $\tilde{F}: I^2 \to E$ be a lifting, $\tilde{F}(0,0) = e_0$.

$\tilde{F}$ is a path-homotopy, $\tilde{F}(0\times I) = \{e_0\}$ $\tilde{F}(1\times I) = \{e_1\}$.



Def: $p: E \to B$ covering, $b_0 \in B$. $e_0 \in E$, $p(e_0) = b_0$.
Given $[f] \in \pi_1(B, b_0)$ let $\tilde{f}$ be a lifting of f to a path in E that begins at e_0. Let $\phi([f])$ be the end point $\tilde{f}(1)$ of $\tilde{f}$.
Then ϕ is a well-defined map

$$\phi: \pi_1(B, b_0) \to p^{-1}(b_0).$$

ϕ is called the lifting correspondence derived from map p.
Depends on choice of e_0.

Thm Let p $\begin{matrix} E \ni e_0 \\ \downarrow \quad \mathrm{I} \\ B \ni b_0 \end{matrix}$. If E is path-connected, the lifting correspondence

$$\phi: \pi_1(B, b_0) \to p^{-1}(b_0) \text{ is surjective.}$$

If E is simply connected, ϕ is bijective

Proof If E is path-connected, for $e_1 \in p^{-1}(b_0)$ $\exists$ path $\tilde{f}$ in E from e_0 to e_1. Let $f = p \circ \tilde{f}$ loop in B at b_0, $\phi([f]) = e_1$.

If E is simply-connected, let $[f], [g] \in \pi_1(B, b_0)$ s.t. $\phi([f]) = \phi([g])$. Let $\tilde{f}, \tilde{g}$ liftings of f, g to paths in E that begin at e_0. Then $\tilde{f}(1) = \tilde{g}(1)$. Since E is simply-connected, $\exists$ a path-homotopy $\tilde{F}$ in E between $\tilde{f}$ and $\tilde{g}$ $\Rightarrow$
$p \circ \tilde{F}$ is a path-homotopy in B between f and g.

Thm $\pi_1(S^1) = \mathbb{Z}$

$p: \mathbb{R} \to S^1$ covering map. $e_0 = 0$, $b_0 = p(e_0)$

$p^{-1}(b_0) = \mathbb{Z}$. set

$\mathbb{R}$ - simply-conn $\Rightarrow$ $\phi : \pi_1(S^1, b_0) \to \mathbb{Z}$ is bijective.

need to show ϕ is a homomorphism

$[f], [g] \in \pi_1(S^1, b_0)$. $\tilde{f}, \tilde{g}$ liftings to paths in $\mathbb{R}$ beg. at 0.

let $n = \tilde{f}(1)$,

$m = \tilde{g}(1)$.

$\phi([f]) = n$

$\phi([g]) = m$

let $\tilde{\tilde{g}}$: $\tilde{\tilde{g}}(s) = n + \tilde{g}(s)$.

$p(x+n) = p$

shift $\tilde{g}$ to start at $\tilde{f}(1)$
to $\tilde{\tilde{g}} = \tilde{g} + n$

$\tilde{f} * \tilde{\tilde{g}}$ is defined, lifts $f * g$, starts at 0.

$\tilde{\tilde{g}}(1) = n + m$ end point

$\phi([f] * [g]) = n + m = \phi([f]) + \phi([g])$.