Topology, fall 2015, Practice Quiz Solutions

1. Mark the boxes that are followed by correct statements.

☐ Any compact space is metrizable.

False: Any finite topological space is compact, but it's metrizable only if the topology is discrete.

☐ Any topology on a 3-element set is Hausdorff.

False

☐ The only homeomorphism from the Cantor set to itself is the identity.

False For instance, there's the reflection homeomorphism (induced by reflecting interval $[0, 1]$ about the midpoint). In fact, homeomorphism group of the Cantor set is huge.

☐ Any subspace of a Hausdorff space is Hausdorff.

True

☐ Any metrizable space is second-countable.

False Take a discrete uncountable space. It's metrizable (for instance, with the metric where all distances $d(x, y) = 1$ for $x \neq y$) but not second-countable.

☐ The space $[0, 1]^\omega$ with the product topology is metrizable.

True We derived a similar result in class for $\mathbb{R}^\omega$ with the product topology (see page 125 Theorem 20.5).

☐ Any finite T_1-space is discrete.

True If you have a finite topological space with n elements, and all $(n - 1)$-element subsets are open, then any subset is open (exercise).

☐ Direct product of two metrizable spaces is metrizable.

True.

☐ If $A, B \subset X$ are both compact then $A \cup B$ is compact.

True. Follows quickly from the definition of a compact set.

☐ A Cantor set C is homeomorphic to a disjoint union $C \sqcup C$ of two copies of itself.
True. This is an important property of C.

☐ The set $\mathbb{N}$ of natural numbers equipped with the discrete topology is metrizable.
True.

☐ In a finite complement topology, any sequence has a convergent subsequence.
True.

☐ Any dense subset of the Cantor set is uncountable.
False. Take the union of boundary points of all the intervals in $[0,1]$ that we're using to define C.

☐ A subspace of $\mathbb{R}$ is connected if and only if it is path-connected.
True: First classify all connected subspaces of $\mathbb{R}$ in the standard topology.

☐ If $A, B \subset X$ are path-connected then $A \cup B$ is path-connected.
False. True if A and B have a nonempty intersection.

☐ Direct product $X \times Y$ of connected spaces X, Y is connected.
True. This is theorem 23.6 on page 150.

☐ $A \subset X$ is connected if and only if $\overline{A}$ is connected.
False. Take $\mathbb{Q} \subset \mathbb{R}$. Unlike $\mathbb{Q}$ itself, its closure $\overline{\mathbb{Q}} = \mathbb{R}$ is connected. The implication the other way is true, see Theorem 23.4 on page 150.

☐ $\mathbb{N}$, in the finite complement topology, is second-countable.
True. In fact, there are only countably many open sets in this

topology.

☐ For any metric space X, map $f : X \times X \times X \longrightarrow \mathbb{R}$ that takes (x, y, z) to

$$f(x, y, z) = d(x, y)d(y, z) - d(x, z)$$

is continuous.
True. Use that $d(x, y) : X \times X \longrightarrow \mathbb{R}$ is continuous.

2. Which of the following spaces are compact?

The Cantor set C. **Compact**

Intersection of the Topologist Sine Curve in $\mathbb{R}^2$ with the closed unit disk. **Compact.** For Topologist Sine Curve we use definition in Munkres, Example 7 on page 156. With this definition the intersection in question is compact.

$\mathbb{Q}$, with the basis of open sets $\{(-\infty, a) | a \in \mathbb{Q}\}$. **Not compact**

Subspace $[0, 1]$ of $\mathbb{R}_\ell$. **Not compact**

Intersection $[0, 1] \cap \mathbb{Q}$, in the standard topology. **Not compact**