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Representation V of algebra A — (affine) Hecke; $\mathcal{U}(\mathfrak{g}), \mathcal{U}_q(\mathfrak{g})$; $\mathcal{U}(\hat{\mathfrak{g}})$; etc.

becomes

I) the Grothendieck group of a category C

$V \simeq K(C)$ $\quad$ A acts by exact functors

or

II) Homology / Cohomology / K-theory of topological space X.

A acts by correspondences

I) categorification

II) geometrization

Grothendieck groups

Category $\mathcal{C} = B$-mod, B a ring

$K(\mathcal{C})$ abelian group with generators

$[M]$, $M \in B$-mod, relations

$[M_2] = [M_1] + [M_3]$

if $\exists$ exact sequence

$$0 \to M_1 \longrightarrow M_2 \longrightarrow M_3 \longrightarrow 0$$

<u>Our example</u>: B a finite-dimensional algebra/field

$\mathcal{C}$ cat. of finitely-generated B-modules

$K(\mathcal{C})$ = free abelian group with basis

$[L_1], \ldots [L_m]$

$L_1 \ldots L_m$ are all irreducible B-modules

(by Jordan-Hölder thm)

A categorification of irreducible sl(2) representations

E, H, F

$[E,F]=H$, $[H,E]=2E$, $[H,F]=-2F$

V_n $(n+1)$-dim irrep of $sl(2)$

Basis $\{v_0, v_1, \ldots, v_n\}$

$Hv_k = (2k-n)v_k$, $Ev_k = (k+1)v_{k+1}$, $Fv_k = (n-k+1)v_{k-1}$

Another basis $\{v^0, v^1, \ldots, v^n\}$

$v^k = \binom{n}{k}^{-1} v_k$

$Hv^k = (2k-n)v^k$, $Ev^k = (n-k)v^{k+1}$, $Fv^k = kv^{k-1}$

n fixed, $Gr(k)$ Grassmannian of k-dim planes in $\mathbb{C}^n$

$$H_k = H^*(Gr(k), \mathbb{Q})$$

$$C_k = H_k\text{-mod}$$

$$C = \bigoplus_{k=0}^{n} C_k$$

$$Fl_{k,k+1} = \left\{ X \subset Y \subset \mathbb{C}^n \ \begin{matrix} \dim X = k \\ \dim Y = k+1 \end{matrix} \right\}$$

$$\begin{tikzcd} & Fl_{k,k+1} \arrow[dl] \arrow[dr] & \\ Gr(k) & & Gr(k+1) \end{tikzcd}$$

$H_{k,k+1} = H^*(Fl_{k,k+1})$ is a module over

$H_k \otimes H_{k+1}$

Define functors

$\mathcal{E}: C_k \longrightarrow C_{k+1}$ $\qquad \mathcal{E}(M) = H_{k,k+1} \otimes_{H_k} M$

$\mathcal{F}: C_k \longrightarrow C_{k-1}$ $\qquad \mathcal{F}(M) = H_{k-1,k} \otimes_{H_k} M$

$\bigoplus_k$: $\quad \mathcal{E}, \mathcal{F}: C \longrightarrow C$

<u>Proposition</u> $\mathcal{E}, \mathcal{F}$ are exact and biadjoint.

$\mathcal{E}, \mathcal{F}$ and $\mathcal{H} = Id^{\oplus(2k-n)}$ satisfy "sl(2) relations"

$$\left.\begin{array}{ll} \mathcal{E}\mathcal{F} \cong \mathcal{F}\mathcal{E} \oplus Id^{\oplus(2k-n)} & \text{if } 2k-n \geq 0 \\ \mathcal{F}\mathcal{E} \cong \mathcal{E}\mathcal{F} \oplus Id^{\oplus(n-2k)} & \text{if } n-2k \geq 0 \end{array}\right\} [E,F]=H$$

$$[\mathcal{H}, \mathcal{E}] \cong \mathcal{E} \oplus \mathcal{E}$$

$$[\mathcal{F}, \mathcal{H}] \cong \mathcal{F} \oplus \mathcal{F}$$

Grothendieck group

$$K(C) \xrightarrow{\simeq} V_n$$

Free modules $\longrightarrow$ basis $\{v_0, \dots, v_n\}$

Simple modules $\longrightarrow$ basis $\{v^0 \dots v^n\}$

$$[H_k] = v_k$$

$$[L_k] = v^k$$

L_k unique simple H_k-module

$$C \xrightarrow{\mathcal{E},\mathcal{F}} C$$

$$\begin{array}{ccc} K(C) & \xrightarrow{[\mathcal{E}],[\mathcal{F}]} & K(C) \\ \cong & & \cong \\ V_n & \xrightarrow{E,\,F} & V_n \end{array}$$

Cohomology rings H_k, $H_{k,k+1}$ are graded.
work with graded H_k-modules.

A - graded ring $\qquad$ A-gmod $\quad$ cat. of graded modules

$K(A\text{-gmod})$ is a $\mathbb{Z}[q,q^{-1}]$-module

$$M \longrightarrow [M]$$

$$M\{1\} \qquad [M\{1\}] = q[M]$$
$$[M\{-1\}] = q^{-1}[M]$$

If A fin. dim alg / field, graded

$$K(A\text{-gmod}) \simeq K(A\text{-mod}) \otimes_{\mathbb{Z}} \mathbb{Z}[q,q^{-1}]$$

$$K(A\text{-mod}) \simeq \langle [L_1], \ldots, [L_s] \rangle$$
free abelian, generated by simple modules.

$$\bigoplus_{k=0}^{n} H_k\text{-gmod} \longrightarrow \text{irrep of } \mathcal{U}_q(\mathfrak{sl}_2)$$

Ariki: Categorification of all irreducible $sl(m)$ representations (and more)

$$H^{aff}_{n,q} \qquad \text{Affine Hecke algebra} \quad q \text{ generic}$$

$$\Big\downarrow \text{quotients}$$

Cyclotomic Hecke algebras
(depend on many parameters)

λ dominant weight $sl(m)$

$$A_\lambda = \bigoplus_{\text{various } n} \left(\text{pieces of cyclotomic Hecke,}\right)$$

V_λ irreducible $sl(m)$ module

$$V_\lambda \simeq K(A_\lambda\text{-mod})$$

Exact functors $\mathcal{E}_i, \mathcal{F}_i : A_\lambda\text{-mod} \longrightarrow A_\lambda\text{-mod}$
biadjoint, $[\mathcal{E}_i] = E_i$, $[\mathcal{F}_i] = F_i$

Indecomposable projective A_λ-modules $\longrightarrow$ Lusztig's basis $(q=1)$ in V_λ

Cyclotomic Hecke algebras

(Ariki, Koike, Broué, Malle)

generators $a_1 \ldots a_n$

relations $\mathcal{H}_n^c$

$$(a_1 - v_1) \ldots (a_1 - v_m) = 0$$

$$(a_i - q)(a_i + q^{-1}) = 0 \qquad 2 \le i \le n$$

$$a_1 a_2 a_1 a_2 = a_2 a_1 a_2 a_1$$

$$a_i a_j = a_j a_i \qquad j - i \geq 2$$

$$a_i a_{i+1} a_i = a_{i+1} a_i a_{i+1} \qquad 2 \le i \le n-1$$

q - generic, v_j - powers of q $\quad v_j = q^{k_j}$

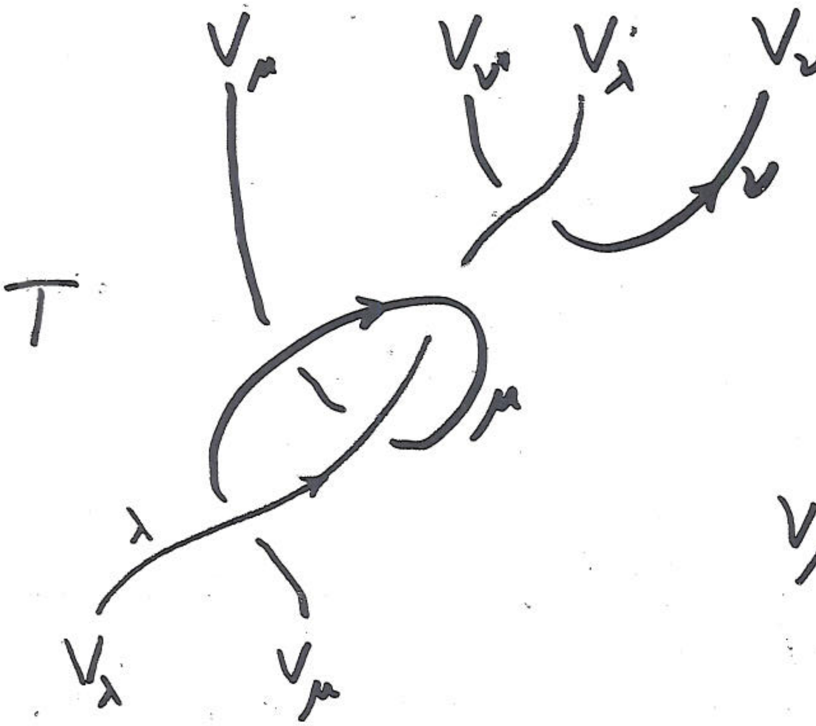
fix $m, k_1, \ldots k_m$.

$$K\Big(\bigoplus_{n \geq 1} \mathcal{H}_n^c\text{-mod} \Big) \simeq V_\lambda \qquad \text{irrep of } gl(\infty)$$

$$\lambda = \omega_{k_1} + \omega_{k_2} + \omega_{k_3} + \ldots + \omega_{k_m}$$

Quantum groups $\longrightarrow$ Invariants of links/tangles
of simple L.A. $\quad \mathcal{U}_q(\mathfrak{g})$ quantum group
$\mathfrak{g}$

(Framed) tangles decorated by irreps of $\mathcal{U}_q(\mathfrak{g})$ / dominant weights λ



$$\begin{array}{c} V_\mu \otimes V_{\nu^*} \otimes V_\lambda \otimes V_\nu \\ \Big\uparrow f(T) \\ V_\lambda \otimes V_\mu \end{array}$$

Functor f: $\quad \mathfrak{g}$-tangles $\longrightarrow \mathcal{U}_q(\mathfrak{g})$ modules

Special case

$\mathfrak{g}$-link $L \longrightarrow f(L) \in \mathbb{Z}[q, q^{-1}]$

Let $\mathfrak{g}$ simply-laced $\quad$ A, D, E

For each $\underline{\lambda} = (\lambda_1 \dots \lambda_n)$

$$V_{\underline{\lambda}} = V_{\lambda_1} \otimes \dots \otimes V_{\lambda_n} \qquad \mathcal{U}_q(\mathfrak{g})\text{-module}$$

want a graded algebra $B_{\underline{\lambda}}$;

an isomorphism

$$K(B_{\underline{\lambda}}\text{-gmod}) \simeq V_{\underline{\lambda}}$$

basis of indecomposable projective modules $\longrightarrow$ Lusztig basis in $V_{\underline{\lambda}}$

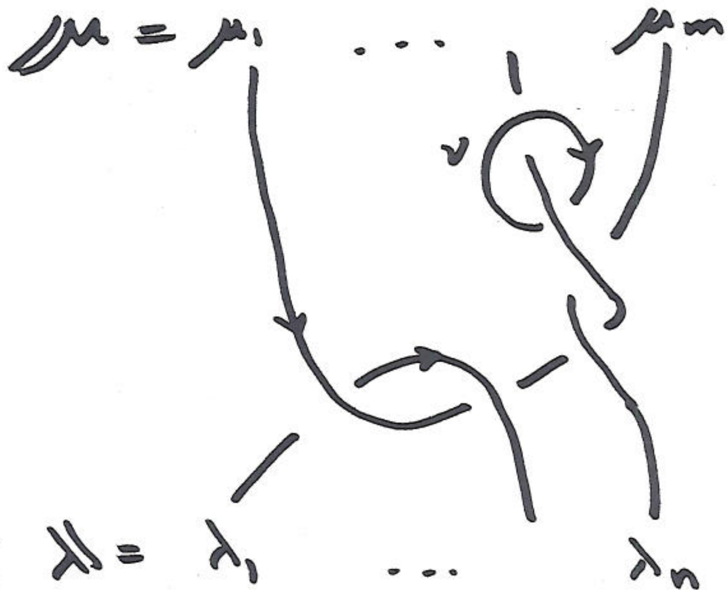
Exact functors

$$\mathcal{E}_\alpha, \mathcal{F}_\alpha : B_{\underline{\lambda}}\text{-gmod} \circlearrowleft$$

α simple root

$$\begin{array}{ccc} K(B_{\underline{\lambda}}\text{-gmod}) & \simeq & V_{\underline{\lambda}} \\ {\scriptstyle [\mathcal{E}_\alpha],[\mathcal{F}_\alpha]}\big\downarrow & & \big\downarrow {\scriptstyle E_\alpha, F_\alpha} \\ K(B_{\underline{\lambda}}\text{-gmod}) & \simeq & V_{\underline{\lambda}} \end{array}$$

For a $\mathfrak{g}$-tangle T



want a functor

$$F(T) : D^b(B_{\underline{\lambda}}\text{-gmod}) \longrightarrow D^b(B_{\underline{\mu}}\text{-gmod})$$

On the Grothendieck group $[F(T)] = f(T)$

For any cobordism $S \subset \mathbb{R}^4$ of $\mathfrak{g}$-tangles T_1, T_2

S oriented surface with corners, decorated by λ's.

a natural transformation

$$F(T_1) \xRightarrow{F(S)} F(T_2)$$

F should be a 2-functor

2-category of $\mathfrak{sl}_2$-tangle cobordisms $\longrightarrow$ 2-category of derived categories

Objects: graded algebras

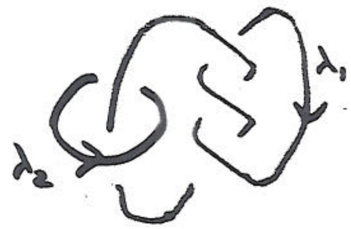
1-morphisms: exact functors
$D^b(B_1\text{-gmod}) \to D^b(B_2\text{-gmod})$

2-morphisms: natural transformation between functors

When tangle is a link L

$\lambda = \mu = \emptyset$

$B_\emptyset = \mathbb{C}$

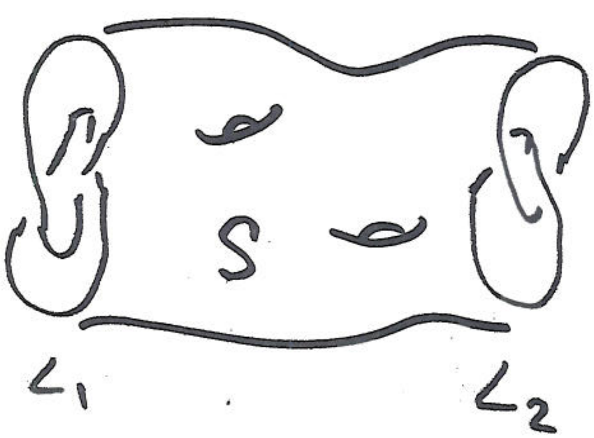


$$F(L) : D^b(\mathbb{C}\text{-gvect}) \longrightarrow D^b(\mathbb{C}\text{-gvect})$$

apply to $0 \to \mathbb{C} \to 0$.

$$F(L)(\mathbb{C}) = \bigoplus_{i,j \in \mathbb{Z}} H^{i,j}(L)$$ (depend on $\mathfrak{g}$-coloring of L)

Link polynomial $f(L) = \sum (-1)^i q^j \dim_{\mathbb{C}} (H^{i,j}(L))$



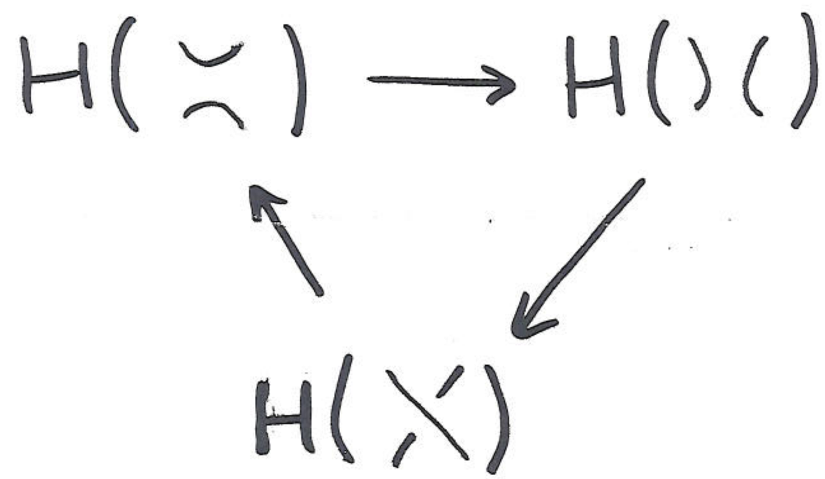
$$H(L_1) \xrightarrow{F(S)} H(L_2)$$

map of homology groups

Such homology theory of links exists for $f(L) =$ Jones polynomial (L)

$\mathfrak{g} = sl_2$, all components of L are labelled by the fundamental (2-dimensional) repres. of $\mathcal{U}_q(sl_2)$.

Built out of exact triangles



$H($  $) \simeq H^*(S^2, \mathbb{Z})$

Extension to tangles

V fund $\mathcal{U}_q(\mathfrak{sl}_2)$ rep.

$$V^{\otimes n} \simeq K(\mathcal{O}_n)$$

$$\mathcal{O}_n = \bigoplus_{k=0}^{n} \mathcal{O}_{k,n-k}$$

$\mathcal{O}_{k,n-k}$ - singular block of $\mathcal{O}$ for $\mathfrak{sl}_n$

integral weight with stabilizer $S_k \times S_{n-k}$

tangle T

$$F(T): D^b(\mathcal{O}_n) \longrightarrow D^b(\mathcal{O}_m)$$

In this approach functors $F(T)$ are hard to define, functor isomorphisms hard to verify.

See J. Bernstein, I. Frenkel, M.K.

A categorification of the Temperley-Lieb algebra ...
arxiv math

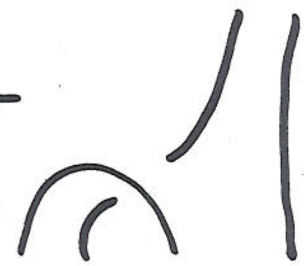
C. Stroppel, recent preprint

Manual approach

even tangles

2m

T



2n

$$f(T): \quad V^{\otimes 2n} \longrightarrow V^{\otimes 2m}$$
$$\quad\quad \cup \quad\quad\quad\quad \cup$$
$$f_0(T): \mathrm{Inv}\left(V^{\otimes 2n}\right) \longrightarrow \mathrm{Inv}\left(V^{\otimes 2m}\right)$$

Construct ring H^n

$$K(H^n\text{-gmod}) \simeq \mathrm{Inv}\,(V^{\otimes 2n}).$$

tangle $T \longmapsto \mathcal{F}(T)$ complex of (H^m, H^n)-bimodules

$\Downarrow$

Functor $F(T): D^b(H^n\text{-gmod}) \to D^b(H^m\text{-gmod})$

$\rotatebox{90}{$\in$}$ M

$$F(T)(M) \overset{\text{def}}{=\!=} \mathcal{F}(T) \otimes_{H^n} M$$

See M.K. A functor-valued invariant of tangles,

Relation between geometrization and categorification

$$\begin{tikzcd}
\text{Representation } V \text{ of algebra } A \arrow[d, "\text{Geometrization}"] \arrow[dd, bend right, "\text{Categorification}"'] \\
V \simeq H^*(X) \arrow[d, "\text{Sheafification}"] \\
\text{Category } \mathcal{C} = \text{Sheaves}(X)
\end{tikzcd}$$

$V \simeq H^*(X)$, X - top. space
A acts by correspondences

Category $\mathcal{C} = \text{Sheaves}(X)$

$V \simeq K(\mathcal{C})$ A acts by functors assigned to correspondences

Sheaves: coherent / constructible / perverse / etc

or $\mathcal{C} \subset \text{Sheaves}(X)$

Example

V_n $(n+1)$ dim irrep of $\mathfrak{sl}_2$

$$\downarrow$$

$$V_n \simeq \bigoplus_{k=0}^{n} H^{mid}\left(T^*Gr(k,n)\right)$$

Nakajima
for $\mathfrak{sl}(2)$

$\mathcal{O}_{k,n}$ structure sheaf of $Gr(k,n) \subset T^*Gr(k,n)$

$$\mathcal{E}xt^*_{T^*Gr}(\mathcal{O}_{k,n}, \mathcal{O}_{k,n}) \simeq H^*(Gr(k,n))$$

$$\text{Coherent sheaves}(T^*Gr(k,n)) \longrightarrow H^*(Gr(k,n))\text{-mod}$$

$$\mathcal{L} \longmapsto R\mathrm{Hom}(\mathcal{O}_{k,n}, \mathcal{L})$$

$C = \bigoplus_k H^*(Gr(k,n))$-mod "sits inside" the derived category of coherent sheaves on quiver variety

$$\coprod_k T^*Gr(k,n)$$