Categories

functors

natural transformations

Grothendieck group

vector spaces or free abelian groups

operators

dimension or Euler characteristic

numbers

<u>Example 1.</u> X - top. space

$$\begin{array}{c} \text{Sheaves}(X) \rightsquigarrow D^b(\text{Sheaves}(X)) \\ \downarrow \\ H^*(X) \text{ or } K(X) \\ \downarrow \\ \chi(X) \quad \text{Euler characteristic} \end{array}$$

need to pick X and $Sh(X)$ carefully, so that

$$\text{Grothendieck group}(Sh(X)) \simeq H^*(X, \mathbb{Z})$$

Example 2. (S. Ariki)

$R_\lambda = \oplus$ blocks of cyclotomic Hecke algebras

$$\begin{array}{c} R_\lambda\text{-mod} \\ \downarrow{\scriptstyle G_0} \\ V_\lambda \\ \downarrow{\scriptstyle \dim} \\ \dim V_\lambda \end{array}$$

irreducible representation of $sl(k)$ of highest weight λ

$$R_\lambda\text{-mod} \xrightarrow{\mathcal{E}_i,\, \mathcal{F}_i} R_\lambda\text{-mod}$$

$\mathcal{E}_i, \mathcal{F}_i$ exact functors

$e_i = [\mathcal{E}_i]$, $f_i = [\mathcal{F}_i]$

G_0 - Grothendieck group

$$V_\lambda \xrightarrow{e_i,\, f_i} V_\lambda$$

e_i, f_i - usual generators of $sl(k)$

relations in $\mathcal{U}(sl(k))$ lift to functor isomorphisms

Grothendieck group

$\mathcal{C}$ - abelian category

$G_0(\mathcal{C})$ - abelian group with generators

$[M]$, $M \in \operatorname{Ob} \mathcal{C}$, relations

$$[M_2] = [M_1] + [M_3] \text{ for each}$$

exact sequence

$$0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow M_3 \longrightarrow 0$$

if $\mathcal{C}$ - category of finite length modules over ring R, then $G_0(\mathcal{C})$ is free abelian with basis

$[L]$ L - simple R-modules

$R = \mathbb{Z}$ basis $[\mathbb{Z}/p]$, p-prime

$R = \mathbb{C}[x]/(x^n)$ $G_0(R\text{-mod}) = \mathbb{Z}$

local ring, has unique simple module $\mathbb{C}$ x acts by 0

Let's try to categorify something
(random attempts at categorification fail in 9 cases out of 10, but pay off in the remaining 1)

Today we'll categorify the polynomial ring $\Bbbk[x]$. It has multiplication and differentiation.

$\Bbbk[x]$ polynomials $\Bbbk$ a field

$\rightsquigarrow$

$A = \mathbb{Z}[x]$ integral structure

∂ and multiplication by x are operators on A, and $\partial x = x\partial + 1$

$$\partial(x f(x)) = x\,\partial(f(x)) + f(x)$$

2 integral lattices

$\mathbb{Z}[x]$ and $\mathbb{Z}\left[\frac{x^n}{n!}\right]$

stable under x and ∂

Have a bilinear form

$$\Bbbk[x] \times \Bbbk[x] \longrightarrow \Bbbk[x]$$

$$(x^n, x^m) = \delta_{n,m}\, n!$$

$$(x \cdot f, g) = (f, \partial(g)) \qquad f, g \in \Bbbk[x]$$

After categorification, want $(,)$ to become $\mathrm{Hom}(,)$ and x, ∂ to become adjoint functors

$$\mathrm{Hom}(XF, G) \simeq \mathrm{Hom}(F, DG)$$

note $(x^n, x^m) = 0$ if $n \neq m$.

also, relative to integral structure

$$\mathbb{Z}[x] \times \mathbb{Z}\left[\frac{x^n}{n!}\right]_{n \geq 1} \longrightarrow \mathbb{Z}$$

$$\underbrace{\{x^n\}}_{\substack{\text{projective} \\ \text{modules}}}, \quad \underbrace{\left\{\frac{x^n}{n!}\right\}}_{\substack{\text{simple} \\ \text{modules}}} \text{ dual bases}$$

For each $n \geq 0$ look for a ring R_n

$$\begin{array}{ccc} X^n & & \dfrac{X^n}{n!} \\ \big\downarrow & & \big\downarrow \\ \text{free module } R_n & & \text{simple module } L_n \end{array}$$

In Grothendieck group $[R_n] = n! [L_n]$.

R_n - local (only one simple module L_n)

if $\dim L_n = 1 \Rightarrow \dim R_n = n!$

You'll find very few such rings

(local, 1-dim simple module, $\dim = n!$)

in the literature:

Cohomology of flag variety

Nilcoxeter algebra

R_n - nil Coxeter algebra / field F

generators $T_1, T_2, \ldots, T_{n-1}$

relations

$$T_i^2 = 0$$

$$T_i T_j = T_j T_i \qquad |i-j|>1$$

$$T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}$$

(compare with group algebra of symmetric group)

Has basis $\{T_w\}_{w \in S_n}$

$T_w = T_{s_1} T_{s_2} \ldots T_{s_k}$ where $w = s_1 s_2 \ldots s_k$ minimal length presentat. as product $s_i = (i, i+1)$

$\dim R_n = n!$

Has unique simple module $L_n = Fv$, $T_i v = 0$

R_n has composition series with $n!$ copies of L_n

$$\underset{\substack{\wr \\ x^n}}{[R_n]} = n! \underset{\substack{\wr \\ \frac{x^n}{n!}}}{[L_n]} \qquad x^n = n! \left(\frac{x^n}{n!} \right)$$

T_i - divided difference operator on polynomials

$f \in F[x_1, \dots x_n]$

$s_i f$ - permute x_i and x_{i+1} in f

$s_1(x_1^3 x_2) = x_2^3 x_1$

$$T_i f = \frac{f - s_i f}{x_i - x_{i+1}}$$

$$T_1(x_1^3 x_2) = \frac{x_1^3 x_2 - x_1 x_2^3}{x_1 - x_2} = x_1 x_2 (x_1 + x_2)$$

$T_i f$ is invariant under $x_i \longleftrightarrow x_{i+1}$ $\Rightarrow$

$T_i^2 f = 0$ $\qquad$ $T_i^2 = 0$

Exercise: check Yang-Baxter relation.

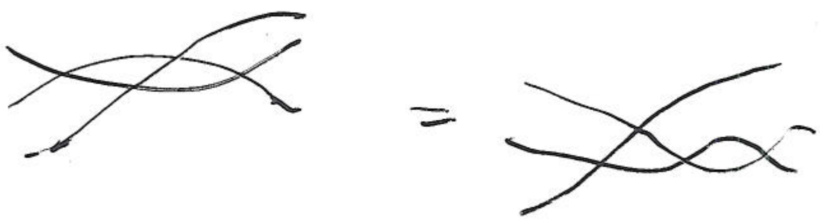
Topological presentation

T_i

$T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}$

 $= 0$

$T_i^2 = 0$

 $= 0$

$T_1 T_2 T_1 T_2 \qquad T_2 T_1 T_2 T_2$

non-minimal presentation $\longrightarrow 0$

$$\mathcal{C} \underset{D}{\overset{X}{\rightleftarrows}} \mathcal{C}$$

adjointness

$$\mathrm{Hom}(XM_1, M_2) = \mathrm{Hom}(X_1, DM_2)$$

$$X(R_n) \simeq R_{n+1}$$

$$x^n \longrightarrow x^{n+1}$$

$[X]$ is multiplication by x

$$[X(L_n)] = (n+1)[L_{n+1}]$$

$$D(R_{n+1}) = R_n^{\oplus n+1}$$

$$x^{n+1} \longrightarrow (n+1)x^n$$

$[D]$ is differentiation

$$D(L_{n+1}) = L_n$$

$$\frac{x^{n+1}}{(n+1)!} \longrightarrow \frac{x^n}{n!}$$

$$\mathrm{Hom}_{\mathcal{C}}(R_n, L_m) = \begin{cases} 0, & n \neq m \\ F, & n = m \end{cases}$$

$$\left(x^n, \frac{x^m}{m!}\right) = \delta_{n,m}$$

$$\mathrm{Hom}_{\mathcal{C}}(P, M) = ([P], [M])$$ P-projective

$R_n \subset R_{n+1}$

Induction and restriction functors

$$R_n\text{-mod} \underset{\Delta = \mathrm{Res}}{\overset{X = \mathrm{Ind}}{\rightleftarrows}} R_{n+1}\text{-mod}$$

restriction is an exact functor

Induction - exact since R_{n+1} is a free R_n-module

$$C = \bigoplus_{n \geq 0} R_n\text{-mod}$$

R_n-mod is the category of finite-dimensional R_n-modules

Grothendieck group

$G_0(C)$ is free abelian, basis $[L_n]$, $n \geq 0$

$$C \xrightarrow{X,\, \Delta} C \qquad G_0(C) \xrightarrow{[X],\, [\Delta]} G_0(C)$$

$[X]$ takes $[M] \in G_0(C)$ to $[X(M)]$

$$DX \simeq XD \oplus \mathrm{Id} \qquad \partial x = x\partial + 1$$

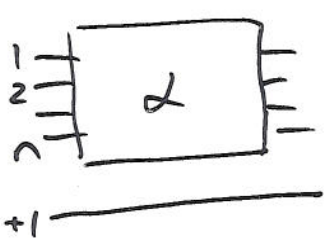
as functors $\quad R_n\text{-mod} \longrightarrow R_n\text{-mod}$

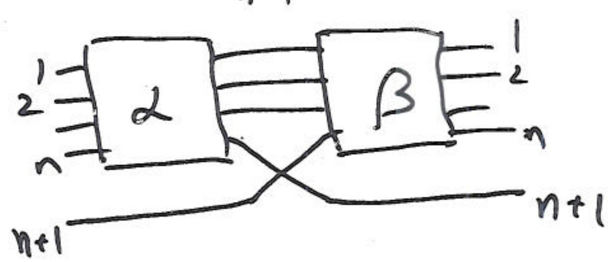
$$DX = \mathrm{Res} \circ \mathrm{Ind} \qquad R_n \longrightarrow R_{n+1} \longrightarrow R_n$$

$$DX(M) \simeq R_{n+1} \otimes_{R_n} M$$

DX is tensor product with R_{n+1}, viewed as an R_n-bimodule

$$R_{n+1} \simeq \underset{\substack{\uparrow \\ \text{no } T_n}}{R_n} \oplus \underset{\substack{\wr\| \\ R_n \otimes_{R_{n-1}} R_n}}{R_n T_n R_n}$$





R_n-bimodule isomorphism

$$R_{n+1} \simeq R_n \oplus \left(R_n \otimes_{R_{n-1}} R_n\right)$$

$$\Downarrow$$

$$DX \simeq XD \oplus \mathrm{Id}$$

$\alpha, \beta \in R_n$

α, β can exchange $T_1, \dots T_{n-2}$, but not T_{n-1}

Nilcoxeter algebra $T_i^2=0$ $\xrightarrow{\text{Grothendieck group}}$ $\mathbb{Z}\left[\frac{x^n}{n!}\right]$

$\downarrow$

Symmetric group $T_i^2=1$ $\longrightarrow$ Ring of symmetric functions

$\downarrow$

Hecke algebra $(T_i+1)(T_i-q)=0$ q-generic $\longrightarrow$ Ring of symmetric functions

$\downarrow$

Hecke algebra $q^N=1$ $\longrightarrow$ Level one integrable representation of $\widehat{sl}(N)$ [LLT]

Affine Hecke algebra, q-generic $\longrightarrow$ $\mathcal{U}^+(sl(k))^*$ [BZ] Ariki Grojnowski

$\downarrow$ quotient

Ariki-Koike cyclotomic Hecke algebras $\longrightarrow$ All irreducible representations of $sl(k)$ Ariki

0-Hecke algebra $q=0$ $\longrightarrow$ Ring of noncommutative symmetric functions

$V = \mathbb{C}^k$

$$S_n \longrightarrow V^{\otimes n} \longleftarrow GL(V)$$

dual pair

$$\mathbb{C}[S_n] \longrightarrow V^{\otimes n} \longleftarrow \mathcal{U}(gl(k))$$

$$\Big\downarrow \text{quantization}$$

3D topology
Jones, HOMFLY-PT
link polynomials
$q^N = 1 \to$ 3-manifold invariants

$$\mathcal{H}_{n,q} \longrightarrow V^{\otimes n} \longleftarrow \mathcal{U}_q(gl(k))$$

Hecke algebra
q - generic

$$\Big\uparrow \text{Grothendieck group}$$

$$\mathbb{H}_{n,q} \longrightarrow \mathcal{V}^n \longleftarrow \mathbb{U}_q$$

4D topology
link homology
χ = Jones, Homfly-PT.
Categorification at $q^N = 1$ is still a mystery

$\mathcal{V}^n$ - triangulated category

$G_0(\mathcal{V}^n) \simeq V^{\otimes n}$

exact functors act on $\mathcal{V}^n$ lifting the action of Hecke algebra and quantum group on $V^{\otimes n}$

$\mathcal{V}^n = D^b(\bigoplus$ pieces of category $\mathcal{O}$ for $gl(n))$

∞-dim reps of $gl(n)$

$\mathcal{V}^n$ = category of cohomologically constructible complexes of sheaves on $\amalg$ partial flag varieties

Geometrization

Algebra A acts on vector space V

V becomes homology / cohomology / K-theory / equivariant K-theory / etc of some topological space X

generators / basis elements of A act by correspondences

$$\begin{array}{c} Y \\ \cap \\ X \times X \end{array} \qquad\qquad \begin{tikzcd} & Y \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ X & & X \end{tikzcd}$$

$$\begin{tikzcd} H^*(X) \arrow[r, "p_1^*"] & H^*(Y) \arrow[d, "\text{Poincare duality } \simeq"'] & H^*(X) \\ & H_*(Y) \arrow[r, "p_{2*}"] & H_*(X) \arrow[u, "\simeq"'] \end{tikzcd}$$

X, Y closed oriented manifolds

Geometrization can often be lifted to categorification

Geometrization $\longrightarrow$ Categorification

(Co)homology or K-theory of X	$\longrightarrow$	derived category of sheaves on X coherent / constructible / equivariant sheaves

$$\begin{tikzcd} K(X) & D^b(sh(X)) \arrow[l] \arrow[dl] \\ H^*(X) & \end{tikzcd}$$

Two types of categorification:

I) Via rings and bimodules

II) Via sheaves and correspondences

I) Work with R-modules. A bimodule N gives a functor

$$R\text{-mod} \xrightarrow{N \otimes} R\text{-mod}$$
$$M \longmapsto N \otimes_R M$$

if N is projective/flat as right R-module, $\Rightarrow$ $\otimes$ with N is exact. Get large supply of exact functors. Sometimes need to pass to complexes of modules and bimodules to get the functors we want

II) Work with sheaves on X. There are almost no interesting exact functors on $Sh(X)$. Pass to $D^b(Sh(X))$ right away.

Need enormous amount of bookkeeping:

to describe a functor $D^b(Sh(X)) \circlearrowleft$

choose a sheaf or complex of sheaves N on $X \times X$

$$\begin{tikzcd} & X \times X \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ X & & X \end{tikzcd}$$

$$D^b(Sh(X)) \longrightarrow D^b(Sh(X))$$

$$\text{"}\mathcal{F} \longmapsto \mathcal{N} \underset{X}{\otimes} \mathcal{F}\text{"}$$

$\mathcal{N}$ - kernel

$$\begin{tikzcd} & p_1^* \mathcal{F} \otimes \mathcal{N} \arrow[dr] & \\ \mathcal{F} \arrow[ur] & & p_{2!}(p_1^* \mathcal{F} \otimes \mathcal{N}) \end{tikzcd}$$

for modules

$$\mathrm{Hom}(\mathcal{N} \underset{R}{\otimes} M_1, M_2) \simeq$$
$$= \mathrm{Hom}(M_1, \mathrm{Hom}_R(\mathcal{N}, M_2))$$

adjoint pairs

$$(p^*, p_*)$$
$$(\mathcal{N} \otimes, \underline{Hom}(\mathcal{N}, \ ))$$
$$(p_!, p^!)$$

Poincaré dual

X measure space

$$K(x,y) \in \mathcal{L}^2(X \times X, \mu_X \times \mu_X)$$

$$\mathcal{L}^2(X) \longrightarrow \mathcal{L}^2(X)$$

$$\begin{tikzcd}
f(x) \arrow[r, mapsto] \arrow[d, "\text{categorification}"'] & \displaystyle\int_X f(x) K(x,y)\, d\mu_x \\
\mathcal{F} \arrow[r, mapsto] & p_{2!}(p_1^* \mathcal{F} \otimes \mathcal{N})
\end{tikzcd}$$