Categorification of quantum groups

$\mathfrak{g}$ simple Lie algebra $\longleftrightarrow$ compact Lie group

$\downarrow$

$\mathcal{U}(\mathfrak{g})$ universal enveloping algebra

$\downarrow$ Drinfeld, Jimbo

$\mathcal{U}_q(\mathfrak{g})$ quantum deformation

$\downarrow$ Beilinson, Lusztig, MacPherson

$\dot{\mathcal{U}}_q(\mathfrak{sl}_n)$ add idempotents 1_λ

$\dot{\mathcal{U}}_q(\mathfrak{g})$ Lusztig

Representation theory of $\mathcal{U}_q(\mathfrak{g})$ $\longrightarrow$ Quantum invariants of knots, links, 3-manifolds

Reshetikhin-Turaev

Witten

...

Quantum groups $\longrightarrow$ 3D TQFT

Conjecture (I. Frenkel, L. Crane) 1993

Categorified quantum groups $\overset{?}{\longleftrightarrow}$ 4D TQFT

Quantum group = Grothendieck ring of some category

Conjecture (I. Frenkel, ≈ 1994)

There exists a categorification of $\dot{\mathcal{U}}_q(sl_2)$.

Lusztig canonical basis vectors $\longleftrightarrow$ special objects of categorification, perhaps simple objects.

Motivations:

G. Lusztig Geometric realization of $U_q^-(\mathfrak{g})$

C. Ringel Canonical basis elements $\longleftrightarrow$ simple perverse sheaves

Beilinson-Lusztig-MacPherson, Lusztig $\dot{\mathcal{U}}_q(\mathfrak{g})$

+ motivations from topology

Recent results: see arXiv

A. Lauda — Categorification of $\dot{\mathcal{U}}_q(sl_2)$

A. Lauda, M. K. — Categorification of $\dot{\mathcal{U}}_q(sl_n)$

Categorification of $\mathcal{U}_q^-(\mathfrak{g})$ for any Kac-Moody $\mathfrak{g}$

J. Chuang, R. Rouquier — $sl(2)$ categorifications — 2004

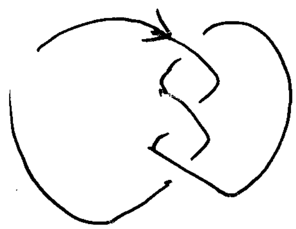
R. Rouquier — Higher representation theory of Kac-Moody algebras, work in progress.

Topological motivations

$\mathfrak{g}$ simple L.A.

V_λ irrep of $\mathcal{U}_q(\mathfrak{g})$ $\qquad \lambda \in X_+$ positive integral weight

L link in $\mathbb{R}^3$, components colored by X_+



λ

$P(L) \in \mathbb{Z}[q, q^{-1}]$

WRT invariant

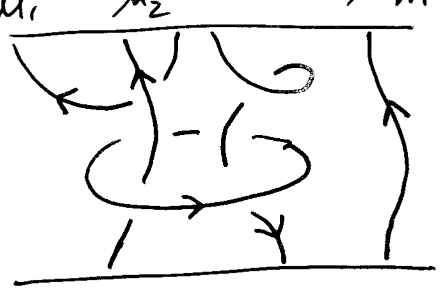
$\mathfrak{g} = sl_2$ (colored) Jones polynomial

$\mathfrak{g} = sl_n$, fund. rep $\qquad$ specializations of HOMFLYPT polynomial

Extends to tangles

$\mu_1 \quad \mu_2 \quad \mu_m$

T



$\lambda_1 \quad \lambda_n$

$$\begin{array}{c} V_{\mu_1} \otimes \ldots \otimes V_{\mu_m} \circlearrowleft \mathcal{U}_q(\mathfrak{g}) \\ \uparrow f(T) \qquad \text{RT invariant} \\ V_{\lambda_1} \otimes \ldots \otimes V_{\lambda_n} \circlearrowleft \mathcal{U}_q(\mathfrak{g}) \end{array}$$

$f(T)$ intertwines $\mathcal{U}_q(\mathfrak{g})$ action. Functor

$\mathfrak{g}$-colored tangles $\longrightarrow$ $\mathcal{U}_q(\mathfrak{g})$-modules

Categorification

$H(L)$ bigraded homology theory of links

$\chi(H(L)) = P(L)$

Euler characteristic = Reshetikhin-Turaev invariant

functorial (link cobordisms)

$g = sl(2)$ M.K. manual construction

J. Bernstein, I. Frenkel, M.K.
C. Stroppel } via highest weight categories for $sl(k)$, all k

P. Seidel, I. Smith via Fukaya-Floer categories quiver varieties

J. Kamnitzer, S. Cautis via coherent sheaves on convolutions of affine Grassmannian

$g = sl(n)$ L. Rozansky, M.K via matrix factorizations

J. Sussan via highest weight categories

C. Stroppel, V. Mazorchuk

C. Manolescu via Fukaya-Floer categories

J. Kamnitzer, S. Cautis coherent sheaves, aff. Grassmannian

M. Mackaay, M. Stosic, P. Vaz via foams in $\mathbb{R}^3$.

T - tangle

$$\begin{array}{c} V_{\underline{\mu}} = V_{\mu_1} \otimes \ldots \otimes V_{\mu_m} \\ \uparrow f(T) \\ V_{\underline{\lambda}} = V_{\lambda_1} \otimes \ldots \otimes V_{\lambda_n} \end{array}$$

$C_{\underline{\lambda}}$ - triangulated category. Grothendieck group

$$K_0(C_{\underline{\lambda}}) \otimes \mathbb{C} \simeq V_{\underline{\lambda}}$$

$K_0(C_{\underline{\lambda}})$ a $\mathbb{Z}[q, q^{-1}]$-module. q becomes additional grading.

tangle T $\longrightarrow$ exact functor $F(T)$

$$\begin{tikzcd} C_{\underline{\mu}} \arrow[r, "K_0"] & V_{\underline{\mu}} \\ C_{\underline{\lambda}} \arrow[u, "F(T)"] \arrow[r, "K_0"] & V_{\underline{\lambda}} \arrow[u, "f(T)"'] \end{tikzcd} \qquad \text{or} \qquad \begin{tikzcd} \mathrm{Inv}(V_{\underline{\mu}}) \\ \mathrm{Inv}(V_{\underline{\lambda}}) \arrow[u] \end{tikzcd}$$

tangle colordism $\longrightarrow$ natural transformation

need categorification of $\otimes$ of irreps :

S. Ariki, H. Nakajima

J. Sussan

H. Zheng Categorification of integrable reps of q. groups, arXiv.

$\mathcal{U}_q(\mathfrak{g})$ acts on $V_{\underline{\lambda}}$, action intertwines $f(T)$

Categorified $\mathcal{U}_q(\mathfrak{g})$ should act on $\mathcal{C}_{\underline{\lambda}}$

$$V_{\underline{\lambda}} = \bigoplus_{\mu} V_{\underline{\lambda}}(\mu) \quad \text{weight decomposition}$$

$$\mathcal{C}_{\underline{\lambda}} = \bigoplus_{\mu} \mathcal{C}_{\underline{\lambda}}(\mu)$$

$$V_{\underline{\lambda}}(\mu) \underset{F_i}{\overset{E_i}{\rightleftarrows}} V_{\underline{\lambda}}(\mu+\alpha_i)$$

$$\mathcal{C}_{\underline{\lambda}}(\mu) \underset{\mathcal{F}_i}{\overset{\mathcal{E}_i}{\rightleftarrows}} \mathcal{C}_{\underline{\lambda}}(\mu+\alpha_i)$$

add idempotents 1_μ of projection onto μ-weight space to $\mathcal{U}_q(\mathfrak{g})$

get $\dot{\mathcal{U}}_q(\mathfrak{g})$

$$1_\lambda 1_\mu = \delta_{\lambda,\mu} 1_\mu \qquad \sum_{\lambda} 1_\lambda = 1$$

$\mathcal{U}_q(sl_2)$ $\qquad E, F, K, K^{-1}$

$$KE = q^2 EK \qquad KF = q^{-2} FK$$

$$EF - FE = \frac{K - K^{-1}}{q - q^{-1}}$$

$$\mathcal{U}_q(sl_2) \longrightarrow \dot{\mathcal{U}}_q(sl_2) = \dot{\mathcal{U}}$$

$1 \longmapsto$ Collection of idempotents $1_n,\ n \in \mathbb{Z}$

$K 1_n = q^n 1_n \quad \to \quad K$ vanishes

$$E 1_n = 1_{n+2} E = 1_{n+2} E 1_n$$

$$F 1_n = 1_{n-2} F = 1_{n-2} F 1_n$$

$$\begin{tikzcd} n+2 \arrow[d, bend left, "F"] \\ n \arrow[u, bend left, "E"] \arrow[d, bend left, "F"] \\ n-2 \arrow[u, bend left, "E"] \end{tikzcd}$$

$$(EF - FE) 1_n = [n] 1_n$$

$$[n] = \frac{q^n - q^{-n}}{q - q^{-1}} = q^{n-1} + \ldots + q^{1-n}$$

$\dot{\mathcal{U}}$ has a basis $\{E^a F^b 1_n\}$ $\quad n \in \mathbb{Z},\ a, b \geq 0$

non-unital ring

$\dot{\mathcal{U}}$ is a category: objects $n \in \mathbb{Z}$,

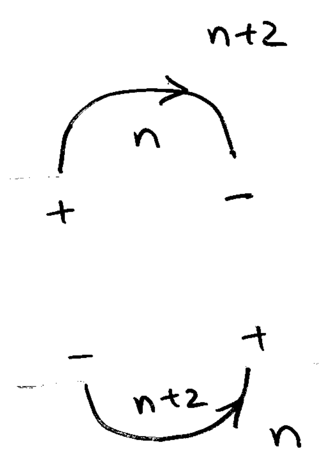
morphisms $n \to m$ are $1_m \dot{\mathcal{U}} 1_n$

In examples of categorifications of $\mathfrak{U}$-representations, E and F become biadjoint functors $\mathcal{E}$ and $\mathcal{F}$

$$\begin{tikzcd} n \arrow[r, bend left, "\mathcal{E}"] & n+2 \arrow[l, bend left, "\mathcal{F}"] \end{tikzcd} \qquad n \xrightarrow{\mathbb{1}_n} n$$

identity functor

biadjointness has a topological interpretation, via isotopies of planar diagrams



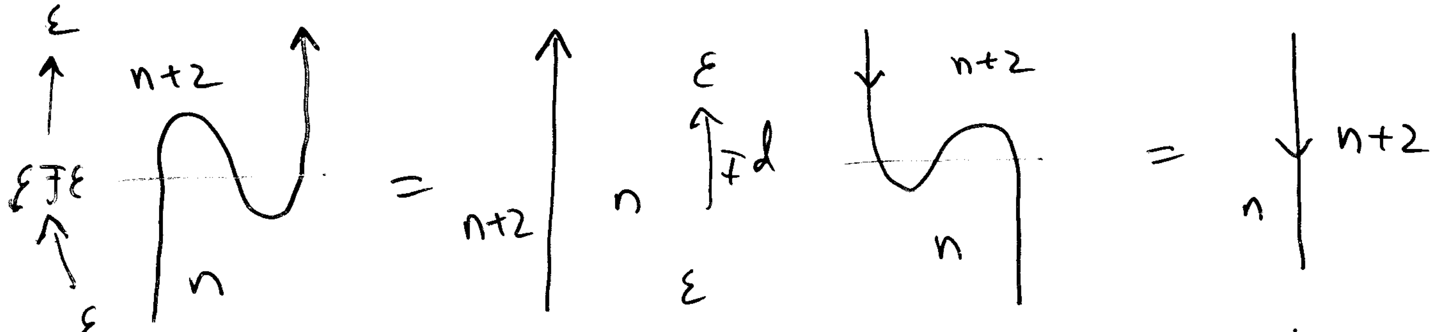
$$\mathbb{1}_{n+2} \leftarrow \mathcal{E}\mathcal{F}\mathbb{1}_{n+2}$$

$$\mathcal{F}\mathcal{E}\mathbb{1}_n \leftarrow \mathbb{1}_n$$

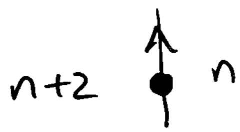
$\mathcal{E}$ is a left adjoint of $\mathcal{F}$

$$\mathrm{Id} \leftarrow \mathcal{E}\mathcal{F}$$

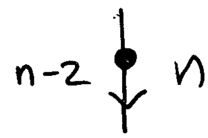
$$\mathcal{F}\mathcal{E} \leftarrow \mathrm{Id}$$



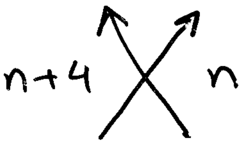
+ opposite orientation $\longrightarrow$ biadjointness



$$\begin{array}{c}\mathcal{E}\mathbb{1}_n\\ \uparrow \\ \mathcal{E}\mathbb{1}_n\end{array}$$

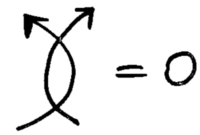


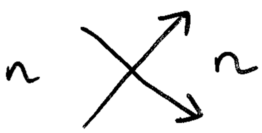
$$\begin{array}{c}\mathcal{F}\mathbb{1}_n\\ \uparrow \\ \mathcal{F}\mathbb{1}_n\end{array}$$



$$\begin{array}{c}\mathcal{E}^2\mathbb{1}_n\\ \uparrow \\ \mathcal{E}^2\mathbb{1}_n\end{array}$$

$$0 \to \mathcal{E}^{(2)} \to \mathcal{E}^2 \to \mathcal{E}^{(2)} \to 0$$





$$\begin{array}{c}\mathcal{F}\mathcal{E}\mathbb{1}_n\\ \uparrow \\ \mathcal{E}\mathcal{F}\mathbb{1}_n\end{array}$$

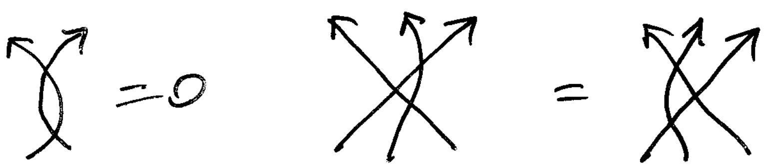
$$(EF - FE)1_n = [n]1_n$$

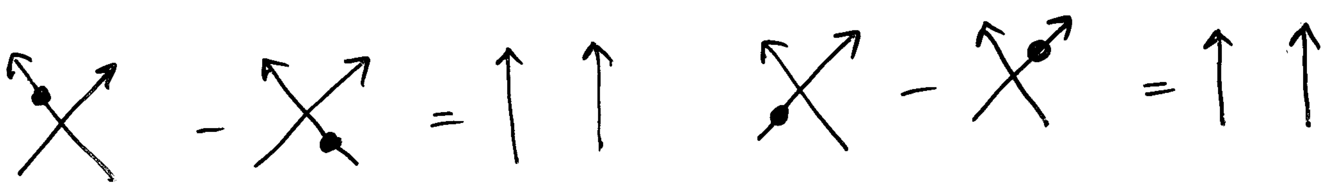
$$EF1_n = FE1_n + [n]1_n$$

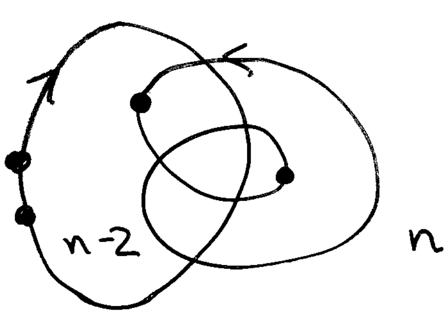
$$\mathcal{E}\mathcal{F}\mathbb{1}_n \simeq \mathcal{F}\mathcal{E}\mathbb{1}_n \oplus \mathbb{1}_n^{[n]}$$

$n \geq 0$

Some relations



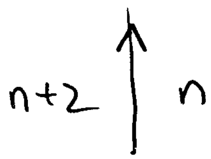




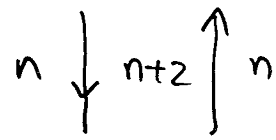
$$\begin{array}{c}\mathbb{1}_n\\ \uparrow \\ \mathbb{1}_n\end{array}$$

Closed diagrams give elements of $\mathrm{End}(\mathbb{1}_n)$

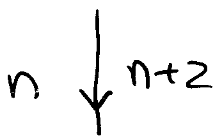
1_n



$$\begin{array}{c} \mathcal{E}\mathbb{1}_n \\ \uparrow \mathrm{Id} \\ \mathcal{E}\mathbb{1}_n \end{array}$$



$$\begin{array}{c} \mathcal{F}\mathcal{E}\mathbb{1}_n \\ \uparrow \\ \mathcal{F}\mathcal{E}\mathbb{1}_n \end{array}$$



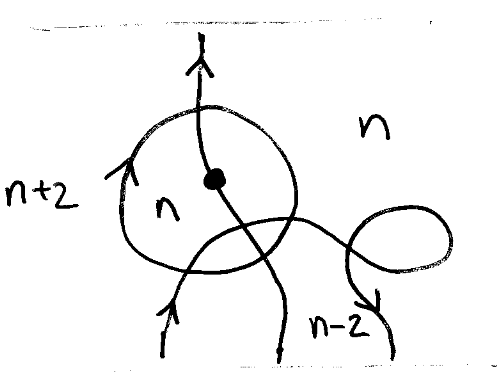
$$\begin{array}{c} \mathcal{F}\mathbb{1}_{n+2} \\ \uparrow \mathrm{Id} \\ \mathcal{F}\mathbb{1}_{n+2} \end{array}$$

2-category $\mathcal{U}'$

objects $n \in \mathbb{Z}$

1-morphisms words $\mathcal{F}\mathcal{E}\mathcal{F}\mathcal{F}\mathcal{E}\mathbb{1}_n$

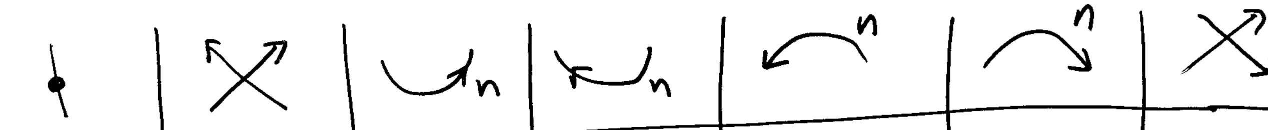
2-morphisms planar diagrams modulo relations



$$\begin{array}{c} \mathcal{E}\mathbb{1}_n \\ \uparrow \\ \mathcal{E}\mathcal{E}\mathcal{F}\mathbb{1}_n \end{array}$$

allow arbitrary isotopies $\longleftrightarrow$ biadjointness of $\mathcal{E}$ and $\mathcal{F}$

Degree of a diagram

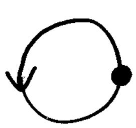


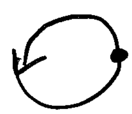
deg	2	-2	$n+1$	$1-n$	$n+1$	$1-n$	0

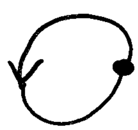
allow arbitrary isotopies of diagrams

all relations are homogeneous

A closed diagram of negative degree $=0$

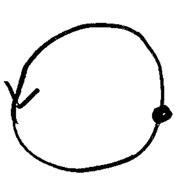
 n $\qquad \deg = 2(n+1)+2 = 2n+4$

if $n<-2$,  $^{n} = 0$

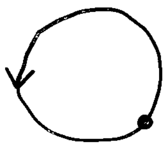
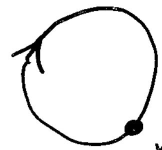
$n=-2$  $^{-2} = -2$

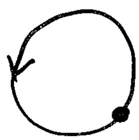
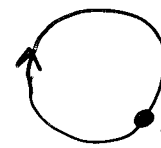
 a $= \Big($  $\Big)^{a}$

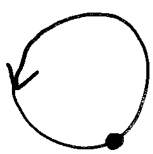
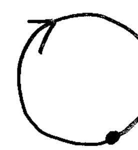
To write the rest of the defining relations in a convenient form, introduce <u>fake bubbles</u>

 a ^{n} $\qquad \deg \geq 0, \quad a<0$

Fake bubbles

 $n \geq 0$, $-n-1+\ell$  $n \leq 0$, $n-1+\ell$ $\quad \ell \geq 0$

 $n \geq 0$, $-n-1$ $= n$  $n \leq 0$, $n-1$ $= n$ $\quad \ell = 0$

$\Big(\sum_{a \geq 0}$  n, $-n-1+a$ $\; t^a \Big) \Big(\sum_{b \geq 0}$  n, $n-1+b$ $\; t^b \Big) = 1$

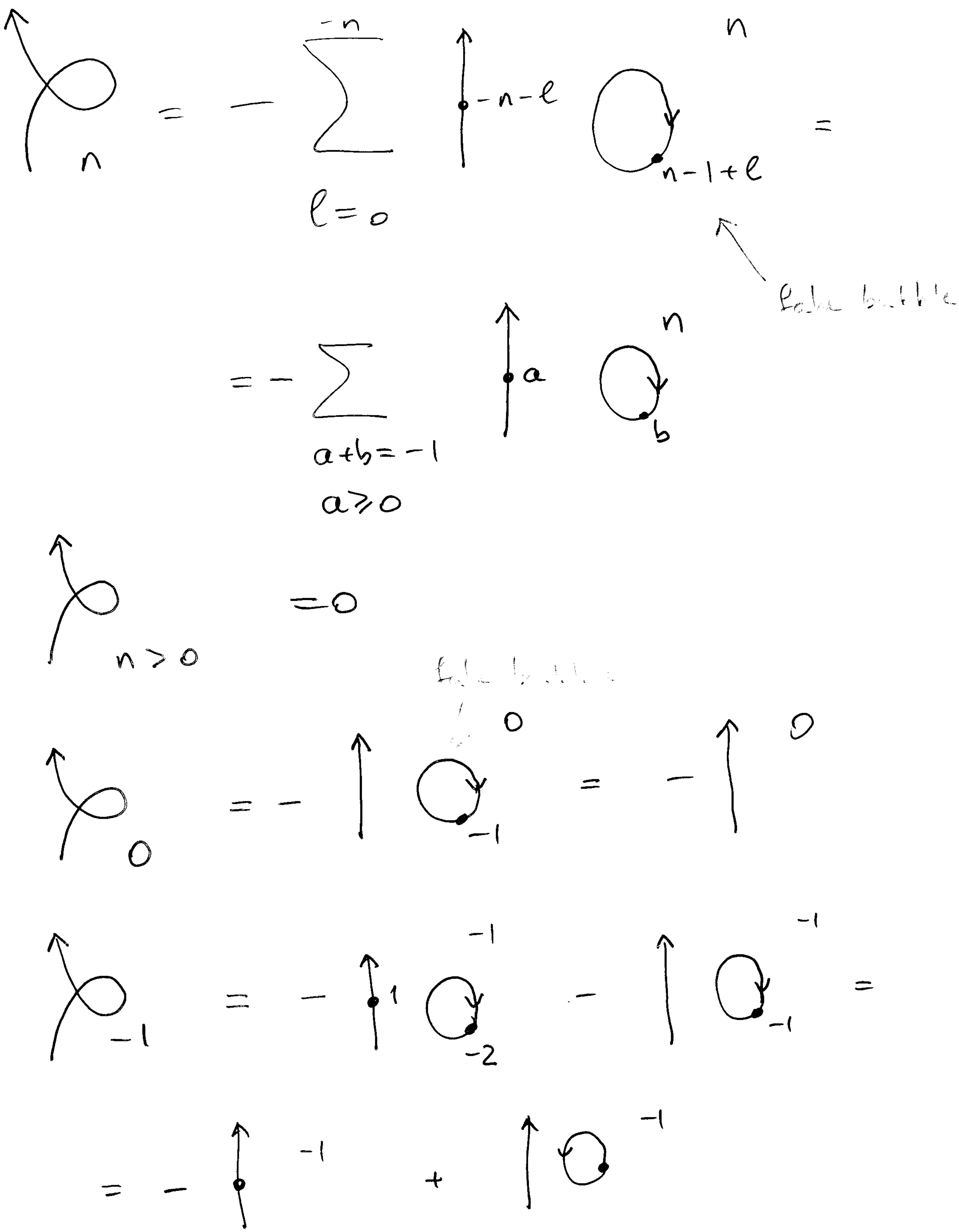
analogous to defining relations in $H^*(Gr(\infty,\infty))$

$\lim Gr(m, m)$

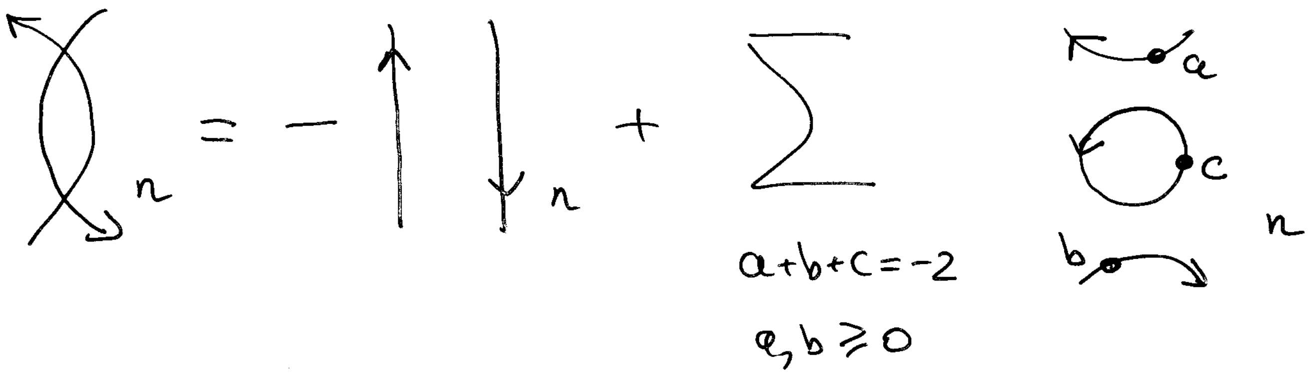
$\mathbb{C}^m \subset \mathbb{C}^{2m}$

$$(1 + x_1 t + x_2 t^2 + \ldots)(1 + y_1 t + y_2 t^2 + \ldots) = 1$$

$$x_1 + y_1 = 0, \quad x_2 + x_1 y_1 + y_2 = 0, \ldots$$

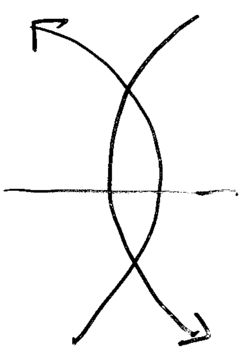


n
−n
−n−ℓ
n−1+ℓ
ℓ=0
fake bubble
a
b
a+b=−1
a≥0
n>0
=0
fake bubble
0
−1
−1
1
−2
−1
−1
−1



if $n \leq 0$

$$\mathrm{Id}_{\mathcal{E}\mathcal{F}\mathbb{1}_n} = -\left(\mathcal{E}\mathcal{F}\mathbb{1}_n \longrightarrow \mathcal{F}\mathcal{E}\mathbb{1}_n \longrightarrow \mathcal{E}\mathcal{F}\mathbb{1}_n \right)$$



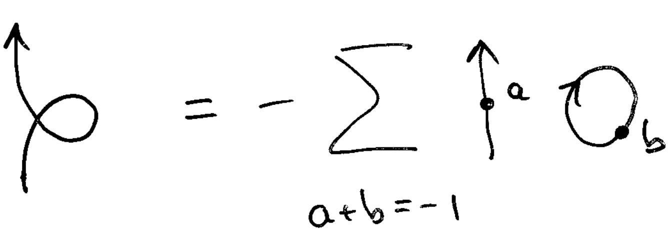
$$\begin{tikzcd}
\mathcal{E}\mathcal{F}\mathbb{1}_n \\
\mathcal{F}\mathcal{E}\mathbb{1}_n \arrow[u] \\
\mathcal{E}\mathcal{F}\mathbb{1}_n \arrow[u] \arrow[uu, bend right, "-\mathrm{Id}"']
\end{tikzcd}$$

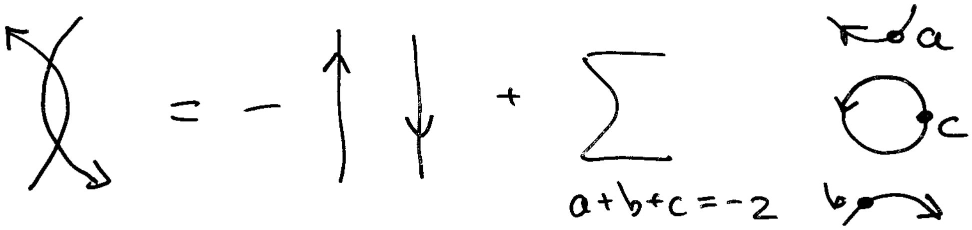
$$\mathcal{F}\mathcal{E}\mathbb{1}_n \simeq \mathcal{E}\mathcal{F}\mathbb{1}_n \oplus \mathbb{1}_n^{-[n]} \qquad n \leq 0$$

$$[E,F]1_n = [n]\,1_n$$

$$[n] = \frac{q^n - q^{-n}}{q - q^{-1}}$$

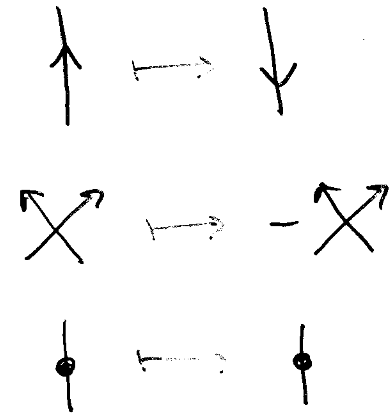
Reidemeister-like relations



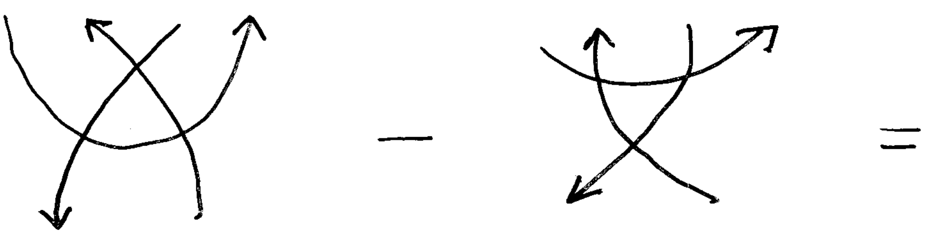


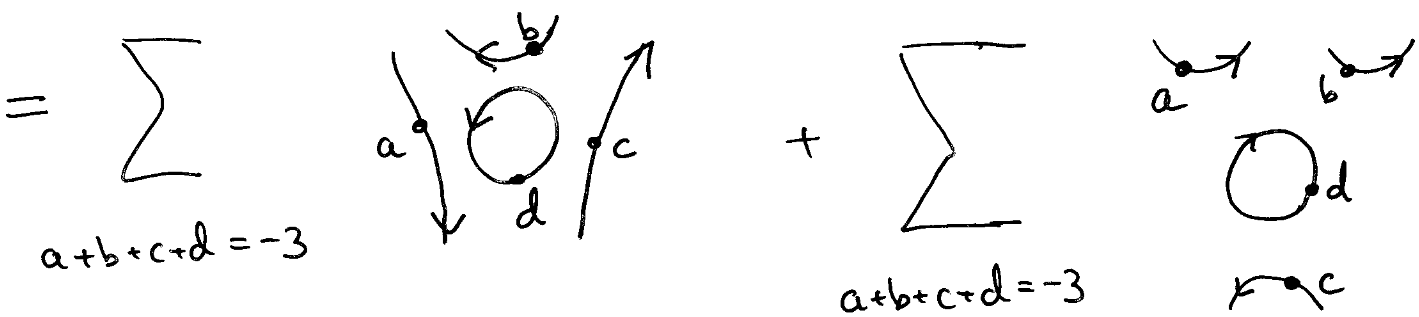
symmetry

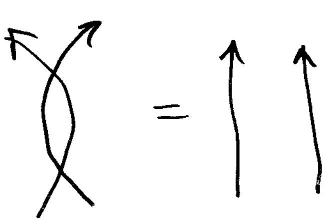
$n \mapsto -n$

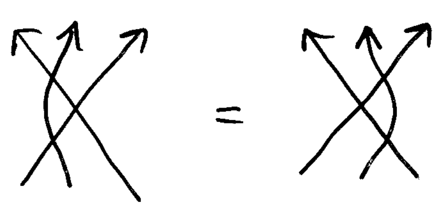


all bubbles in these relations are fake!









<u>Theorem</u> (Aaron Lauda, arxiv 0803.3652)

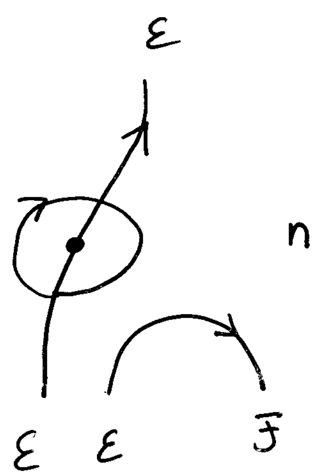
This graphical calculus is consistent and categorifies $\dot{\mathcal{U}}(sl_2)$.

$\dot{\mathcal{U}}(sl_2) \simeq$ Grothendieck ring / category of this 2-category.

Objects: $n \in \mathbb{Z}$

1-morphisms: $m \; \underset{\mathcal{E}}{\overset{+}{\bullet}} \quad \underset{\mathcal{E}}{\overset{+}{\bullet}} \quad \underset{\mathcal{F}}{\overset{-}{\bullet}} \; n$

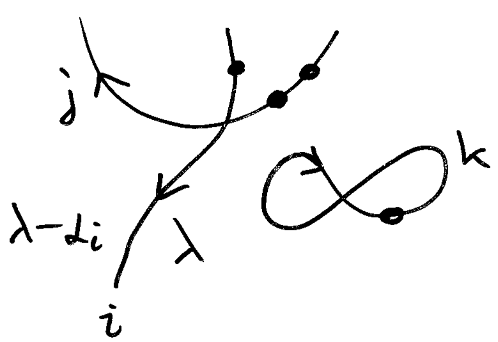
2-morphisms:



add idempotents (Karoubi envelope)
grading

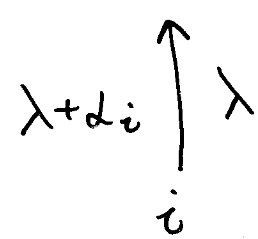
Categorification of $\dot{\mathcal{U}}(\mathfrak{sl}_2)$ + Categorification of U^+ for any Kac-Moody Lie algebra

Categorification of $\dot{\mathcal{U}}(\mathfrak{sl}_n)$



regions are labelled by integral weights of $\mathfrak{sl}_n$

$\lambda \in X$



strands are labelled by simple roots

+ dots on strands

+ local relations.

<u>Theorem</u>: Grothendieck ring of this 2-category is $\dot{\mathcal{U}}(\mathfrak{sl}_n)$.

For any Kac-Moody Lie algebra there is a surjective homomorphism

$$\dot{\mathcal{U}}(\mathfrak{g}) \twoheadrightarrow K_0$$

Grothendieck group

To show that $\twoheadrightarrow$ is an isomorphism, it suffices to check that the graphical calculus is nondegenerate (that obvious spanning sets in $\mathrm{Hom}(\mathcal{E}_{i_1}\mathcal{F}_{j_1}\ldots 1\!\!1_n, \mathcal{E}_{i_1'}\mathcal{F}_{j_1'}\ldots 1\!\!1_n)$ are bases).

A special case of nongeneracy in $\mathfrak{sl}(2)$ case:

For $n \geq 0$, $\mathrm{Hom}(1\!\!1_n, 1\!\!1_n)$ is a polynomial algebra on generators

, ...

Potential applications of quantum group categorifications:

1) Combinatorics of 2-morphisms

2) Categorified $\mathcal{U}$ should act on various categories:

- highest weight categories
- coherent sheaves on quiver varieties — S. Cautis, J. Kamnitzer, T. Licata
- Fukaya-Floer for quiver varieties
- modules over cyclotomic Hecke algebras — J. Brundan, A. Kleshchev
- coherent sheaves on convolution varieties
- Zheng's categorification of tensor products.

3) Higher representation theory of $\mathcal{U}$ — J. Chuang, R. Rouquier

4) Categorification of Witten-Reshetikhin-Turaev invariants