$G_0(\mathcal{C})$ is naturally a biring

(bialgebra over $\mathbb{Z}$)

Nil Coxeter algebra $T_i^2 = 0$

Symmetric group S_n $T_i^2 = 1$

$$\mathcal{C}' = \bigoplus_{n \geq 0} \mathbb{C}[S_n]\text{-mod}$$

$G_0(\mathcal{C}') \simeq$ Ring of symmetric functions

L. Geissinger 1977

$$S_n \longrightarrow GL(n, \mathbb{F}_q)$$

A. Zelevinsky 1981

$S_n \longrightarrow$ Hecke algebra

$$(T_i+1)(T_i-q)=1$$

$$q = e^{\frac{2\pi\sqrt{-1}}{N}}$$

G_0 = Level one integrable representation of $\widehat{s\ell}(N)$

A. Lascoux, B. Leclerc, J. Y. Thibon 1996

Cyclotomic Hecke algebras

Ariki-Koike

Cherednik

G_0 = Any irreducible representation of $s\ell(N)$ or integrable representation of $\widehat{s\ell}(N)$

S. Ariki 1996

$x, \partial \longrightarrow$ generators E_i and F_i of $s\ell(N)$

$\mathfrak{g}$ simple Lie algebra

$\mathfrak{g} = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_-$ triangular decomposition

$U(\mathfrak{g}) \longrightarrow U_q(\mathfrak{g})$ quantum group

Hopf algebra deformation

Drinfeld, Jimbo

$U(\mathfrak{n}_+) \longrightarrow U_q(\mathfrak{n}_+)$

Tower of algebras R'_m, $m \geq 0$

$$G_0\Big(\bigoplus_{m \geq 0} R'_m\text{-mod}\Big) \simeq U_q(\mathfrak{n}_+)$$

A. Lauda, M.K. 2008, arxiv

J. Brundan, A. Kleshchev

R. Rouquier

After categorification, quantization parameter q becomes a grading shift

R $\mathbb{Z}$-graded ring

R-gmod category of graded R-modules

$$M \longmapsto M\{1\} \quad \text{grading shift}$$

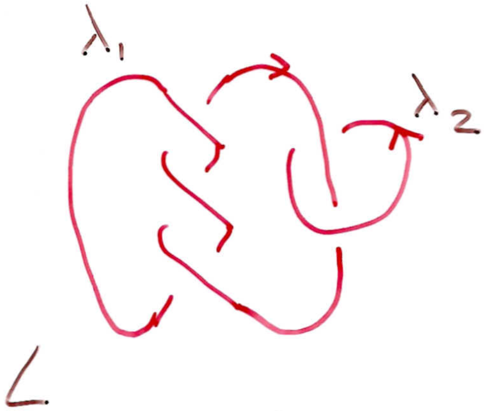
$G_0(R\text{-gmod})$ is a $\mathbb{Z}[q, q^{-1}]$-module

$$[M\{1\}] = q[M]$$

$$[M\{-1\}] = q^{-1}[M]$$

Reshetikhin-Turaev invariants

$\mathfrak{g}$ - simple Lie algebra



Label components of L by irreducible representations of $\mathfrak{g}$

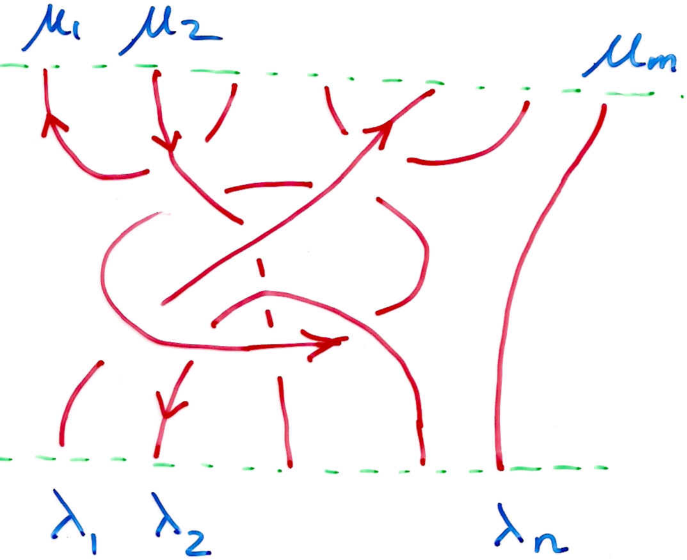
(by positive integral weights)

$P(L, \mathfrak{g}) \in \mathbb{Z}[q, q^{-1}]$

invariant of colored links

comes from representation theory of $U_q(\mathfrak{g})$

Tangles



$T \subset \mathbb{R}^2 \times [0,1]$

each component is labelled by a positive integral weight

Induced orientation on boundary points.

$$\begin{tikzcd}
V_{\mu_1}\otimes V_{\mu_2}\otimes \cdots \otimes V_{\mu_m} \\
V_{\lambda_1}\otimes V_{\lambda_2}\otimes \cdots \otimes V_{\lambda_n} \arrow[u, "f(T)"']
\end{tikzcd}$$

$V_{\mu_1}\otimes V_{\mu_2}\otimes \cdots \otimes V_{\mu_m}$: $\mathcal{U}_q(\mathfrak{g})$-module

$f(T)$: RT invariant of T

$V_{\lambda_1}\otimes V_{\lambda_2}\otimes \cdots \otimes V_{\lambda_n}$: $\mathcal{U}_q(\mathfrak{g})$-module

f is a functor

$\mathfrak{g}$-labelled tangles $\longrightarrow$ $\mathcal{U}_q(\mathfrak{g})$-modules

When T is a link

$f(T) : \mathbb{C} \longrightarrow \mathbb{C}$

a scalar, $f(T) \in \mathbb{Z}[q, q^{-1}]$

$$\begin{array}{c} C_{\mu_1, \ldots, \mu_m} \\ \Big\uparrow F(T) \\ C_{\lambda_1, \ldots \lambda_n} \end{array}$$

Categorification of $f(T)$ must be a functor.

Grothendieck group

$$G_0(C_{\lambda_1, \ldots, \lambda_n}) \simeq V_{\lambda_1} \otimes \ldots \otimes V_{\lambda_n}$$

or at least

$$G_0(C_{\lambda_1,\dots,\lambda_n}) \supset \mathrm{Inv}(V_{\lambda_1} \otimes \dots \otimes V_{\lambda_n})$$

Categories $C_{\lambda_1,\dots,\lambda_n}$ will be triangulated.

<u>Example</u>: Category of complexes of modules over a ring R modulo chain homotopies

$$\xrightarrow{d} M^i \xrightarrow{d} M^{i+1} \xrightarrow{d}$$

Case $\mathfrak{g} = sl(k)$, each component of T is colored by the fundamental k-dimensional representation

$\lambda_1, \ldots \lambda_n \in \{\omega_1, \omega_{k-1}\}$

Multiple constructions of $C_{\lambda_1, \ldots, \lambda_n}$ are known in this case.

Tangle $T \longrightarrow$ Functor $F(T)$

Tangle cobordism $\longrightarrow$ Natural transformation of functors

Recent developments

- Categorification of arbitrary tensor products $V_{\lambda_1} \otimes \ldots \otimes V_{\lambda_n}$

Hao Zheng arxiv, March 2008

- Categorification of $\dot{\mathcal{U}}_q(\mathfrak{sl}(2))$

Aaron Lauda arxiv, March 2008

- Categorification of $\dot{\mathcal{U}}_q(\mathfrak{sl}(n))$

Aaron Lauda, M.K. arxiv, July 2008

- 2-Kac-Moody algebras

Raphael Rouquier, arxiv, several days ago

L. Crane, I. Frenkel

Categorification of quantum groups and their representations $\overset{?}{\Longrightarrow}$ 4-dimensional topology

$\uparrow$

Representation theory of Hopf algebras and quantum groups $\Longrightarrow$ 3-dimensional topology, invariants of knots, links, and 3-manifolds

Geometric representation theory $\longrightarrow$ Categorification

Algebras and their representations become (co)homology groups, or K-theory, or equivariant K-theory of special varieties X

$\longrightarrow$

Algebras and their representations become Grothendieck groups of categories of sheaves on X

G. Lusztig, D. Kazhdan, H. Nakajima, I. Grojnowski,