Categorification of quantum groups and link invariants

Mikhail Khovanov

Joint Mathematics Meetings
Washington, DC
6 January 2009

Categorification

Louis Crane and
Igor Frenkel

Four-dimensional topological quantum field theory, Hopf categories, and the canonical bases,

Journal of Mathematical Physics, 1994.

Early users:

John Baez, M.K.

WHAT IS CATEGORIFICATION?

(I) LIFTING INTEGERS TO VECTOR SPACES OR FREE ABELIAN GROUPS

V

$a = \dim V \in \mathbb{Z}_+ = \{0, 1, 2, \dots\}$

$b = \dim W$

$a + b$	$V \oplus W$
ab	$V \otimes W$

More structure and information:
Operators between vector spaces.

Linear algebra is a categorification of $\mathbb{Z}_+$

Analogue of $a - b$

$V - W$?

A complex

$$0 \longrightarrow \underset{0}{V} \xrightarrow{\partial} \underset{1}{W} \longrightarrow 0$$

Euler characteristic

$$\chi = \dim V - \dim W = a - b$$

Example M nice topological space / compact manifold

$$\chi(M) \in \mathbb{Z}$$

Does not depend on choice of triangulation

Homology and cohomology groups

$$H_*(M) = \bigoplus_{n \geq 0} H_n(M)$$

$$\chi(M) = \sum_{n \geq 0} (-1)^n \, \mathrm{rk}\, H_n(M)$$

Advantages of homology over Euler characteristic:

- carry more information about M
- defined for any topological space M

- functoriality

A continuous map

$$f: M \longrightarrow N$$

induces a homomorphism

$$H_*(f): H_*(M) \longrightarrow H_*(N)$$

- fine structure:

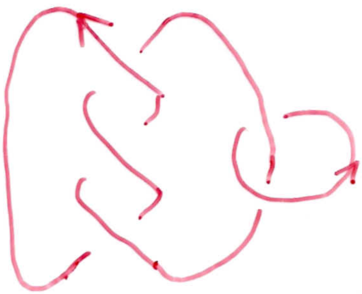
  cohomological operations

  Extraordinary (co) homology theories : K-theory, cobordisms, ...

- leads to modern algebraic topology

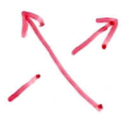
The Jones polynomial

L oriented link in $\mathbb{R}^3$



$$J(L) \in \mathbb{Z}[q, q^{-1}]$$

$$J(\bigcirc) = q + q^{-1}$$

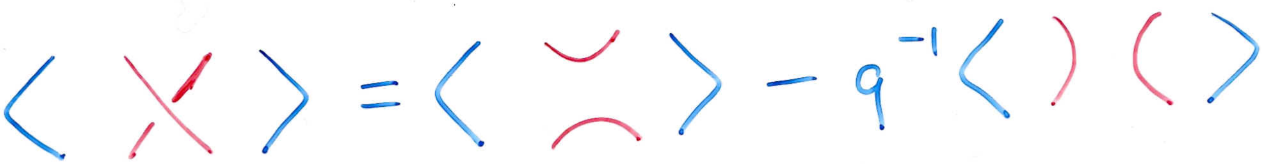
$$q^2 J\left(\right.$$  $$\left. \right) - q^{-2} J\left(\right.$$  $$\left. \right) = (q - q^{-1}) J\left(\right.$$  $$\left. \right)$$

$$J(\underbrace{\bigcirc \bigcirc \ldots \bigcirc}_{k}) = (q + q^{-1})^k$$

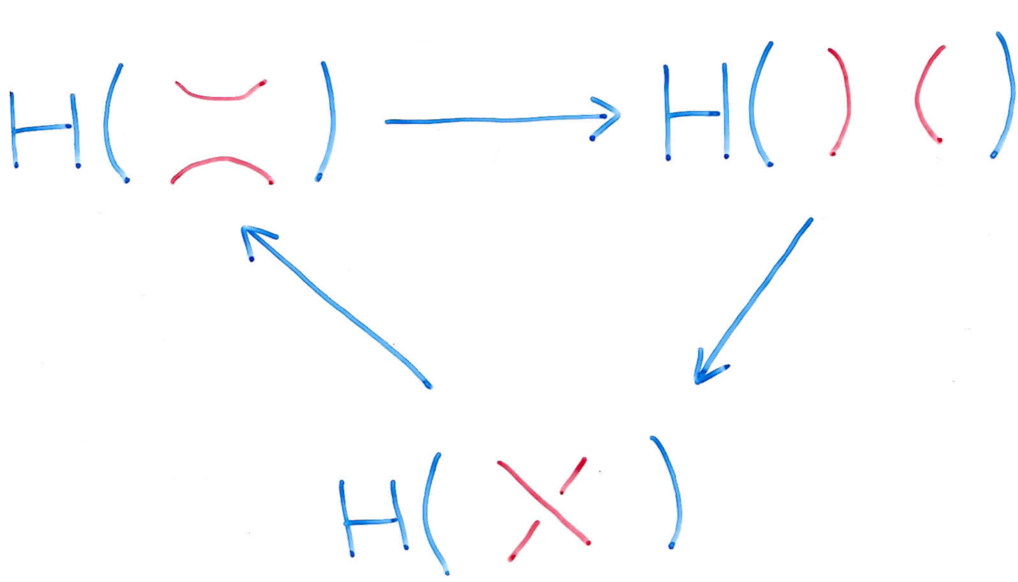
Theorem Jones polynomial admits a categorification

$$H(L) = \bigoplus_{i,j\in\mathbb{Z}} H^{i,j}(L)$$

$$J(L) = \sum_{i,j\in\mathbb{Z}} (-1)^i q^j \, \mathrm{rk}\, H^{i,j}(L)$$



becomes



Kauffman bracket

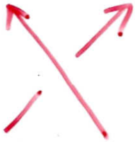
D planar diagram of L

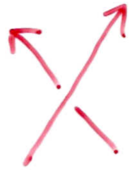
$$\langle D \rangle \in \mathbb{Z}[q, q^{-1}]$$

$\langle$  $\rangle = \langle \asymp \rangle - q^{-1} \langle)\,(\rangle$

$$\langle \underbrace{\bigcirc \; \bigcirc\!\!\!\!\!\!\bigcirc}_{k} \rangle = (q+q^{-1})^{k}$$

$$J(L) = \langle D \rangle (-1)^{x(D)} \, q^{-y(D)+2x(D)}$$

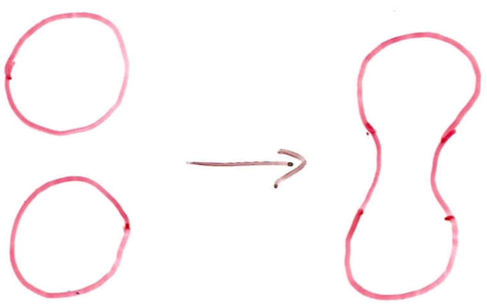
 $x(D)$

 $y(D)$

$$J(\bigcirc) = q + q^{-1}$$

$$H(\bigcirc) = \mathbb{Z}[X]/(X^2) = H^*(S^2, \mathbb{Z})$$

	X		1
basis		in degrees	
	$\mathbb{1}$		-1



multiplication m

comultiplication

$$\Delta(\mathbb{1}) = \mathbb{1} \otimes X + X \otimes \mathbb{1}$$

$$\Delta(X) = X \otimes X$$